The Long-Run Relationship between Real GDP, Money Supply, and Price Level: Unit Root and Cointegration Tests with Structural Changes

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Many studies have attempted to find a stable money demand function, which is a prerequisite, among other things, for monetary targeting to work effectively as a means of monetary policy. An error correction model has increasingly become popular as a means of finding such a stable function; however, its shortcomings have also been pointed out. This paper examines prerequisites for the application of the traditional unit root and cointegration tests and emphasizes the importance of structural changes in the deterministic trend as well as the distinction between deterministic and stochastic cointegration. It presents a time series model with a deterministic trend consisting of multiple linear and nonlinear parts as the appropriate model for Japan's postwar real GDP, money supply (M_1), and the GDP deflator series, which exhibit structural changes. The unit root test of this model produces evidence against the presence of a unit root in the real GDP and the GDP deflator. This indicates that the cointegration between the three variables, which is supported by previous studies on the money demand function, arises from a misspecification of the time series model, and that the instability of the money demand function arises from the non-stationarity of the M_1 series.

Key words: Structural change; Deterministic cointegration; Stochastic cointegration; Money demand function

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I. Introduction

In the 1970s, many countries used money supply measures such as M_1 and M_2+CDs as the intermediate target for monetary policy. It was believed that they could control money supply, and that a stable relationship existed between money supply and the ultimate target of monetary policy such as the general price level and output. In the 1980s, however, money demand functions were found to be unstable, and the effectiveness of the monetary targeting approach started to be questioned. In the meantime, many studies have attempted to find a stable money demand function, including an error correction model, with scant success.

The purpose of this paper is to reexamine the assumptions of a cointegration model on which an error correction model is based, and to investigate whether a stable relationship exists between money supply, output, and price level, as found in earlier studies. The paper particularly emphasizes the importance of treating a deterministic trend properly, which has often been neglected in the past. It also focuses on the issue of structural changes as well as the distinction between the two types of cointegration: that is, deterministic and stochastic cointegration.

Traditional cointegration tests have estimated a linear deterministic trend model without considering the possibility of structural changes in the data. Many macroeconomic and financial data appear to exhibit a kink in the trend due to a structural change in the potential growth rate. Perron (1989) proposed a unit root test for a time series model with a structural change in the deterministic trend. His test indicated the stationarity of some long-run U.S. time series data that had been judged as nonstationary by previous studies. Since then, it has become apparent that the empirical results of many time series tests critically depend on the assumption of a deterministic trend. Kunitomo (1995) proposed a cointegration test for a multivariate time series model with structural changes. He found that if a structural change is assumed in the Japanese growth trend in the early 1970s, postwar times series data for real GDP and real private final consumption expenditure turn out to be stationary around a linear deterministic trend with a structural break, and that the long-run relationship between the two variables depends on maintaining the stable relationship before and after the structural change in the deterministic trend, rather than the cointegration between stochastic trends. This suggests the risk of a "spurious unit root" and a "spurious cointegration" arising from a misspecification of a deterministic trend when the traditional time series model is applied without appropriate caution.

In addition to the problem of structural changes, there is another problem concerning a deterministic trend in the cointegration test. In general, a nonstationary variable with a growth trend can be expressed as the sum of deterministic and stochastic trends. The long-run relationship between deterministic trends is not necessarily identical to that between stochastic trends. Ogaki and Park (1992) have called the identical case between the two relationships "deterministic cointegration" and the unidentical case "stochastic cointegration." Earlier cointegration tests proposed by Engle and Granger (1987) and Johansen (1991) dealt only with a deterministic cointegration model. However, in the case of the money demand function which includes the interest rate in a cointegration space in addition to money supply, output, and price level, the interest rate is the only variable that is not a growth variable. Therefore, it is likely that the relationship between deterministic trends is not identical to that between stochastic trends. In this case, a stochastic cointegration model must be used. Even if a cointegration space does not include the interest rate, it is not appropriate to assume deterministic cointegration as the null hypothesis. In particular, when structural changes exist in the deterministic trend, deterministic cointegration holds only under the strict condition that the pattern and the timing of the structural change are identical among all variables.

In the case of macroeconomic variables with a growth trend such as real GDP and money supply, whether they are stationary or nonstationary, a large part of the variations can be explained by a deterministic trend. Therefore, one must take into consideration the above-mentioned two problems in constructing a model for empirical time series analysis.

This paper presents a time series model that deals with more complicated structural changes than a simple kink in the growth trend or a jump in the level, which are empirically tested in Perron (1989) and Kunitomo (1995). It shows that although previous studies have found that the price variable is an $I(2)$ process, their results depend on the model specification of a deterministic trend. The test results indicate that real GDP and the GDP deflator are stationary and that M_1 is nonstationary. Previous studies that do not consider structural changes have argued that these variables are all nonstationary and cointegrated, and have obtained a stable money demand function by error correction model. These results, however, are likely to be the consequence of two misspecifications—ignoring structural changes in the deterministic trends and assuming *a priori*, deterministic cointegration.[1] The present paper argues that the deterioration in the forecast performance of the money demand function is due to the absence of a stable relationship between the variables implied by cointegration, because money supply is nonstationary while real GDP and price level are stationary.

The rest of the paper is as follows: Section II surveys previous studies on the method of a cointegration test from the viewpoint of structural changes in the deterministic trend and types of cointegration. It then points out that the postwar Japanese time series of M_1 and the GDP deflator exhibit structural changes, and shows by a stability test for a cointegrating vector that a linear deterministic trend is inappropriate for the unit root and cointegration tests. Section III presents a time series model with complicated deterministic trends found in the Japanese M_1 data, and conducts a unit root test that applies the method of Kunitomo (1995). Section IV considers the implications of the test results. The Appendix discusses the

1. It has been pointed out that the stability of an error correction model is spurious and depends on the model specification that involves short-term factors that are not part of the cointegration space. For example, Hess, Jones, and Porter (1994) have emphasized this point by showing the deterioration of the out-of-sample performance of the error correction model (Baba, Hendry, and Starr [1992]), which argued for the existence of a stable U.S. M_1 demand function.

cointegration test of Kunitomo (1992, 1995), which deals with a multivariate time series model with structural changes by generalizing the exogenous variables.

II. Previous Cointegration Tests and Their Problems

A. Classification of Previous Studies

First, I will classify the previous studies on the cointegration test according to the treatment of deterministic trends. Earlier studies such as Engle and Granger (1987), Johansen (1988), and Phillips and Ouliaris (1990) assumed no deterministic trend or a linear deterministic trend. Hansen (1992) and Johansen (1994) extended the earlier studies by introducing a higher-order time trend. A structural change in a deterministic trend was first considered in a unit root test by Perron (1989). He showed theoretically that if the data-generating process has a kink or a jump in the deterministic trend, a unit root test that ignores such possibilities tends to have a bias for accepting the null hypothesis of a unit root, and conducted a unit root test on the model that assumes a change in deterministic trends, using long-run data involving kinks such as the Great Depression and the oil crises. He found that many of the variables that had previously been judged as nonstationary were actually stationary. This stimulated many subsequent studies on the unit root test with an unknown structural change and those which set up as the alternative hypothesis a linear deterministic trend with multiple structural changes. Christiano (1992), Zivot and Andrews (1992), Banerjee, Lumsdaine, and Stock (1992), and Ohara (1994) are such examples. Introducing a structural change to a cointegration test based on a maximum likelihood ranking test, Kunitomo (1995) proved theoretically that the traditional test produces a bias toward reducing the rank if the data-generating process has a structural change. He emphasized the risk of "spurious cointegration." He also proposed a cointegration test for the variables with kinked linear deterministic trends, and presented some of the applied examples.

Next, I will review the tests with deterministic trends, keeping in mind the difference between deterministic and stochastic cointegration. In addition to Ogaki and Park (1992), who first distinguished between the two kinds of cointegration, Johansen (1994) and Hansen (1992) also used a cointegration test that makes the distinction between the two.[2] Directly introducing a deterministic trend to the error correction term, Johansen (1994) proposed a testing method for the following two cases: one for the case in which the cointegrating vector is linearly independent from the exogenous variables (consisting of a constant and a linear trend) and another for the case in which the cointegrating vector is linearly dependent on the exogenous variables. Although he did not use the terminology of stochastic cointegration, his case of linear dependence corresponds to stochastic cointegration. Extending the

2. Hatanaka (1994) distinguished a deterministic from a stochastic cointegration model by different introduction of a deterministic trend into the Wold-type time series model. One method is to add a drift term in each period to the Wold-type model, and another is to add a drift term separately from the MA process. The former expresses deterministic cointegration and the latter stochastic cointegration.

estimation method of Phillips and Hansen (1990), Hansen (1992) proposed a stability test for the cointegrating vector based on the Lagrangian multiplier method. His method has the advantage that it can test the stability of the relationship between deterministic trends in addition to a cointegrating vector.

Table 1 summarizes the various tests by the three kinds of deterministic trend and the two types of cointegration. It also presents a representative study for each category.

Table 1 Cointegration Test by Different Kinds of Deterministic Trend

Cointegration	Deterministic trend		
	Linear	Higher order	Structural change
Deterministic cointegration	Engle and Granger (1987)	Johansen (1994)	Kunitomo (1995)
Stochastic cointegration	Ogaki and Park (1992)	Johansen (1994)	———

Earlier cointegration studies assumed a simple deterministic trend in the statistical model for the sake of theoretical convenience. The reliability of their estimation results tends to diminish if they are applied to data that do not satisfy the assumptions of the model. Many cointegration studies on the money demand function investigate the presence of cointegration between real GDP, money supply, and price level, or between real GDP and real money supply. Many others investigate the cointegration between four to seven variables such as the interest rate, the interest spread, the value of financial assets, and an exchange rate in addition to the above three variables. In all cases, it is necessary to properly deal with the choice of a deterministic trend because real GDP, money supply, and price level are growth variables. However, the previous studies assumed a simple linear deterministic trend and deterministic cointegration. Using the stability test for cointegration proposed by Hansen (1992), this section shows that the assumption of a linear deterministic trend without structural changes is inappropriate for the growth variables. Next, observing the variations in the error correction term from the estimated cointegrating vector, it shows that the assumption of deterministic cointegration is also inappropriate for the case in which non-growth variables such as the interest rate are included in the cointegration space.

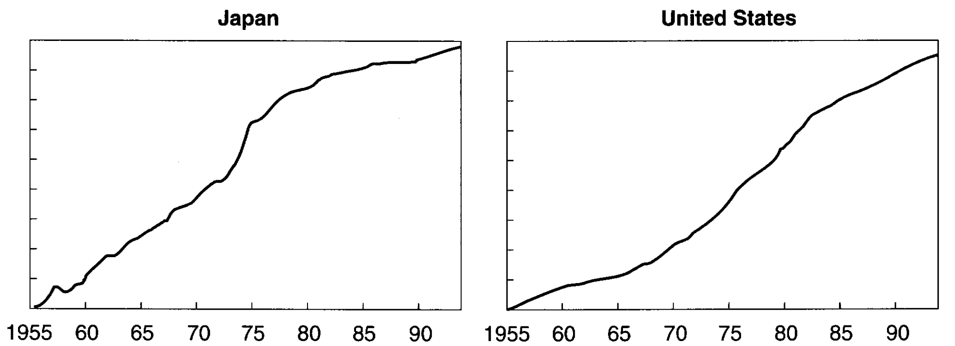
B. Structural Changes

The problem of structural changes in a deterministic trend was first recognized in the unit root test for a univariate time series model. The structural change in Perron (1989) was limited to a jump or a kink in a linear deterministic trend. Applying this model to postwar Japanese macroeconomic data, Soejima (1995) found that real GDP may be stationary under the assumption of structural changes in the linear deterministic trend, but also that such an assumption is inappropriate for nominal variables such as money supply and price level.

To see this, let us look at U.S. and Japanese GDP deflators in Figure 1. This indicates that both data contain a period of smooth and gradual change in the trend, and that therefore a structural change in the linear deterministic trend cannot capture such variables very well. If a change in the linear deterministic trend is imposed on such time series data, it would increase the probability that variations around the trend are judged as nonstationary.

Although one could put forward a deterministic trend of higher degree like a cubic trend, it is hard to justify in terms of economic theory. It is more natural to assume that a historical event has caused a structural change and that the data-generating process has changed as a result. For example, if Japan's GDP deflator can be viewed as having experienced the phases of stable growth, a structural change at the time of the 1973 oil crisis, and a return to stable growth in the 1980s after some years of adjustment, then structural changes may be captured by a model with three deterministic trends: a linear trend, a quadratic trend, and another linear trend, in that order.

Figure 1 Japanese and U.S. GDP Deflators



Note: Logarithm of seasonally adjusted data.

1. An application to money supply

Let us apply the above method to the analysis of money supply. Recent studies have indicated that the logarithms of nominal variables such as money supply and price level are $I(2)$. Even if the logarithms of money supply and price level are $I(2)$, real money supply ($\ln M - \ln P$) may be $I(1)$ as long as the relationship between the two variables remains stable in the long run. With this reasoning, many studies on the money demand function have used a multivariate model under the assumption that real money supply is $I(1)$.[3]

3. Using a single regression equation just like that of Engle and Granger (1987), Stock and Watson (1993) considered a cointegration test for a case involving variables of different integration orders. Johansen (1995) presented an estimation method for a cointegrating vector, which could deal with $I(2)$ processes in a multivariate

However, it may end up losing valuable information by taking differentials twice to make an $I(2)$ process stationary. Figures 2 [1], [2], and [3] show the logarithm of M_1 and its first and second differentials. M_1 is not stationary after the first differential and becomes stationary around the average of zero only after the second differential. However, the first differential is not like a nonstationary process such as a random

Figure 2 [1] Nominal M_1

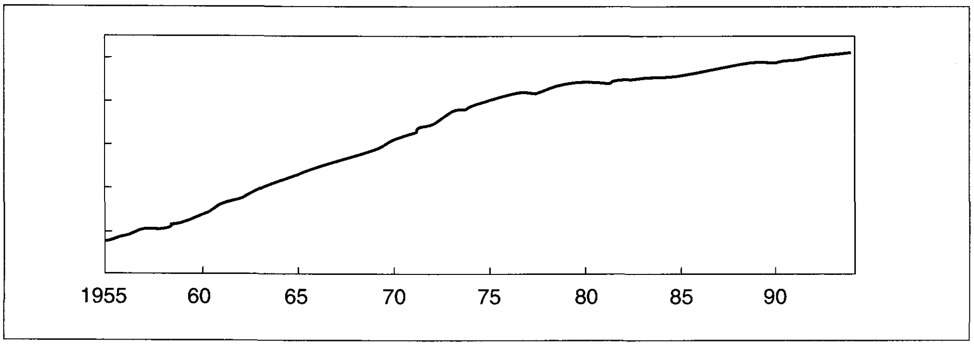


Figure 2 [2] First Differential

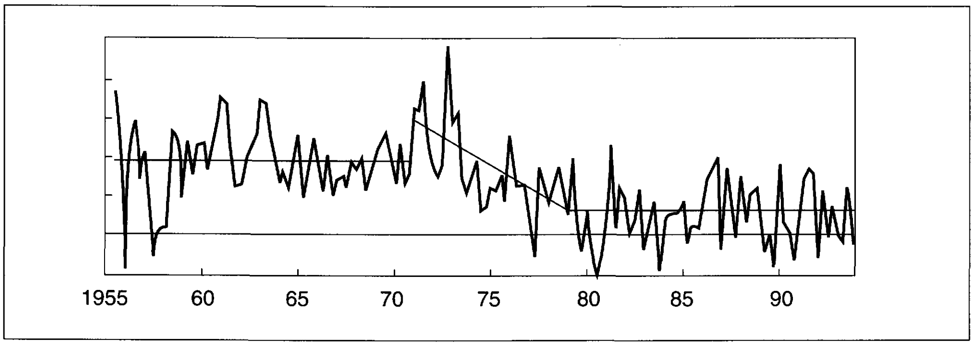
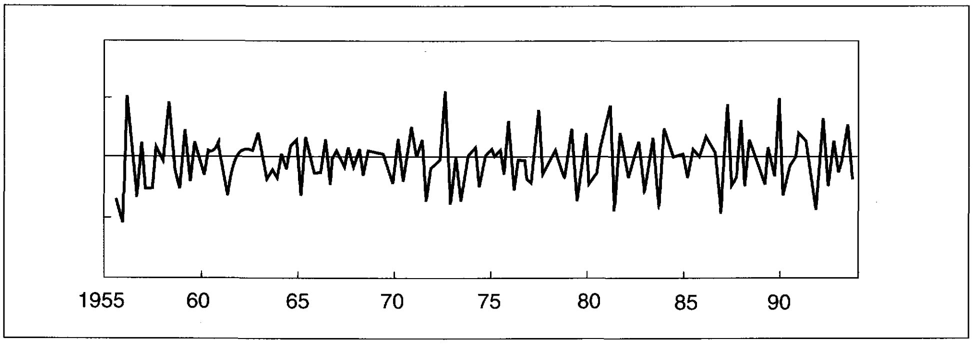


Figure 2 [3] Second Differential



time series model. Juselius (1994) applied the same method to the estimation of a Dutch money demand function under the assumption that nominal money supply is an $I(2)$ process.

variable; instead, just like the GDP deflator with structural changes, it may be viewed as a stationary process with structural changes in the trend as shown in Figure 2 [2]. Therefore, it is possible from the viewpoint of a time series model with structural changes in the deterministic trend that the GDP deflator and M_1 series are stationary or an $I(1)$ process.

2. Stability test for cointegration

Applying Hansen's (1992) stability test for cointegration, this section indirectly examines the validity of a linear deterministic trend. In the case of variables with a linear deterministic trend without structural changes, their linear combination becomes stationary around a constant if deterministic cointegration holds, while it becomes stationary around a linear trend if stochastic cointegration holds. However, in the case of variables with a deterministic trend with structural changes, even if cointegration exists between their stochastic trends, their linear combination based on the cointegrating vector does not necessarily have a constant or linear deterministic trend. If the pattern and timing of structural changes do not coincide, the time series will exhibit a shift in the constant or trend term. As Hansen's (1992) stability test for the cointegrating vector can check the stability of the constant and trend parameters, I will utilize it as a test for structural changes in the deterministic trend.

Hansen's stability test comprises three tests. The first tests the null hypothesis of no changes in the parameters (including the cointegrating vector) during the sample period against an alternative hypothesis of a shift in the parameter at an unknown date. This is appropriate for finding the incident of a sudden structural change. The other two tests assume that each parameter follows a stochastic process and test the null hypothesis of zero variance in the parameters (constant parameters) against an alternative hypothesis of a positive variance (non-constant parameters). They check the possibility that parameters change gradually, and are suitable for testing the stability of a model throughout the sample period. The first test uses SupF as test statistics, and the latter, MeanF and Lc.[4]

This section applies Hansen's test to the following three variables: nominal M_1 (m), real GDP (y), and GDP deflator (p). The ADF test for unit roots, which ignores possible structural changes, supports the non-stationarity of the three variables. The data used are the logarithm of the seasonally adjusted variables, and the sample period is from the second quarter of 1955 to the first quarter of 1994. The estimation results are as follows:

$$m_t = -1.56 - 1.27t + 1.16y_t + 1.18p_t.$$
$$\quad (.549)\quad (.659)\quad (.107)\quad (.124)$$

	p-value
SupF = 229	(.00)
MeanF = 34.0	(.00)
Lc = 1.15	(.01)

4. The MeanF and Lc test are based on the same set of null and alternative hypotheses. However, the specifications of the covariance matrix of the error terms for stochastic parameters (Martingale processes) are different because

The estimated values and standard deviations in parentheses are obtained by the fully modified (FM) estimation method proposed by Phillips and Hansen (1990). The presence of a trend term (t) captures the possibility of stochastic cointegration: if the trend is statistically significant, then it implies stochastic cointegration. In the case of structural changes in the deterministic trend, the parameters of the model may be deemed unstable even if the cointegrating vector is stable because of a change in the trend parameter. In fact, the p-values of the three test statistics reject the stability of the model parameters at a high significance level. However, one cannot be certain of the reason for the rejection: it may be because of structural changes in the deterministic trend or because of a change in the cointegrating vector.

To identify the reason, Figure 3 [1] presents the F-value and MeanF obtained by extending the sample period by one period consecutively. SupF and MeanF imply that at the 1 percent level of significance instability emerged in the early 1970s and after the late 1970s. This matches the timing of trend changes with respect to money supply found in Figure 2 [2] as well as that for real GDP reported in Soejima (1995). Next, assuming structural changes in the deterministic trend in the second quarter of 1972 and the first quarter of 1980, the sample period is divided into the three sub-periods. The same test does not reject the stability of parameters for almost all of the first and third sub-periods, as shown in Figure 3 [2].[5] These results indicate that the parameter instability of the full model results from the model specification that imposes the same linear deterministic trend for the whole sample period. The instability of parameters in the second subsample period may suggest the inadequacy of the assumption of a linear deterministic trend for the GDP deflator and nominal money supply in the 1970s.

C. Stochastic Cointegration

Next, I will point out some important issues in the application of deterministic cointegration to the money demand function. For simplicity, I assume the deterministic trend to be linear and to involve no structural changes. I assume the following simple form of money demand function:

$$m/p = \alpha y + \beta r$$

where m/p is the logarithm of real money supply, y the logarithm of real GDP, and r the nominal interest rate. Each of these variables ($i = m/p, y, r$) consists of a deterministic trend (D_i) and a stochastic trend (S_i). Deterministic cointegration holds if α^* and β^* exist such that $(S_{m/p} - \alpha^* S_y - \beta^* S_r)$ is stationary and $(D_{m/p} - \alpha^* D_y - \beta^* D_r)$ is constant. If deterministic cointegration holds in the true data-generating process, the traditional estimation method of the cointegrating vector produces correct (α^*, β^*). However, if only stochastic cointegration holds, it may not produce

MeanF uses F-values and Lc uses maximum likelihood. SupF and MeanF are obtained excluding the 15 percent at both ends of the sample period.

5. The trend term is not included in the FM estimation for the subsample period of 1955/II–1972/II because the trend term is not significant and the possibility of deterministic cointegration is high.

Figure 3 [1] Stability Test for Cointegration

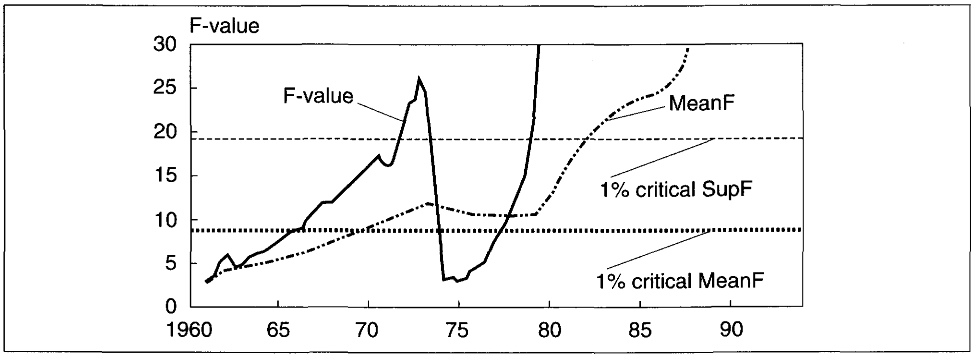
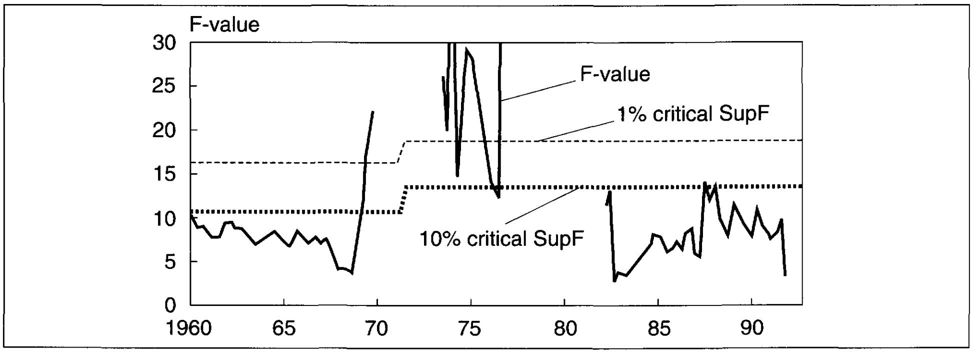


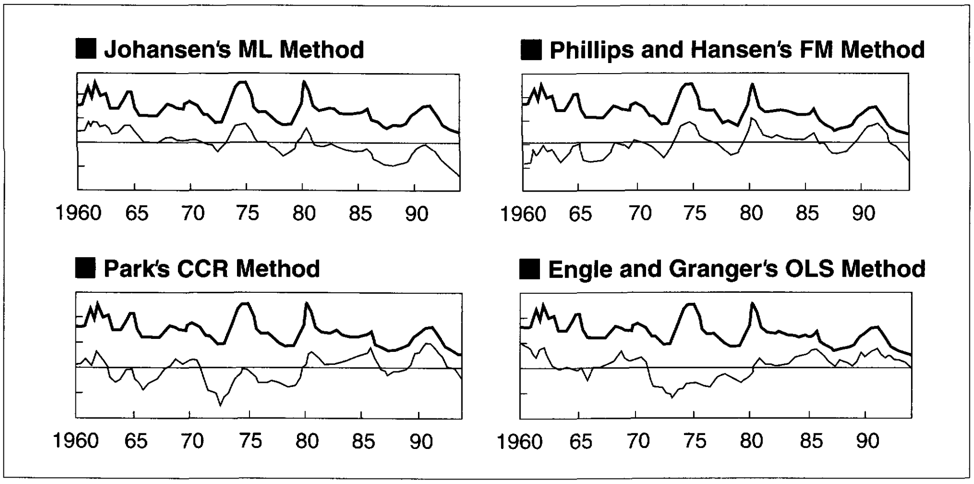
Figure 3 [2] Subsample



the correct (α^*, β^*) but a vector that represents the relationship between the deterministic trends. This is because D_i is greater than S_i in many of the aggregated macroeconomic variables, and $\Sigma y^2 = \Sigma D_y^2$ holds approximately. The estimated β^* is greater than the true β^* because D_r is smaller than $D_{m/p}$ and D_y. Therefore, the error correction term obtained by the traditional estimation method under the assumption of deterministic cointegration tends to strongly reflect the variations in S_r. Figure 4 shows variations in the error correction term and the interest rate in the money demand function estimated by the following four methods: Johansen's maximum likelihood (ML) method, Park's canonical cointegrating regressions (CCR) method, Phillips and Hansen's FM method, and Engle and Granger's OLS method. The data consist of the three variables used in the stability test in the previous section plus the call rate (quarterly average of monthly averages). All graphs show that the variation in the error correction term is closely linked to the variation in the interest rate (represented by the darker lines), which supports the above arguments.[6,7]

6. The fact that the error correction term and the interest rate tend to move parallel to each other when cointegration is applied to a money demand function with the interest rate was pointed out to the author by Professor Yoshihisa Baba (Soka University).
7. Johansen's method produces two cointegrations in a model with three variables. The figure shows the error correction term produced by the cointegrating vector with the larger eigenvalue.

Figure 4 Estimated Error Correction Term and Interest Rate



III. The Model and Empirical Analysis

A. The Model

This section presents a nonstationary time series model that expresses a drift term as a function of time. It can capture such general structural changes as found in M_1 (Section II.B). Using dummy variables DT_{it} and QT_{it} (i = 1, 2), equation (1) expresses a deterministic trend with structural changes at period TB_1 and TB_2 as shown in Figure 5 [1].

$$f(t) = \alpha_0 + \alpha_1 t + (\alpha_2 - \alpha_1)DT_{1t} + \alpha_3(QT_{1t} - QT_{2t}). \qquad (1)$$

$$DT_{it} = \begin{cases} 0 & t \le TB_i \\ t - TB_i & t > TB_i \end{cases} \qquad QT = \begin{cases} 0 & t \le TB_i \\ (t - TB_i)^2 & t > TB_i \end{cases}$$

Using equation (1), a stationary time series model around the trend can be expressed by the following equation (2):

$$\theta(L)(y_t - f(t)) = \varepsilon_t \qquad (2)$$

where $\theta(L)$ is a lag polynomial in the AR model. The absolute value of the roots of $\theta(L) = 0$ is greater than one because of stationarity.

Next, let us consider a case in which equation (2) represents a unit root process. Equation (2) can be transformed into a difference stationary model (3) by using $\theta^*(L) = (1 - L)\theta(L)$:

$$\theta^*(L)\Delta y_t = f(t)^* + \varepsilon_t. \qquad (3)$$

Figure 5 [1] Deterministic Trend: f(t)

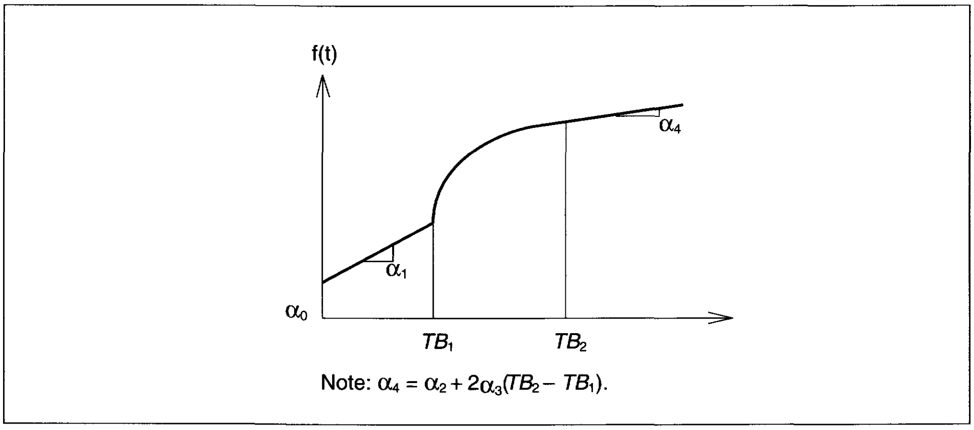
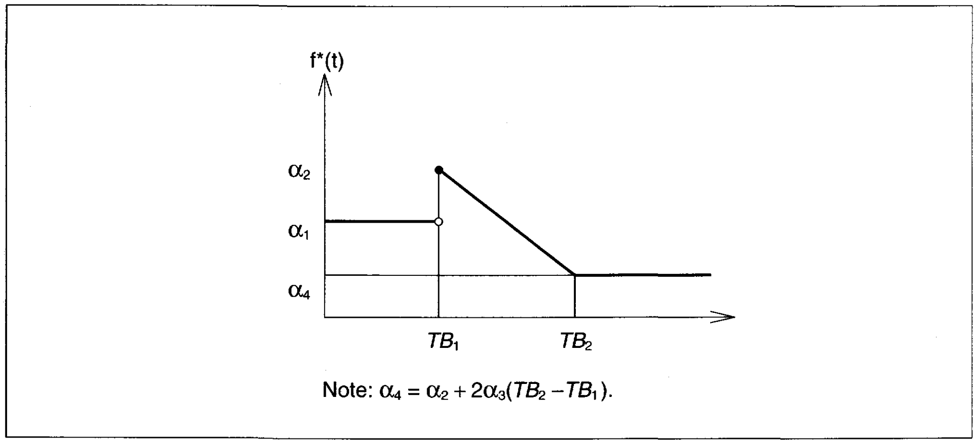


Figure 5 [2] Drift Term: f*(t)



Here the drift term becomes a function of time under the maintained hypothesis of equation (2), while the drift term in the traditional difference stationary model without structural changes is assumed to be constant throughout the entire sample period. The drift term that corresponds to a deterministic trend $f(t)$ becomes $f^*(t)$ in the following equation (4) as shown in Figure 5 [2].

$$f^*(t) = \alpha_1 + (\alpha_2 - \alpha_1)DU_{1t} + 2\alpha_3(DT_{1t} - DT_{2t}). \qquad (4)$$

$$DU_{it} = \begin{cases} 0 & t \leq TB_i \\ 1 & t > TB_i \end{cases}$$

The time series model that incorporates both trend stationary process (2) and difference stationary process (3) can be expressed in the following equation (5):

$$y_t = \alpha_0 + \alpha_1 DU_t + \alpha_2(DT_{1t} - DT_{2t}) + \alpha_3 DT_{2t} + \alpha_4 t + \alpha_5(QT_{1t} - QT_{2t}) \quad (5)$$
$$+ \beta y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \nu_t.$$

With this model, the unit root hypothesis and trend stationarity hypothesis can be expressed in H_0 and H_1:

H_0: $\beta = 1, (\alpha_3, \alpha_4, \alpha_5) = (0, 0, 0)$
H_1: $-1 < \beta < 1, (\alpha_3, \alpha_4, \alpha_5) \neq (0, 0, 0)$.

For the sake of comparison, the deterministic trend and the drift term for the three models of Perron (1989) are shown in Figures 6 [1], [2], and [3]. Equations (1) and (4) for the U.S. GDP deflator are shown in Figure 6 [4].

Figure 6 [1] Jump in Trend

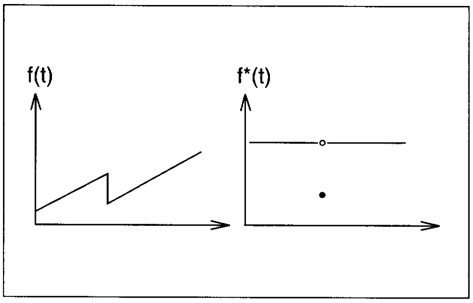


Figure 6 [2] Kink in Trend

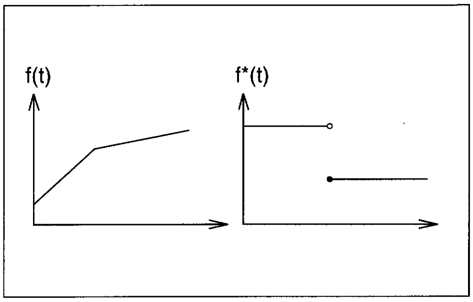


Figure 6 [3] Hybrid Type

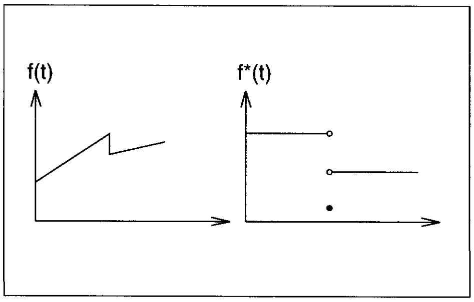
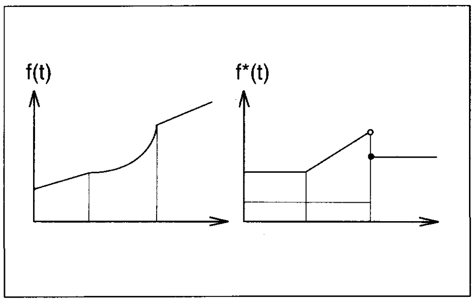


Figure 6 [4] U.S. GDP Deflator



B. Empirical Analysis

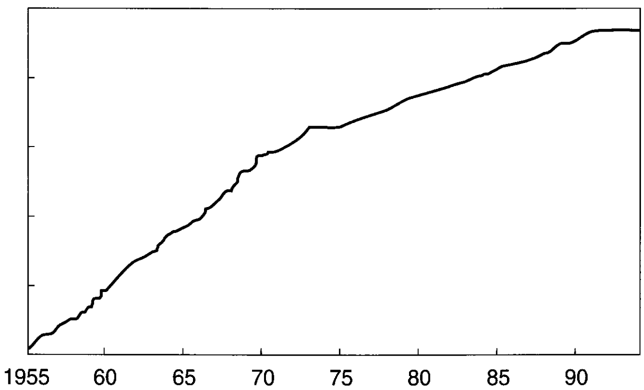
1. Tests of stationarity and non-stationarity

This section conducts a unit root test of real GDP, M_1, and the GDP deflator, using two types of univariate time series models with structural changes in the deterministic trend. Figure 7 shows the logarithm of the real GDP series, which suggests a kink in the growth trend between the earlier high-growth period and the subsequent lower-growth period. Ohara (1994) and Soejima (1995) reject the unit root hypothesis for similar macroeconomic growth variables, using a model with a kink in a linear trend. They are a simple *t*-value test similar to the method of Perron (1989), which does not test the zero restriction on the coefficient of a deterministic trend. Here I conduct a hypothesis test that imposes a zero restriction on the deterministic trend using the following time series model:

$$\Delta y_t = \gamma_0 + \gamma_1 DU(t) + \gamma_2 t + \gamma_3 DT(t) + \beta y_{t-1} + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + v_t. \qquad (6)$$

H'_0: $\beta = \gamma_2 = \gamma_3 = 0$
H'_1: $-2 < \beta < 0, (\gamma_2, \gamma_3) \neq (0, 0)$

Figure 7 Real GDP



Note: Logarithm of seasonally adjusted data.

Kunitomo (1995) presents a cointegration test based on a multivariate time series model, which can handle several kinds of deterministic trends. His cointegration can be applied to a unit root test for a univariate case (Appendix). The present method is an application of a deterministic trend with structural changes (discussed in the previous section) to the model of Kunitomo (1995). The distribution of test statistics is taken from Kunitomo and Sato (1995). The data are seasonally adjusted, and the sample period ranges from the third quarter of 1957 to the first quarter of 1994.[8] The date of a kink in the trend has been tested from the first quarter of 1960 to the fourth quarter of 1979. Table 2 [1] shows the limiting distribution of test statistics

8. As the maximum of nine periods of autoregressive lags is assumed in the model, data from the second quarter of 1955 are actually used.

LR_2.[9] The asymptotic distribution of LR_2 differs depending on the chosen date of a kink in the trend. Therefore, it is estimated for $\delta = 0.1, 0.2, ..., 0.9$.[10]

Table 2 [1] Limiting Distribution of LR_2(ex. 1; $G_0 = 1$, $q = 1$, δ)

$\delta = TB/T$	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99	Mean	S.D.
0.1	2.12	2.65	3.12	3.87	7.50	13.48	15.56	17.82	20.20	8.17	3.89
0.2	2.68	3.18	3.80	4.56	8.55	14.83	16.96	18.80	21.17	9.21	4.13
0.3	3.18	3.74	4.47	5.35	9.53	15.84	18.21	20.35	23.02	10.19	4.26
0.4	3.76	4.45	5.11	5.94	10.00	16.36	18.67	20.77	23.55	10.70	4.24
0.5	3.96	4.57	5.20	6.06	10.21	16.56	18.66	20.67	22.96	10.84	4.16
0.6	3.72	4.31	5.03	5.86	10.11	16.30	18.39	20.65	22.91	10.71	4.17
0.7	3.17	3.82	4.45	5.24	9.33	15.74	17.85	20.05	22.92	10.04	4.21
0.8	2.59	3.20	3.83	4.58	8.55	14.79	16.92	19.30	21.93	9.23	4.16
0.9	2.11	2.70	3.17	3.82	7.45	13.26	15.43	17.52	19.86	8.11	3.83

See Footnote 9 for LR_2(ex. 1; $G_0 = 1$, $q = 1$, δ). Source: Kunitomo and Sato (1995).

Table 2 [2] Limiting Distribution of $\max LR_2$(ex. 1; $G_0 = 1$, $q = 1$)

	0.025	0.05	0.1	0.5	0.9	0.95	0.975	Mean	S.D.
maxLR2	13.46	14.05	14.90	18.57	23.74	25.54	27.56	19.04	3.58

See Footnote 9 for LR_2(ex. 1; $G_0 = 1$, $q = 1$). Source: Kunitomo and Sato (1995).

Figure 8 presents test statistics for the case of five autoregressive lags (a dark line) along with 90 and 95 percent critical values (dotted lines). It shows how they depend on the chosen date of a kink in the trend. For example, if the kink is assumed in 1965–69 or 1971–73, then the unit root hypothesis is rejected at the 10 percent significance level, while it is not rejected for the other periods. This result has been observed irrespective of the length of lags.[11]

If a kink in the deterministic trend is present, the unit root hypothesis becomes less likely to be rejected by a model that ignores the kink. Moreover, even with the model that assumes a kink in the trend, the unit root hypothesis becomes less likely to be rejected if a wrong date for a kink is selected. Therefore, the test that leaves the date of a kink unknown is more appropriate. Kunitomo and Sato (1995) obtain the limiting distribution of $\max LR_2$ under the assumption of an unknown date for a kink in the trend (Table 2 [2]), and conduct a test that does not depend on the selected

9. LR_2 are likelihood ratio statistics. Their asymptotic distribution differs depending on the setting of a deterministic trend, the number of variables in the multivariate time series model, the number of kinks, and their dates. To use the terminology of the Appendix, the present test statistics LR_2 can be expressed as LR_2(ex. 1; $G_0 = 1$, $q = 1$, δ) as the deterministic trend is that of example 1 (ex. 1), the number of the variables (G_0) is one, and the number of kinks (q) is one. δ represents the relative date of a kink (the period of a kink/the whole period = TB/T). As $\max LR_2$ does not depend on δ in the present case, it becomes $\max LR_2$(ex. 1; $G_0 = 1$, $q = 1$).
10. δ represents the relative date of a kink (the period of a kink/the whole period = TB/T).
11. For the lag length of one or two periods, the rejection period is slightly wider than in the case for the lag length of five periods.

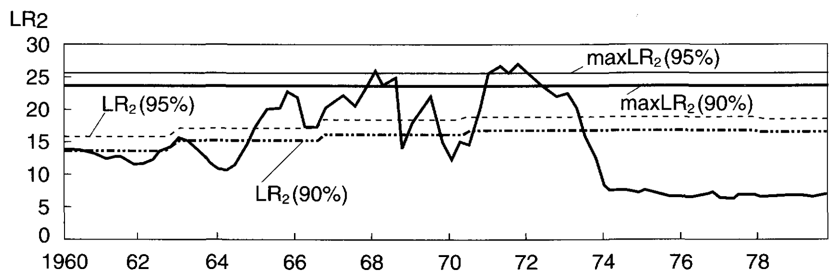
date for a kink. The critical values of maxLR_2 become higher than those of LR_2 because maxLR_2 is the maximum of LR_2 for every date of a kink. As the maximum value of LR_2 is greater than the 95 percent critical value of maxLR_2, the unit root hypothesis is rejected.

Next, I conduct a unit root test for M_1 and the GDP deflator, using equation (5). As is the case with the test for real GDP, the limiting distribution of test statistics differs depending on the selected dates for structural changes TB_1 and TB_2. Therefore, the test assumes the date of a kink to be unknown. The closer the estimated date of structural changes to the true date, the greater the value of test statistics. To avoid an error in selecting the date of structural changes, I take the date of the first structural change (TB_1) to be between the first quarter of 1961 and the fourth quarter of 1974, and the second (TB_2) to be the first quarter of 1978 and the fourth quarter of 1982.

Figure 9 [1] depicts test statistics LR_2 for different combinations of structural change dates for the case of five period lags.[12] Table 3 shows the limiting distribution of maxLR_2. For the GDP deflator, test statistics tend to be greater when the first change is set around 1970–early 1971 or mid-1973, and when the second date is the last half of the assumed period. Test statistics are greater than the 97.5 percent critical value for a wide range of combinations, and the unit root hypothesis is rejected at the 2.5 percent level of significance. As for M_1, test statistics become greater when the first change is in the second quarter of 1972. However, the unit root hypothesis is not rejected at the 10 percent level of significance for any selection of the second date (Figure 9 [2]). This result is robust with respect to choice of lag length.

The logarithms of real GDP and GDP deflator are stationary around the deterministic trend. Therefore, if the date of structural change TB in real GDP is the same as TB_1 in the GDP deflator, then the logarithm of nominal GDP (the sum of the two) will have a deterministic trend and be stationary around the trend. Figure 9 [3] depicts the test statistics for nominal GDP, using model (5) with a deterministic trend. The unit root hypothesis is rejected at the 2.5 percent level of significance.

Figure 8 Test Statistics LR_2(ex. 1; G_0 = 1, q = 1, δ)



See Footnote 9 for LR_2(ex. 1; G_0 = 1, q = 1, δ).

12. Test statistics LR_2 are based on example 2 in the Appendix and can be expressed as LR_2(ex. 2; G_0 = 1, δ_1, δ_2). This has a different limiting distribution from LR_2(ex. 1; G_0 = 1, q = 1, δ). Here δ_1 and δ_2 represent the relative

Figure 9 [1] LR_2(ex. 2; G_0 = 1, δ_1, δ_2) for GDP Deflator

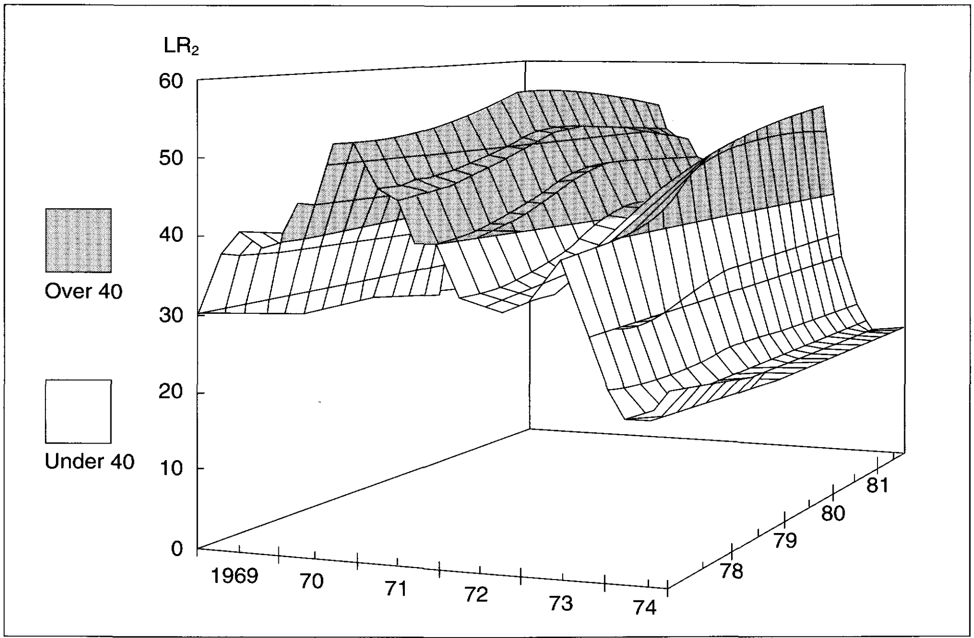
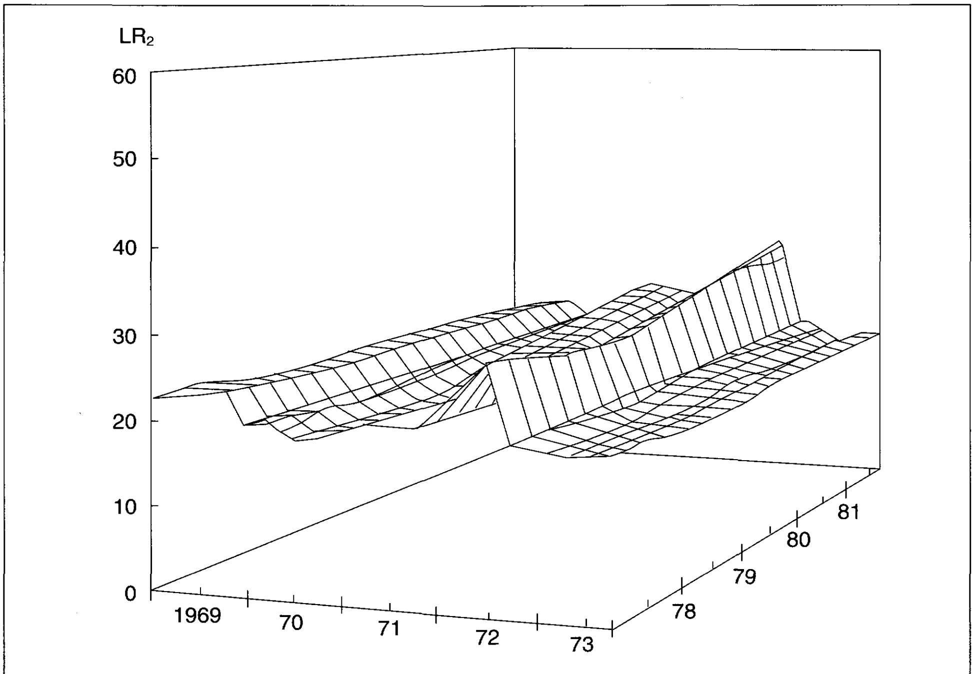


Figure 9 [2] LR_2(ex. 2; G_0 = 1, δ_1, δ_2) LR_2 for M_1



periods of structural change (TB_1/T, TB_2/T).

Figure 9 [3] LR_2(ex. 2; G_0 = 1, δ_1, δ_2) for Nominal GDP

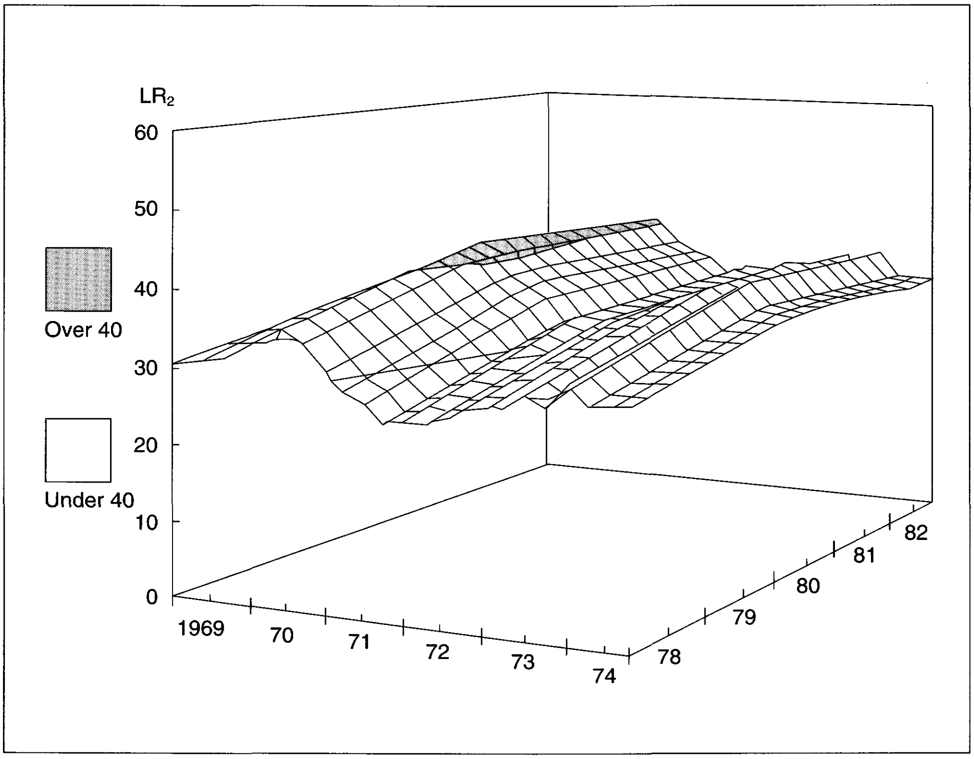


Table 3 Limiting Distribution of $\max LR_2$(ex. 2; G_0 = 1, δ_1, δ_2)

	0.025	0.05	0.1	0.5	0.9	0.95	0.975	Mean	S.D.
$\max LR_2$	23.30	24.26	25.02	30.54	36.74	38.65	40.76	31.07	4.25

See Footnote 12 for LR_2(ex. 1; G_0 = 1, δ_1, δ_2). Source: Kunitomo and Sato (1995).

2. Test results and the selection of a deterministic trend

Although the unit root hypothesis is not rejected for M_1 even if the possibility of structural changes is taken into account, real GDP and the GDP deflator (which were once thought to be nonstationary according to traditional tests) are found to be stationary around the trend. Many of the previous studies have assumed from unit root and cointegration tests that those variables are nonstationary, that there exists a cointegration between the three variables or those plus the interest rate, and that the cointegrating vector provides the parameter values of the long-run money demand function. However, their findings are likely to be the result of misspecification of the model that did not take structural changes into proper consideration. This claim is supported by the empirical analysis of this paper. The instability of the relationship between the three variables is likely to produce instability in the money demand function.

However, it should be remembered that the two types of deterministic trends assumed in the present analysis are based on a general view of economic growth, and are therefore to some extent arbitrary. In decomposing the variation in a time series variable into deterministic and stochastic parts, the former is given from outside a model and thus relies on the judgment of the model builder. Therefore, it is desirable that the hypothesis to be tested should not include the parameters that are related with structural changes.[13] Moreover, exogenous structural changes in a deterministic trend are equivalent to continuous exogenous shocks that form a stochastic trend. In addition, making a deterministic process stationary by introducing many structural changes is equivalent to the assumption that the process itself is nonstationary. In view of these implications, it is important to remember that the present conclusion holds only under the condition that the assumption of a deterministic trend is valid.

3. An extracted stochastic process

The cointegration test of Kunitomo (1995) suffers from the problem that the test for the number of cointegrating vectors, which make a stochastic trend stationary, is not separated from the test for the number of vectors that make a linear combination of deterministic trends a constant.[14] Because of this problem, it cannot deal with stochastic cointegration explicitly. In the above-mentioned univariate model, two types of a deterministic trend are used, and the dates of structural changes are not necessarily identical. As a result, the deterministic trends for the three variables are linearly independent. Therefore, even if they are nonstationary, possible cointegration is limited to stochastic cointegration. Thus, extracting a stochastic process from a deterministic trend used in a univariate model, one can tentatively test the unit root and cointegration hypotheses.

Figure 10 depicts the residuals of the regression of each variable on a deterministic trend, which is a stochastic process around the zero mean.[15,16] The dates of structural changes in the deterministic trend are chosen so that the unit root test is most likely to reject the non-stationarity hypothesis. The estimation period for the call rate is

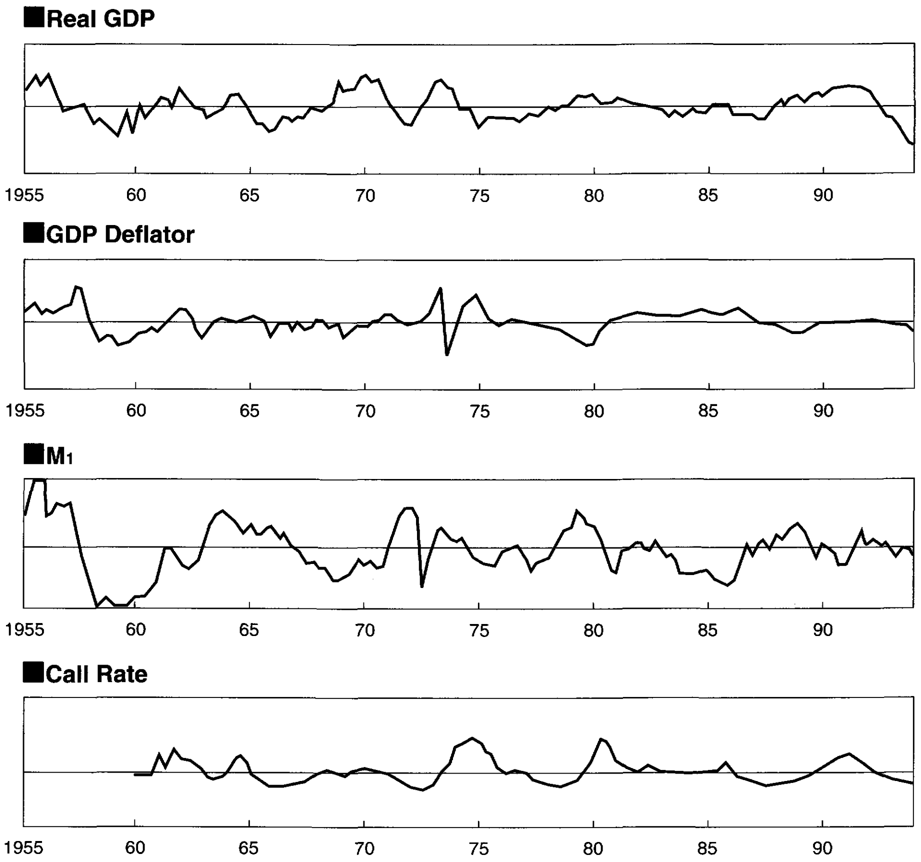
13. The null hypothesis (H_0) and alternative hypothesis (H_1) do not involve parameters α_1 and α_2, which are related with structural changes.
14. See hypothesis H_5 in the Appendix. In dealing with stochastic cointegration in the model of Kunitomo (1995), the consideration of the following hypothesis is possible:

$$\text{H: rank } (B^*) = r \text{ and rank } (\Gamma_2) = r'.$$

where B^* is the matrix of the coefficients of error correction terms when the multivariate time series model is transformed into an error correction form, and Γ_2 is the matrix of the coefficients of the vector of exogenous variables.

15. When a stochastic process is a unit root process or AR process with a root of the characteristic equation close to one, there is a better method than OLS for estimating the coefficient of the deterministic trend. Canjels and Watson (1994) argue that the Prais-Winstern method (one of the four GLS methods) is the best among OLS, the difference transformation, and the four kinds of the GLS method. The present analysis uses OLS because the aforementioned results of the unit root test indicate the stationarity of the two variables.
16. Another method decomposing a variable into trend and cyclical parts is the filter method, which is often used in the business cycle analysis. The Hodrick-Prescott (HP) filter is the most representative example. It differs from the Box-Jenkins time series method in that it does not regard the trend part as deterministic. Therefore, it does not require structural changes to be exogenously given. However, it suffers from the problem of artificial variations in the cyclical part. When it is applied to the same data, the variation around the trend exhibits a very similar movement and the variance tends to become smaller.

Figure 10 Extracted Stochastic Process



Note: Logarithm of seasonally adjusted data, except for the call rate.

from the first quarter of 1960 to the first quarter of 1994, and the period for the others is from the second quarter of 1955 to the first quarter of 1994. The deterministic trend for the call rate is assumed to have no structural changes. M_1 and the GDP deflator exhibit spikes in 1972 and 1973, producing large variations in the residuals around the deterministic trend. They are, however, limited to a short period of time and are much smaller than the variations that would have been produced by the trend without consideration of any structural changes.

Table 4 shows the moments and autocorrelations of the residuals. As the standard deviation of M_1 is greater than that of others, the coefficient of M_1 in the cointegrating vector should be relatively smaller if cointegration holds. As the growth rates of real GDP and real M_1 are similar, the traditional estimation of deterministic cointegration tends to produce a cointegrating vector close to $(\ln Y, \ln M, \ln P)$ = (1, –1, 1), which means the stationary velocity of M_1. These estimations, however, are likely to be affected by the ratio of the deterministic trend in view of the nature of the extracted stochastic process.

Table 4 Moments and Autocorrelations of Residuals

	Moments			Autocorrelations					
	S.D.	Skewness	Kurtosis	(–1)	(–2)	(–3)	(–4)	(–5)	(–6)
Real GDP	0.0213	0.099	–0.12	0.827	0.677	0.493	0.296	0.121	–0.033
GDP deflator	0.0171	0.258	1.38	0.711	0.448	0.256	0.169	0.160	0.150
M_1	0.0415	–0.037	0.09	0.870	0.712	0.550	0.411	0.311	0.215
Call rate	0.0194	0.995	1.12	0.886	0.718	0.518	0.287	0.082	–0.079

If the decomposed stochastic process is nonstationary, then stochastic cointegration may hold. ADF and PP tests are applied to the residuals, which appear to be a stochastic process around the zero mean. Table 5 shows the test results, which reject the non-stationarity hypothesis irrespective of the length of lags in the AR model. This is consistent with the test results of the previous section in the case of real GDP and the GDP deflator. The non-stationarity of M_1, on the other hand, is rejected by the present test, which separates a stochastic process, while it was not rejected by the previous test. The reason may be found in the two-stage estimation method (which estimates the separated stochastic process), and also in the reliability of the size and power of the test, which involves structural changes.[17] Either way, it has been shown that the non-stationarity of those series found in previous cointegration studies resulted from the oversimplification of the deterministic trend.

Table 5 Unit Root Test of Residuals

	ADF test					PP test				
Lag order	$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$
Real GDP	–2.90	–2.63	–3.52	–4.29	–4.53	–2.93	–2.89	–3.14	–3.35	–3.49
GDP deflator	–5.66	–5.98	–5.73	–4.98	–4.48	–5.77	–5.93	–6.01	–6.00	–5.99
M_1	–3.44	–3.88	–4.29	–4.26	–3.71	–3.25	–3.40	–3.53	–3.56	–3.51
Call rate	–2.65	–3.70	–4.36	–5.32	–4.67	–2.74	–3.12	–3.38	–3.60	–3.68

Note: 1% critical value –2.60, 5% critical value –1.95 in ADF test.
1% critical value –2.58, 5% critical value –1.95 in PP test.
Source of critical values from Fuller (1976).

IV. Conclusion

This paper has examined the validity of the cointegration approach to the money demand function, which assumes a stable relationship between output, money stock, and price level. Such a stable relationship is a prerequisite, for example, for the

17. Here the stationarity test of a stochastic process is conducted under the assumption that the assumed deterministic trend is the true model. If this assumption does not hold, the stationarity tests such as the ADF test designed for an independent stochastic process cannot be applied. In this case, it becomes necessary to make adjustments in the distribution of test statistics.

effectiveness of the monetary targeting approach in monetary policy. The paper has pointed out that the traditional time series models suffer from a reliability problem because they do not deal properly with the problem of structural changes. The paper has presented a model that incorporates structural changes and has tested the unit root hypothesis. The results suggest that the variables that have been judged as nonstationary in earlier studies may actually be stationary, and that therefore the error correction models of the money demand function need to be reexamined. In this case, the instability of the estimated money demand function may reflect the fact that the cointegration method is applied to the variables that consist of both stationary and nonstationary ones.

The paper finds that real and nominal GDP series can be understood as a stationary process around a deterministic trend with structural change, while M_1 series are nonstationary with large variations. There are some hypotheses that find the source of the non-stationarity of the M_1 series in the unstable velocity of money. For example, Bordo and Jonung (1987) point out that financial innovation at intermediaries has reduced the amount of money required for a given volume of transactions and has shifted the means of settlement from narrow money such as cash to broad money, reducing the measured velocity of narrow money. Moreover, the growth of financial assets such as large time deposits is affected by many factors (other than the growth of the economy) such as changes in the financial system, the development of direct finance, and changing demographics. Therefore, money does not necessarily grow in line with nominal GDP. If such developments occur randomly rather than at a constant rate, the relationship between money and the real economy will become unstable. Further study of the role of money as an intermediate target in the presence of such instability is thus required.

Appendix: Unit Root and Cointegration Tests with Structural Changes

This section introduces the cointegration test of Kunitomo (1992, 1995) which incorporates structural changes in the exogenous variables of a multivariate time series model. In the case of a univariate model, this test can be applied to the unit root test, as has been done in this paper.

A. The Unit Root and Cointegration Hypotheses in a Multivariate Time Series Model

Kunitomo (1992) starts from the following G-dimensional vector (y_t) multivariate time series model with the vector (z^*_t) of exogenous variables:

$$y_t = \Gamma z^*_t + A_1 y_{t-1} + \ldots.. + A_p y_{t-p} + \nu_t \tag{A.1}$$

where z^*_t represents the $K^* \times 1$ exogenous vector, G the $G \times K^*$ coefficient matrix, A_i the $G \times G$ coefficient matrix, and ν_t the vector of error terms. The G variables in y_t are assumed to be either stationary or nonstationary with unit roots, and to have no exploding elements. This is identical to assuming that the absolute values of the roots of the characteristic function of the autoregressive part in equation (A.1) are equal to or less than one. That is:

$$\left| \lambda^p I_G - \sum_{i=1}^{p} \lambda^{p-i} A_i \right| = 0 \tag{A.2}$$

where $| \lambda_i | \leq 1$ for all eigenvalues $\lambda (i = 1, 2, \ldots, pG)$.

The advantage of this model is that the various time trends can be expressed by exogenous variables. To write the unit root hypothesis, let us rewrite the equation as follows:

$$y_t = \Gamma z^*_t + B_1 y_{t-1} + \sum_{i=2}^{p} B_i \Delta y_{t-(i-1)} + \nu_t \tag{A.3}$$

where Δ is the difference operator and the coefficient matrices B_1 and B_j are given by:

$$B_1 = \sum_{i=1}^{p} A_i, \quad B_j = \sum_{i=j}^{p} A_i, \quad (j = 2, \ldots, p).$$

Now the unit root hypothesis can be expressed using equation (A.3). Let us divide the exogenous vector into two as follows: $z^{*\prime}_t = (z^{*\prime}_{1t}, z^{*\prime}_{2t})$, where z^*_{1t} is a $K^*_1 \times 1$ vector and z^*_{2t} a $K^*_2 \times 1$ vector. The corresponding coefficient matrix is also divided into two matrices: $\Gamma = (\Gamma_1, \Gamma_2)$, where Γ_1 is a $G \times K^*_1$ matrix and Γ_2 a $G \times K^*_2$ matrix. Dividing the elements of the exogenous vector z^*_{2t} such that the zero restrictions are imposed under the unit root hypothesis, we can write the unit root hypothesis as follows:

$$H_2: \Gamma_2 = 0 \text{ and } B_1 = I_G.$$

Now even a complicated trend can be expressed by introducing dummy variables in the exogenous variables. For example, in the case of a univariate model ($G = 1$) of equation (A.2), the unit root null hypothesis (H_0) can be expressed in the form of H_2 with $\Gamma_2 = (\alpha_3, \alpha_4, \alpha_5)$ and $B_1 = \beta$. Similarly, the unit root hypothesis of Dickey and Fuller (1979) can be expressed as the case of z^*_{1t} being a constant and z^*_{2t} being a linear time trend in the univariate model of equation (A.3); and that of Perron (1989) as the case of z^*_{1t} being a constant and a constant dummy and z^*_{2t} being a linear time trend and its dummy.

Next, let us consider the cointegration hypothesis. Deducting y_{t-1} from equation (A.3), we can obtain the following equation (A.4):

$$\Delta y_t = \Gamma z^*_t + B^* y_{t-1} + \sum_{i=2}^{p} B_i \Delta y_{t-(i-1)} + v_t \tag{A.4}$$

which takes the form of a multivariate time series model used in the Johansen test together with the addition of generalized exogenous variables. It is assumed that the rank of the coefficient matrix B* of y_{t-1} is r: that is, rank (B*) = r. Depending on the value of r, the multivariate model of equation (A.1) can take the following three cases:

(1) When $r = 0$, equation (A.1) becomes a multivariate model of differential series $\{\Delta y_t\}$.
(2) When $r = G$, $\{y_t\}$ becomes a stationary series without a unit root.
(3) When $0 < r < G$, the number of cointegration is r.

Until Johansen (1988) formulated cointegration in a multivariate model, it had been a common practice in dealing with nonstationary data to conduct time series analysis to the first differences $\{\Delta y_t\}$ if they are $I(1)$. This corresponds to the case in which rank (B^*) = r = 0 in equation (A.4), or case (1) above. However, B^* is not necessarily a zero matrix and r can take any value between 0 and G.

It can be shown by the use of the characteristic equation that equation (A.1) does not have a unit root if all of the row vectors of B^* are linearly independent (case [2] above).[18] Therefore, the stationary model can be used for the estimation.

The last case, (3), is the one that was considered in Johansen (1988). If the G nonstationary series in $\{y_t\}$ becomes stationary when it is linearly combined by a nonzero matrix B, then the cointegration proposed by Engle and Granger (1987)

18. The characteristic equation of (A.1) can be written with B^* and B_j as follows:

$$A(\lambda) = \lambda^p I_G - \sum_{i=1}^{p} \lambda^{p-i} A_i$$

where

$$A(\lambda) = -\lambda^{p-1} B^* + (\lambda - 1)\,[\lambda^{p-1} I_G - \sum_{i=2}^{p} \lambda^{p-(i-1)} B_i].$$

Substituting $\lambda = 1$, one can obtain $|A(\lambda)| = |-B^*| \neq 0$. It follows that the characteristic equation does not have a unit root unless $r = G$.

exists among the elements of $\{y_t\}$. In this case, the rank of B^* becomes smaller than G.[19] Therefore, the cointegration hypothesis for a multivariate model with no exogenous variables involving a time trend will be expressed in terms of the rank of the coefficient matrix as follows:[20]

$$H_3: \text{rank}\,(B^*) = r, \quad 0 < r < G.$$

Kunitomo (1992) proposes a likelihood ratio statistics on the rank of the matrix of regression coefficients presented in hypothesis H_3, which differs from Johansen's method. By generalizing exogenous variables from a constant and a linear trend to more complicated ones, Kunitomo's method can be applied to various specification of the hypothesis. It also has the advantage of making it easy to calculate test statistics as well as to construct their distribution. Therefore, I introduce the test of Kunitomo (1992) to the rank of the matrix of regression coefficients, and then explain the testing method of Kunitomo (1995), which formulates the cointegration hypothesis in terms of a model with exogenous variables involving trends.

B. Test Statistics on Rank

To express test statistics in a generalized form, we rewrite equation (A.4) as follows:

$$\Delta y_t = \beta z_t + \nu_t \tag{A.5}$$

where z_t is a $K \times 1$ vector of predetermined variables, including z^*_t, y_{t-1}, Δy_{t-1}, ..., $\Delta y_{t-(p-1)}$, and $K = K^* + pG$. β is a $G \times K$ matrix of coefficients.

Let us divide the predetermined vector z_t into a $K_1 \times 1$ vector z_{1t} and a $K_2 \times 1$ vector z_{2t}, and the corresponding matrix of coefficient β into a $G \times K_1$ matrix β_1 and a $G \times K_2$ matrix β_2. Now, the cointegration hypothesis of H_3 can be expressed by a hypothesis on the rank of β_2 because β_2 corresponds to B^* and β_1 corresponds to $(\Gamma, B_2, ..., B_p)$ when the predetermined vectors are set as follows:

$$z_{2t}' = y_{t-1}', \quad z_{1t}' = (z_t^{*\prime}, \Delta y_{t-1}', ...\Delta y_{t-(p-1)}').$$

The next section formulates the cointegration hypothesis with structural changes in the time trend. But before that, we specify test statistics on the rank of β_2.

Anderson and Kunitomo (1992) consider test statistics of the rank condition on regression coefficients, which is used in Kunitomo (1992). Test statistics are given by the following likelihood ratio statistics:[21]

19. Let us write the cointegrating relationship of $\{y_t\}$ as $B^*y_t = \varepsilon_t$ where $\{\varepsilon_t\}$ is a stationary process. Then B is not full rank, because it has no inverse matrix. If B^* has an inverse matrix, then the stationary process $B^{*-1}\varepsilon_t$ becomes equivalent to a nonstationary process y_t. This is a contradiction.

20. When the rank of B^* drops, there exist such $G \times r$ matrices α and β that $B^* = \alpha\beta'$ holds. There exist r cointegrations because the linear combination of $\{y_t\}$ made by r row vectors in β becomes stationary.

21. In addition to likelihood ratio statistics, Lagrangian multiplier statistics and the Wald statistics are derived.

$$\mathrm{LR}_1(G_0) = T\sum_{i=1}^{G_0} \log(1 + \lambda_i^*) \quad \text{(A.6)}$$

where λ^* represents the smallest roots of the characteristic function (A.7) up to G_0. Y, Z, and Z_1 in the characteristic function are given by a $T \times G$ matrix $\{y_t\}$, a $T \times K$ matrix $\{z_t\}$, and a $T \times K_1$ matrix $\{z_{1t}\}$, respectively. P_Z and P_{Z1} are given by $Z(Z'Z)^{-1}Z'$ and $Z_1(Z_1'Z_1)^{-1}Z_1'$, respectively.[22] G_0 is assumed to be $(G - r)$. $\mathrm{LR}_1(G_0)$ where $G_0 = 1, ..., G$ are the test statistics for the null hypothesis $r = G - G_0$ against an alternative hypothesis $r = G$.[23]

$$\left| \Delta Y'(P_Z - P_{Z1})\Delta Y - \lambda^*\Delta Y'\bar{P}_Z\Delta Y \right| = 0. \quad \text{(A.7)}$$

C. Time Trends and Structural Changes

When a multivariate time series model has exogenous variables involving a time trend, a rank such as H_3 cannot be interpreted directly as the cointegration hypothesis. For example, take a constant and a linear trend as an exogenous vector z^*_t. With $z^*_t = (1, t)$ and nonstationary variables in model (A.4), z^*_t plays the role of a drift term in the random walk process. In this case, if z^*_t includes a linear time trend, $\{y_t\}$ will have a quadratic trend. Therefore, in the model of a nonstationary process around a linear trend, the coefficients of all exogenous variables must be zero except for a constant term. In view of this, Kunitomo (1992) expressed the cointegration hypothesis with exogenous variables as follows:

$$H_4: \text{rank}\,(B^*) = r \text{ and } \Gamma_2 = 0.$$

Here Γ_1 and Γ_2 are the coefficient matrices corresponding to z^*_{1t} and z^*_{2t}, respectively, with z^*_{2t} corresponding to the time trend in the above example. However, Kunitomo (1995) shows that H_4 is inadequate for the cointegration hypothesis involving structural changes in the trend. This is illustrated the following examples:

- **Example 1:** The case of a kink in the linear time trend

Let DT_{it} be a dummy variable for a linear time trend as follows:

$$z^{*\prime}_{2t} = (1, DT_{it}, \ldots\ldots, DT_{qt}) \quad \text{(A.8)}$$

$$DT_{it} = \begin{cases} 0 & 0 < t \leq TB_i \\ t - TB_i & TB_i < t \leq T \end{cases}$$

where TB_i is the date of kink i (i = 1, ..., q) in the trend. The linear trend in a

22. P_Z is a projection factor on the space spanned by the column vectors of a matrix Z. The projection factor perpendicular to Z is given by $P^*_Z = I_T - P_Z$.

23. There is another statistic that uses the maximum eigenvalue λ_{G0}: this for the null hypothesis of $r = G - q$ against an alternative hypothesis of $r = G - q + 1$.

nonstationary model can be expressed as a constant drift term as follows:

$$z^{*\prime}_{1t} = (1, DU_{it}, \ldots.., DU_{qt}). \tag{A.9}$$

$$DU_{it} = \begin{cases} 0 & 0 < t \leq TB_i \\ 1 & TB_i < t \leq TB_j \end{cases}$$

If we write $z^{*\prime}_t = (z^{*}_{1t}{}', z^{*}_{2t}{}')$, $z^{*}_{1t}{}' = (1, DU_{1t}, \ldots, DU_{qt})$, and $z^{*}_{2t}{}' = (t, DT_{1t}, \ldots, DT_{qt})$, the cointegration hypothesis may appear to be expressed as H_4. However, there exists a vector that produces a linear combination of the elements of z^{*}_{2t} stationary just as there exists a vector that produces a linear combination of the elements of y_t stationary. If we can find these vectors, $\Gamma_2 z^{*}_{2t} + B^{*}y_t$ becomes stationary. Unlike H_4, the rank of Γ_2 only has to be less than r and does not have to be zero. Therefore, the cointegration hypothesis can be expressed as follows:

$$H_5\text{: rank } (\Gamma_2, B^{*}) = r$$

where z^{*}_{1t} is set to be equal to Δz^{*}_{2t}.

Test statistics of this hypothesis H_5 are given by equation (A.6) by setting vectors as follows:

$$z_{2t}{}' = (z^{*\prime}_{2t}, y_{t-1}{}'), \quad z_{1t}{}' = (z^{*\prime}_{1t}, \Delta y_{t-1}{}', \ldots.., \Delta y_{t-(p-1)}{}') \tag{A.10}$$

in equation (A.5).[24] The asymptotic distribution of the test statistics will be different from the case of $LR_1(G_0)$ with no structural changes because z_{2t} contains z^{*}_{2t}, a part of z^{*}_t. These test statistics can be expressed as $LR_2(\text{ex. } 1; G_0, q, \delta)$ because they differ depending on the relative kink period δ_i (= TB_i/T) and on the number of kinks; q.

• **Example 2:** The case of a trend consisting of a linear trend and a quadratic trend

For the multivariate model of equation (A.5), the cointegration hypothesis can be stated as H_5 if the exogenous vector z^{*}_t is set as follows:

$$z^{*}_{1t} = (1, DU_{1t}, (DT_{1t} - DT_{2t}))$$
$$z^{*}_{2t} = (t, (QT_{1t} - QT_{2t})).$$

24. Nevertheless, when the rank of (Γ_2, B^{*}) is r, the rank of B^{*} is not necessarily r. And the following two cases are possible:

rank $(B^{*}) = r$ and $0 <$ rank $(G_2) < r$

rank $(\Gamma_2) = r$ and $0 <$ rank $(B^{*}) < r$.

Therefore, there can be a rare case of no cointegration ($B^{*} = 0$) even if H_5 holds. This point was not made explicit in Kunitomo (1995).

The test statistics LR_2(ex. 2; G_0, δ_1, δ_2) can be obtained by setting z_{1t} and z_{2t} in the same way as (A.10) in Example 1.

D. Application of the Cointegration Test to the Unit Root Test

Perron's (1989) unit root test for a variable with structural changes utilizes the t test to judge whether the AR part has a unit root. It remains an imperfect test because it does not contain zero restrictions on the coefficient of the time trend in the unit root hypothesis. For example, in the case of a univariate model of equation (2), it tests only $\beta = 1$ in the unit root hypothesis H_0, and does not test $(\alpha_3, \alpha_4, \alpha_5) = (0, 0, 0)$. Kunitomo (1995) shows that the univariate case of $G = 1$ in the multivariate model (A.1) can be applied to the testing of a unit root.

The three cases (1), (2), and (3) on the rank of B^* discussed in Section A of this Appendix become (1) or (2) when $G = 1$. In other words, the rank (r) of B^* becomes either zero or G. In the former case, it becomes a unit root process and in the latter a stationary process. If rank (Γ_2, B^*) in the cointegration hypothesis H_5 is zero, the coefficient of the time trend also becomes zero. And, therefore, both the unit root and cointegration hypothesis can be expressed in terms of the rank hypothesis on the coefficient matrix. Moreover, in the case of a univariate model, test statistics take a very simple form. Because $\Delta Y' P_Z D \Delta Y$ and $\Delta Y' P_{Z1} \Delta Y$ in equation (A.7) are the squared sum of residuals when z_t and z_{1t} are regressed on Δy_t and $z_{1t} = z_t - z_{2t}$, they correspond to the unrestricted regression sum of squares (URSS) and the restricted regression sum of squares (RRSS). Therefore, the root λ of equation (A.7) can be written as:

$$\lambda = \frac{\text{RRSS} - \text{URSS}}{\text{URSS}}$$

and in this case the likelihood ratio statistics take the following simple form:

$$\text{LR} = T \log \left(\frac{\text{RRSS}}{\text{URSS}}\right).$$

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