The Theory of Exchange Rate Determination in a Multi-Currency World

—An Analysis of the Supply of and Demand for Foreign Exchange in a Multiple Reserve Currency System—

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I. Summary

In addition to the dollar, international transactions have increasingly been denominated in other currencies such as the German mark, the Japanese yen, and the Swiss franc. The weight of these different currencies in the reserve assets of each country is also increasing (See Table 1). As a result, a one dimensional perspective of looking at the Japanese yen only in terms of the U.S. dollar has been losing its validity. To analyze the yen-U.S. dollar exchange rate, it has become increasingly necessary to adopt a multi-dimensional approach of examining the exchange rates of the Japanese yen with other major currencies such as the German mark, the Swiss franc, and the U.K. pound in addition to the U.S. dollar. This point has been briefly explained in Fukao (2). Here, the theory of exchange rate determination between multiple currencies will be explained in detail utilizing a model of real exchange risk (Fukao (3), pp. 33-40)

In analyzing the supply and demand of foreign exchange in a multiple currency world, the concepts of substitutability and complementarity among currencies are important. In Part II, it will be shown that the substitutability and complementarity

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among different currencies are determined by the correlation among exchange rates of these currencies.

In Part III, the micro foundations of the analysis in Part II will be presented rigorously by using the capital asset pricing model (CAPM). Assuming investors are risk averse, the demand function for different currency denominated assets will be derived. Next, it will be shown that the amount of foreign currency denominated assets held by each investor depends upon such factors as the interest rates in each country, current exchange rate levels, and the previously mentioned expected future correlation between exchange rates (mathematically speaking, variance and covariances).

In Part IV, the supply and demand of foreign exchange in a multicurrency world is analyzed from the macro point of view by making use of the international balance of indebtedness table (Fukao (1)) enlarged to n currencies and n countries. An equilibrium equation for supply and demand in the foreign exchange market in which transactions take place among n currencies is introduced. Moreover, an exchange rate determination equation is derived for the case where no intervention occurs in the foreign exchange market by combining this equilibrium equation with the demand function for foreign currency denominated assets presented in Part III. By using this exchange rate determination equation, it is shown that deviations from the long run equilibrium exchange rate as determined by purchasing power parity between the ith country and the arbitrarily chosen base currency country consist of real interest rate differentials between the two countries and the risk premium arising from real exchange rate fluctuations (This risk premium is an additional return which compensates for the risk of holding foreign currency denominated assets). Furthermore, it is shown that this risk premium is proportional to the weighted total value of the world's cumulative current account balance excluding that of the base country.

In Part V, sterilized exchange rate intervention by monetary authorities is introduced in the model. The main conclusions of this analysis are as follows.

1) Intervention changes the amount of different currency denominated assets held by the private sector. Through changes in the risk premium, intervention affects the exchange rate.

2) In order to conduct support buying of the yen, selling dollars and selling marks affects the yen dollar exchange rate differently, with the difference depending on the degree of substitutability and complementarity currencies.

3) When Japan intervenes in the foreign exchange market by selling dollars and buying yen, this affects the dollar-mark exchange rate. Similarly, when Germany intervenes by selling dollars and buying marks, the yen-dollar exchange rate is affected.

4) Cooperative intervention supporting the yen and the mark through simultaneous selling of dollars by both Japan and West Germany has a greater impact on the

yen-dollar exchange rate than does independent intervention because of the above stated influence on the third country currency. In other words, comparing the case in which the Bank of Japan sells dollars alone in order to support a weak yen with the case where this occurs in conjunction with the Bundesbank selling dollars and buying marks, the effect of strengthening the yen is greater in the latter case.

Finally, since rather elaborate mathematical exposition is needed to treat the problem of determining n-1 independent exchange rates in an n currency world, all mathematical proofs are relegated to the Appendices.

Table 1 The Currency Composition of Official Foreign Exchange Reserves

(percents, end of the quarter values)

	1973 : I	1975 : IV	1976 : IV	1977 : IV	1978 : IV	1979 : IV*	1980 : IV*
U.S. dollar	84.5	85.2	86.7	85.2	82.8	78.9	73.1
U.K. pound	5.9	4.1	2.1	1.8	1.6	2.0	3.0
West German mark	6.7	6.6	7.3	8.3	10.1	11.3	14.0
French franc	1.2	1.3	1.0	0.8	1.0	1.0	1.3
Swiss franc	1.4	1.7	1.6	2.2	2.1	3.2	4.1
Dutch guilder	0.4	0.6	0.5	0.5	0.5	0.7	0.9
Japanese yen	–	0.6	0.7	1.2	1.9	2.8	3.7
Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0

*Europen currency units issued against the U.S. dollar are included, while those issued against gold are not. (Source: Ministry of Finance.)

II. Substitutability and Complementarity amongst Different Currency Denominated Assets

In recent years, connected by a global information network, foreign exchange markets have truly become international. The exchange rates of the major currencies such as the dollar, the mark, the yen, the pound, and the Swiss franc reflect the economic and political situations of all countries. Hence, looking at fluctuations in the yen-dollar exchange rate for example, these fluctuations are not determined by the political and economic state of affairs in only the U.S. and Japan. The yen-dollar exchange rate is influenced by political and economic situations in other countries and fluctuations of other major currencies such as the mark and the pound with the dollar. In this Part, the basic concepts of substitutability and complementarity between currencies are introduced in order to analyze the influence of fluctuations in the supply and demand of one country's currency on the supply and demand of other countries' currencies.

A. Substitutability and Complementarity amongst Assets

Normally, if two or more consumption goods fulfill the same need, they are said to be substitutes (For example, an automobile produced by Company A and an automobile produced by A's rival Company B). On the other hand, if the simultaneous consumption of two or more goods is necessary to fulfill the same need, these goods are called complements. (For example, an automobile and gasoline). When two goods are substitutes, an increase in the supply of one (an A's car) causes a decrease in the demand for the other (a B's car). For complementary goods, an increase in the supply of one (an automobile) causes an increase in the demand for the other (gasoline).

When expanding this concept of substitutes and complements to include risky assets such as stocks and foreign currency denominated bonds, the following can be said to be true.

Substitute assets---- a positive correlation is expected between the future value (the principal and the yield) of two assets.

Complements------- a negative correlation is expected between the future value of two assets.

For example, considering the stock of Company A and Company B, the two companies produce the same type of products and both are export industries. Consequently, both stocks tend to be influenced in the same way by changes in such factors as fluctuations in exports or domestic consumption expenditures. As a result, the future value of these two stocks is expected to have a positive correlation and they are therefore substitutes. In contrast, in the case of an automobile stock and a refinery stock, the former is weakened by a strong yen while the latter's price is strengthened by a strong yen. Hence, the future value of these two stocks is expected to exhibit a negative correlation. As a result, these two stocks are complementary assets.

When two assets are substitutes in the above meaning, an increase in the amount of one asset held in an investor's portfolio causes an increase in the risk exposure of that investor. Hence, that investor's demand for the substitute asset decreases. Similarly, when two assets are complements in the above sense, if the amount of one asset in an investor's portfolio increases, in order to hedge against the increased risk, that investor's demand for the other asset increases.[1]

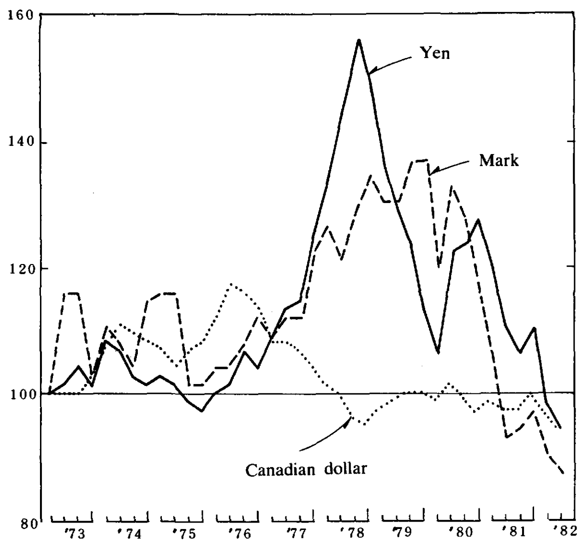
1. Usually, in the theory of consumer behavior, substitutes and complements are defined holding the utility level of the consumer constant. That is, if increasing the price of one good while augmenting the comsuner's income to hold his utility level constant causes the amount of the other good demanded to increase, these two goods are said to be substitutes. Similarly, if the same process causes the amount of the other good demanded to decrease, the two goods are said to be complements. However, in this paper, in order to simplify the analysis which follows,

B. Substitutability and Complementarity of Currencies

The definition for substitute and complement assets as stated in section A can similarly be applied to foreign currency denominated assets and debt. Here, as an example, consider the case of four currencies consisting of the Japanese yen, the West German mark, the Canadian dollar, and the U.S. dollar. Chart 1 shows past fluctuations in the real exchange rates for the Canadian dollar, the mark, and the yen from the point of view of the U.S. investor to whom the U.S. dollar is a riskless asset (In other words, to this investor the U.S. dollar is the habitat currency).[2,3]

Chart 1 Exchange Rate Fluctuations from the Perspective of the U.S.

(Indices of real exchange rates of each currency against the U.S. dollar, with the first quarter of 1973 set equal to 100)



substitutability and complementarity is considered only as the relationship between the amount of two different assets demanded. In other words, when defining substitutability and complementarity here, the investor's utility level is not held constant.

2. In order to simplify the analysis, only the risk associated with exchange rate fluctuations is considered. Risk in holding finacial assets associated with such factors as changes in future interest rates, inflation, and default are ignored. As a result, for investors in each country, assets denominated in own currency are perfectly riskless while risky assets are those assets denominated in a foreign currency.

3. The real exchange rate was adopted in order to analyze the supply of and demand for exchange by means of a real exchange risk model which abstracts from risk arising from inflation. See

As can be seen from the correlation coefficients in Table 2a, the yen-dollar exchange rate is positively correlated with the mark-dollar exchange rate (+. 7) If this past relationship is expected to continue in the future, the yen and the mark can be considered substitute assets from the perspective of the U.S. investor. On the other hand, the Canadian dollar-U.S. dollar exchange rate and the yen-U.S. dollar exchange rate exhibit a negative correlation (-.46) as does the Canadian dollar-U.S. dollar exchange rate and the mark -U.S. dollar exchange rate (-.19). Hence the yen and the Canadian dollar, and the mark and the Canadian dollar are complementary assets.

Now consider the degree of substitutability of the home currency with foreign currencies from the point of view of the U.S. investor. This degree of substitutability is calculated from the extent to which the dollar denominated foreign currency exchange rates fluctuate. Looking at the standard deviations of the foreign currency-dollar real exchange rates in Table 2a, it can be seen that this standard deviation of the Canadian dollar (6%) is much less than that of the yen (13%) or the mark (12%). Consequently, for the U.S. investor, the exchange risk for the Canadian dollar is extremely small compared to that of the yen or the mark (see Chart 1 for reference), and as a result, the Canadian dollar is a better substitute asset than either the yen or the mark for the U.S. investor.

Looking at fluctuations in real exchange rates from the perspective of the Japanese investor to whom the yen is a riskless asset, movements in the yen-U.S. dollar, the yen-mark, and the yen-Canadian dollar rates are shown in Chart 2. As can be seen from Table 2b, the correlation coefficients of any two of the three foreign currency -yen exchange rates are positive for the observation period. If these relationships are expected to continue in the future, these foreign currencies are substitutes. In particular, the U.S. dollar and the Canadian dollar are very good substitute assets (correlation coefficient of .95).

Now consider the substitutability of these three foreign currencies with the yen which is a riskless asset for a Japanese investor. The standard deviation of the mark-yen exchange rate is 9% (Tabel 2b), the smallest amongst the standard deviations of foreign currency-yen rates. Thus, in comparison with the other two foreign currencies, the mark is the best substitute asset for the yen.

The interested reader can make similar observations for the cases when the mark and the Canadian dollar are home currencies using Charts 3 and 4, and Tables 2c and 2d.

This section has explained the concepts of substitutability and complementary

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Fukao (2, pp2-5) for a definition of the risk of exchange rate fluctuations in a real exchange risk model.

amongst currencies based on coefficients of correlation between past exchange rates. However, the substitutability and complementarity amongst assets depends essentially on expected *future* correlations and cannot be judged soley on the basis of correlations in the past. For example, until 1977-78, the English pound-U.S. dollar real exchange rate moved almost parallel with the mark-U.S. dollar exchange rate. After this time, however, the pound-dollar real exchange rate, reflecting such factors as the discovery of North Sea oil, moved independently of the mark-dollar real exchange rate (see Chart 5). Factors which cause changes in the fundamental economic or trade structures of a country such as the discovery of oil naturally affect the correlation between currencies.

In spite of this qualification, the correlation of exchange rate fluctuations in the past is one important factor influencing an investor's selection of his portfolio of different currency denominated assets. This is because past exchange rate correlations well reflect continuous factors such as the fundamental industrial and trade structures of each country as well as various socio-political relationships. In other words, these correlations posses continuity. For instance, looking at countries such as the U.S. and Canada or France and West Germany which are closely tied together economically by trade, the real exchange rates between their respective currencies do not exhibit much fluctuation. This is true for both the U.S. dollar-Canadian dollar rate which floats as well as for the French franc- West German mark rate which follows an adjustable peg system. These correlations are especially strong amongst currencies which belong to a strongly unified economic sphere, and such currencies are very good substitutes for one another. Also, currencies for countries such as West Germany and Japan which have similar industrial and trade structures and similar stances towards the management of the economy are also relatively good substitutes.

In contrast to this, exchange rate fluctuations are negatively correlated for countries having mutually complementary industrial and trade structures such as Japan which imports energy and raw materials and Canada which exports these products. As a result, the yen and the Canadian dollar tend to be complementary assets (Table 2a). For instance, the impact on Japan of the First Oil Crisis was extremely large, but the impact on Canada, which is a net energy exporter, was small. Consequently, at this time, the yen fell against the U.S. dollar while the Canadian dollar rose against the U.S. dollar.

Substitutability and complementarity between currencies as defined in this section is decided by investors' subjective expectations of the future correlation between exchange rates. Because of this, substitutability and complementarity cannot be directly measured at a particular moment in time. However, everyday, investors such as firms and life insurance companies, have to manage their foreign currency denominated assets and liabilities depending on their future expectations of exchange rate correlations which, in turn, are based on past experienced correlations. In the next

Part, under the assumption that an investor's future expectations of these correlations are given, the behavior of investors towards different currency denominated assets will be analyzed using a capital asset pricing model.[4]

Table 2 Correlation between Real Exchange Rates

a) Correlation Coefficients Between Dollar Denominated Real Exchange Rates

	yen	mark	Canadian dollar
yen	1.00		
mark	0.70	1.00	
Canadian dollar	−0.46	−0.19	1.00
(standard deviation)	0.13	0.12	0.06

b) Correlation Coefficients Between Yen Denominated Real Exchange Rates

	U.S. dollar	mark	Canadian dollar
U.S. dollar	1.00		
mark	0.47	1.00	
Canadian dollar	0.95	0.51	1.00
(standard deviation)	0.13	0.09	0.16

c) Correlation Coefficients Between Mark Denominated Real Exchange Rates

	U.S. dollar	yen	Canadian dollar
U.S. dollar	1.00		
yen	0.30	1.00	
Canadian dollar	0.92	0.09	1.00
(standard deviation)	0.12	0.09	0.14

d) Correlation Coefficients Between Canadian Dollar Denominated Real Exchange Rates

	U.S. dollar	yen	mark
U.S. dollar	1.00		
yen	0.71	1.00	
mark	0.56	0.81	1.00
(standard deviation)	0.06	0.16	0.14

*Correlation coefficients and standard deviations are calculated from natural logarithms of real exchange rates. Consequently, standard deviations show the magnitude of fluctuations in real exchange rates measured approximately by percentages. For instance, the standard deviation of the yen real exchange rate against the mark in b) is
0.09 = 9%

4. In the theoretical analysis of the behavior of investors, it is not necessary to assume that past exchange rate correlations are the same as the future correlations expected by investors. It is possible to consider that various factors can change these expected correlations. However, because it is difficult to model the process of the formation of investors' expectations, it is standard for empirical analysis to consider that these past correlations represent investors' expectations.

Chart 2 Exchange Rate Fluctuations from the Perspective of Japan

(Indices of real exchange rates of each currency against the yen, with the first quarter of 1973 set equal to 100)

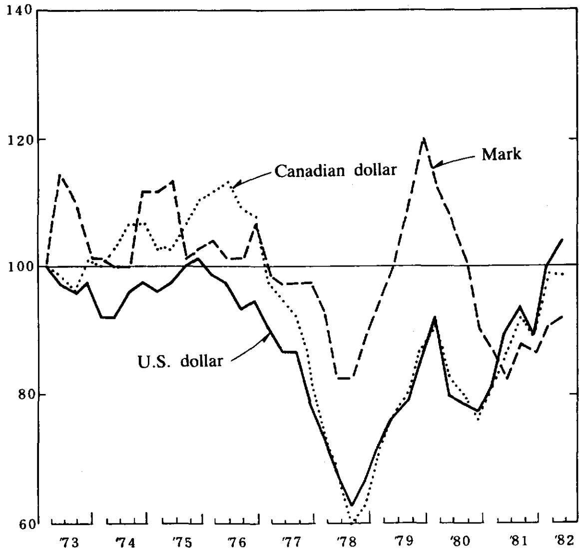


Chart 3 Exchange Rate Fluctuations from the Perspective of West Germany

(Indices of real exchange rates of each currency against the mark, with the first quarter of 1973 set equal to 100)

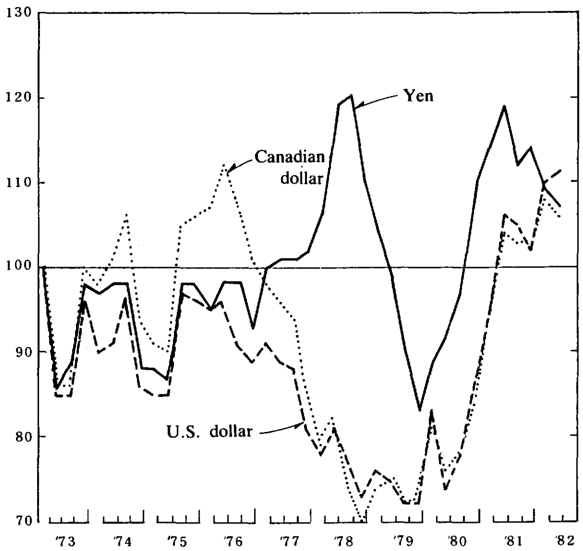


Chart 4 Exchange Rate Fluctuations from the Perspective of Canada

(Indices of real exchange rate fluctuations of each currency against the Canadian dollar, with the first quarter of 1973 set equal to 100)

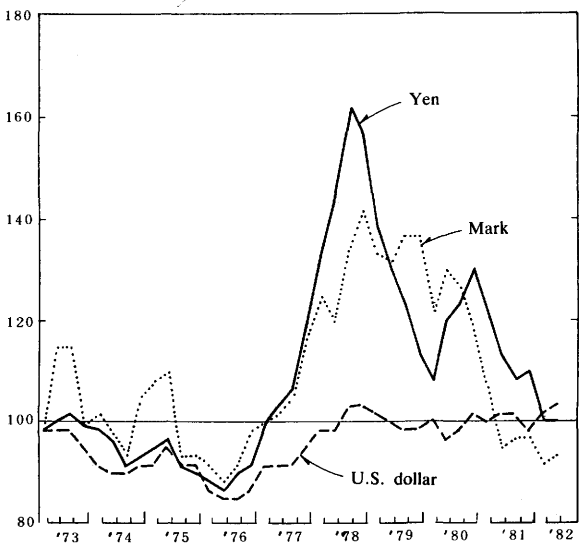
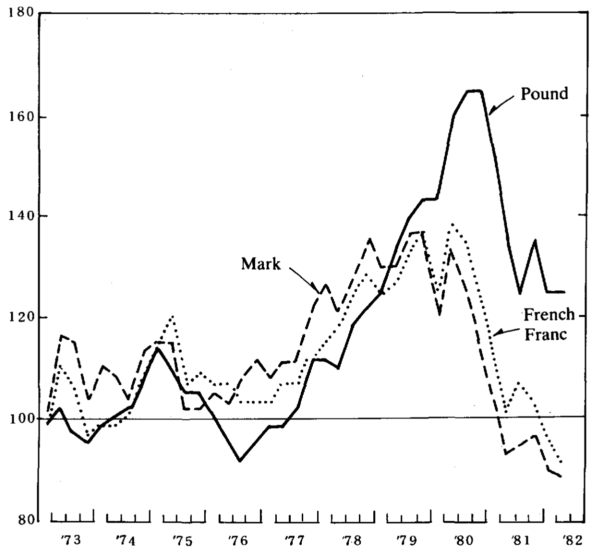


Chart 5 Indices of the Real Exchange Rates of the pound, the mark, and the franc with the U.S. dollar

(The first quarter of 1973 is set equal to 100)



III. Investors' Behavior in a Multiple Reserve Currency System —Analysis based upon the Capital Asset Pricing Model—

In Part II, the relationship between the substitutability and complementarity of currencies was analyzed. Here, the portfolio decision behavior of investors will be analyzed by use of the capital asset pricing model.[5] Next, in the framework of a real exchange risk model, demand function for foreign currency denominated assets will be derived for risk averse investors in the foreign exchange market.

A. Assumptions of the Model

Assume a world consisting of n countries, each of which has its own currency which floats against other currencies. Furthermore, assume no intervention by monetary authorities. Choose one particular country as a base country against which all foreign exchange rates are calculated and call this the nth-country. Hereafter, consider investors in this nth country.

In order to apply the CAPM to the analysis of foreign exchange markets, the following six assumptions are made.

(A-1) Points of time considered in the model are only two, $t=0$ and $t=1$. Accordingly, all liabilities and assets are assumed to have a one period term to maturity.

(A-2) Investors in the nth country are risk averse and act so as to maximize their end period expected utility as defined by the following equation.[6,7]

$$u = a_n \mu_w - b_n \sigma_w^2 \tag{1}$$

u : expected utility of all investors in the nth country

μ_w : expected end period value of assets ($t=1$)

σ_w^2 : variance of end period value of assets

$a_n>0$, $b_n>0$: Constants

5. The CAPM is a theory which analyzes the behavior of investors faced with multiple risky assets. To date, the CAPM has largely been used to analyze the price of stocks. This theory assumes that investors try to increase the rate of return of their portfolio while considering the risk involved. For a detailed explanation of the CAPM model, see surveys by Jensen (9) and Ross (12).

6. In order to simplify the equations which follow, an aggregate utility function for the nth country's investors is used. However, it would be quite simple to derive demand functions for foreign currency denominated assets from the utility functions of each individual investor.

7. See Appendix 2 for the justification of writing the utility function in this particular form.

In other words, this equation states that the greater is the expected value of assets μ_w at the end of the time period (t=1) and the smaller is the variance σ_w^2 which shows the degree of risk, the higher is the investors' utility.

(A-3) There is no inflation in any country, so the domestic value of each currency is constant.

(A-4) Investors estimate end of the period exchange rates as a probability distribution. Moreover, the expected values, of exchange rates between all currencies at the end of the time period are equal to the long run equilibrium exchange rates as determined by purchasing power parity.[8,9]

(A-5) Interest rates for each country are determined exogenously.[10]

(A-6) Foreign exchange markets are competitive, and investors act using spot exchange rates as given.

Of these assumptions, A-3 is adopted only to facilitate ease of analysis. In the final equation for exchange rate determination, this assumption is removed. In its place, the assumption that inflation exists but that future inflation rates are foreseen with certainty is introduced.

As can be understood from these assumptions, nth country investors choose a portfolio at the beginning of the period (t=0) of own currency denominated assets which are safe assets and assets denominated in any of n-1 foreign currencies which are subject to exchange risk. The basis of this selection is the expected utility function given in equation (1). The investors' investment horizon (assumed in A-1 to be of one period) is sufficiently long. At the end of this period, investors can only foresee

8. Although it is easy to define purchasing power parity for the exchange rate between two countries, when multilateral trade occurs amongst n countries, it is quite difficult to define purchasing power parity for the resulting n-1 independent exchange rates. In this paper, the system of real exchange rates which cause equilibrium in the long run in the current account balances of all countries is called the system of long run equilibrium real exchange rates. It is assumed that these rates are constant. Purchasing power parity exchange rates are defined as the nominal exchange rates corresponding to these long run equilibrium real exchange rates. For details, see Appendix 3.

9. Here, it is assumed that investors have a fairly long investment horizon of one to three years. As shown in Fukao (4), Table 1, uncovered investment in short term foreign currency denominated assets is much more risky than long term inversment. Hence, in the case of investors with a long time horizon, the elasticity of demand for foreign currency denominated assets and liabilities with respect to expected exchange rate changes is much greater than the case of investor with a short time horizon. Thus, long term investors play a more important role in determining exchange rates than do short term investors. Porter (11) has shown that investors with a horizon of two years are important in determining the Canadian dollar-U.S. dollar exchange rate.

10. Theoretically speaking, exchange rates and interest rates should be determined simultaneously in asset markets. However, as shown in Fukao and Ōkubo (4), even if interest rates are determined endogenously, the results of the following analysis are largely the same.

exchange rates imperfectly, but exchange rates are expected to be equal on average to the long run equilibrium rates determined by purchasing power parity. Moreover, risk in holding foreign currency denominated assets is assumed to consist only of the risk from fluctuations in real exchange rates. Risk from uncertainty in the value of currency due to inflation is not considered.[11]

B. Demand Functions for Foreign Currency Denominated Assets

From the assumptions in the last section, demand functions for foreign currency denominated asssets and liabilities for investors in the nth country can be derived through the maximization of the expected utility function of these investors who are faced with exchange rate uncertainty at the end of the period. This section explains the demand functions that are obtained as a result of this maximization, with mathematical proofs being relegated to the Applendix 1. (For a detailed explanation of the symbols used in this section, see the Appendix 1).

In order to understand the properties of these demand functions, assume a simplified world consisting only of Japan, West Germany, and the U.S. In this case, the demand functions for foreign currency denominated assets for investors in the U.S. which is the base country is composed of two demand functions, one for yen denominated assets and the other for mark denominated assets.

$$X_1^3 = \left(\frac{c_3 M_{22}^3}{\triangle}\right)\beta_1 - \left(\frac{c_3 M_{12}^3}{\triangle}\right)\beta_2 \tag{2}$$

$$X_2^3 = -\left(\frac{c_3 M_{21}^3}{\triangle}\right)\beta_1 + \left(\frac{c^3 M_{11}^3}{\triangle}\right)\beta_2 \tag{3}$$

X_i^3 : The amount of assets denominated in yen (i=1) or marks (i=2) demanded by U.S. investors. (Converted into dollars using the purchasing power parity rate. If $X_i^3 < 0$, this is a liability).

M_{ij}^3 : The variance of the dollar denominated yen exchange rate (i=1) or the dollar denominated mark exchange rate (i=2) expected at the end of the period. This is always positive. (Exchange rates here and below are in logs.)

$M_{ij}^3 (i \neq j)$: The covariance of the yen and the mark dollar denominated exchange rates. This can be either positive or negative. (In the former case, the yen and the mark are substitutes while in the latter case they are complements).

11. This is a real exchange risk model. For reference, see Fukao (3).

$\triangle = M^3_{11} M^3_{22} - M^3_{12} M^3_{21}$: $\triangle$ is always positive. (see Lidgren (10, p 463) for reference).
For other notation, see Appendix 1.

These amounts demanded depend upon the expected rate of return differentials defined in the following equations for the yen and the mark against the dollar, where β_1, β_2 represent additional return to compensate for the risk accompanying the holding of foreign currency denominated assets. This additional return is called the risk premium.

$$\beta_1 = r_1 + (\bar{f}^3_1 - e^3_1) - r_3 \quad (4)$$

$$\beta_2 = r_2 + (\bar{f}^3_2 - e^3_2) - r_3 \quad (5)$$

r_1 : yen interest rate
r_2 : mark interest rate
r_3 : dollar interest rate
e^3_1 : yen spot exchange rate (in U.S. dollars, in logarithm)
e^3_2 : mark spot exchange rate (in U.S. dollars, in logarithm)
$\bar{f}^3_1$: expected value of the yen spot exchange rate at the end of the period (in U.S. dollars, in logarithm)
$\bar{f}^3_2$: expected value of the mark spot exchange rate at the end of the period (in U.S. dollars, in logarithm)

For example, the expected rate of return differential between the yen and the dollar (β_1) consists of the yen interest rate (r_1) plus the rate of the expected increase of the yen against the dollar ($\bar{f}^3_1 - e^3_1$) minus the dollar interest rate (r_3). The coefficients inside the parentheses in equations (2) and (3) which are multiplied by β_1 and β_2 depend upon the degree of risk aversion of U.S. investors ($c_3 > 0$) and the variance and covariance of the yen exchange rate and the mark exchange rate against the dollar (M^3_{ij}, i, j=1,2). As will be explained below, this shows the degree of substitutability and complementarity between the currencies.

Examine in detail the demand function for yen assets for U.S. investors given by equation (2). As is shown by this equation, the demand for yen assets X^3_1 depends not only on the expected rate of return differential between yen and dollar assets (β_1) but also upon the expected rate of return differential between mark and dollar assets (β_2). Since c_3, M^3_{22}, and $\triangle$ are all positive, if β_1 increases either by (i) a rise in the yen interest rate r_1, (ii) a decrease in the dollar interest rate r_3, or (iii) a decrease in the yen spot rate e^3_1, which causes an increase in the expected rate of increase of the yen ($\bar{f}^3_1 - e^3_1$), then the demand for yen assets will increase. (In (iii) above, regressive expectations in which the current exchange rate moves towards the future purchasing power parity rate are assumed).

On the other hand, the effect of β_2 on the demand for yen assets (X^3_1) depends upon the sign of M^3_{12}. If it is expected that, at the end of the period, the correlation between the yen-dollar exchange rate and the mark-dollar exchange rate will be positive, the covariance becomes positive[12], and yen and mark assets are substitutes. In this case, the coefficient on β_2 becomes negative. If the expected rate of return of the mark increases against the dollar interest rate (β_2 increases), then the demand of U.S. investors shifts from yen to marks and the demand for yen assets X^3_1 decreases. If U.S. investors expect the future correlation between the yen and the mark to be very close to 1, then $\triangle$ is very close to zero[13], and the absolute values of the coefficients on β_1 and β_2 become extremely large. In other words, for U.S. investors, when the yen and the mark are highly correlated, they become very good substitute assets and even with a small change in the expected rate of return, a large shift in demand occurs.

In contrast, if it is expected that at the end of the period the yen-dollar and the mark-dollar rates are negatively correlated, the covariance M^3_{12} becomes negative and mark and yen assets become complements. In this case, the coefficient on β_2 in equation (2) for the yen asset demand function becomes positive. An increase in the expected rate of return of the mark (β_2 increases) causes not only an increase in the demand for mark assets through equation (3) but also increases the demand for yen assets. That is, because the yen and the mark are negatively correlated, in order to hedge against part of the increased mark exchange position, the demand by investors for yen increases.

The demand by U.S. investors for own currency denominated assets is obtained from the balance sheet constraints since the total of the net holdings of the dollar, yen, and mark assets is equal to the U.S. private sector's cumulative savings minus cumulative real investment.

In this section, only the demand function for foreign currency denominated assets for U.S. investors has been examined. However, demand functions of Japanese and German investors for foreign currency denominated assets can be derived in a similar fashion. Through inserting these demand functions in the equilibrium equa-

12. Denoting the coefficient of correlation between the yen-U.S. dollar and the mark-U.S. dollar exchange rates as ρ_{12}, the following equation holds.

$$\rho_{12} = \frac{M^3_{12}}{\sqrt{M^3_{11}\ M^3_{22}}}$$

Concerning this equation, see Lindgren (13, p135).

13. From the equation in footnote 12 and the relationship, $M^3_{12} = M^3_{21}$, we have

$$\triangle = M^3_{11}\ M^3_{22}\ (1 - \rho_{12}^2)$$

Hence, when ρ_{12}^2 approaches 1, $\triangle$ approaches zero.

tion for supply and demand in the foreign exchange market derived in Part IV and solving for the spot exchange rates, equations for exchange rate determination in the multi-currency case can be obtained.

To date, this section has dealt with a three country model. The equation for the general n country model is as follows.[14]

$$\underline{X}^n = c_n [\underline{M}^n]^{-1} [(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)] \tag{6}$$

The left hand side of this equation expresses as a vector the amount of n-1 foreign currency denominated assets and liabilities held by nth country investors. c_n on the right hand side is the positive coefficient showing the degree of risk aversion of nth country investors. The larger the degree of risk aversion, the smaller is c_n, and the absolute value of the demand for foreign currency assets and debt shrinks. M^n expresses as a variance-covariance matrix the degree of risk from fluctuations in exchange rates expected at the end of the period by investors. The superscript -1 indicates the inverse of this matrix. Lastly, $(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)$ is a vector showing the differential in the expected rate of return between the n-1 foreign currency denominated assets and assets denominated in the nth country's currency for investors of the nth-country.

IV. The Equation for Exchange Rate Determination and the Supply and Demand of Foreign Exchange in a Multi-Currency World

An analysis of exchange rate determination and the supply and demand of exchange in a two country world has been conducted in a previous paper using the international balance of indebtedness table (Fukao (1) (2)). Here, that analysis is extended to the multi-currency, multi-country case, and the simultaneous determination of exchange rates amongst multiple currencies is analyzed.

14. Rewriting equation (6) in matrix notation, the following is obtained.

$$\begin{bmatrix} X_1^n \\ X_2^n \\ \vdots \\ X_{n-1}^n \end{bmatrix} = c_n \begin{bmatrix} M_{11}^n & M_{12}^n & \cdots & M_{1\,n-1}^n \\ M_{21}^n & M_{22}^n & \cdots & M_{2\,n-1}^n \\ \vdots & & & \vdots \\ M_{n-1,1}^n & M_{n-1,2}^n & \cdots & M_{n-1,\,n-1}^n \end{bmatrix}^{-1} \left[\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \end{pmatrix} - r_n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \begin{pmatrix} \bar{f}_1^n \\ \bar{f}_2^n \\ \vdots \\ \bar{f}_{n-1}^n \end{pmatrix} - \begin{pmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{n-1}^n \end{pmatrix} \right]$$

For details, see Appendix1.

A. Analysis of the Supply and Demand of Foreign Exchange Based on the International Balance of Indebtedness

First, an example of the international balance of indebtedness table for the three country, three currency case is shown in Table 3. In this table, for simplification, each country is aggregated into one sector. It is assumed that monetary authorities hold no foreign currency denominated assets or debt and that all government deficits are financed through own currency denominated debt. Analysis of the effect of intervention which is assumed away here is conducted in Part V.

Table 3 International Balance of Indebtedness in the Multi-Currency, Multi-Country Case

(in U.S. dollars)

	Japan (1)	West Germany (2)	the U.S. (3)	Total
Yen (1)	$E_1^3\ Y_1^1$	$E_1^3\ Y_1^2$	$E_1^3\ Y_1^3$	0
Mark (2)	$E_2^3\ Y_2^1$	$E_2^3\ Y_2^2$	$E_2^3\ Y_2^3$	0
Dollar (3)	$E_3^3\ Y_3^1$	$E_3^3\ Y_3^2$	$E_3^3\ Y_3^3$	0
Total	B_1	B_2	B_3	0

*In this table, liabilities payable and credits receivable accompanying the conclusion of forward exchange contracts and import-export contracts are included in the assets and liabilities of each sector. For details, see Fukao (1).

E_i^j: Exchange rate of the ith-currency denominated in terms of the jth-currency. (For instance, E_2^3 is the dollar denominated mark exchange rate).

Y_i^j: Balance of foreign assets denominated in the ith-currency held by the jth-country. (If $Y_n^j > 0$, this is the asset balance, and if $Y_i^j < 0$, this is the balance of foreign debt).

B_j: Total net quantity of foreign assets of the jth-country (balance of indebtedness).

Looking at this table vertically, we can see the net external assets (or liabilities) for each country. The column sum B_j which is the quantity of net foreign financial assets held by each country equals the cumulative total of domestic savings minus domestic real investment. These are equal to the cumulative current account balances adjusted for capital gains and losses which accompany exchange rate changes (Fukao 2, p. 16). These realtionships can be thought of as balance sheet constraints accompanying the holding of foreign assets by each country and are expressed in the following three equations[15]

$$E_1^3 Y_1^1 + E_2^3 Y_2^1 + E_3^3 Y_3^1 = B_1$$

$$E_1^3 Y_1^2 + E_2^3 Y_2^2 + E_3^3 Y_3^2 = B_2 \qquad (7)$$

$$E_1^3 Y_1^3 + E_2^3 Y_2^3 + E_3^3 Y_3^3 = B_3$$

The supply of and demand for assets denominated in each currency can be seen from the rows of this table. In other words, the external liabilities of one country in each currency are held as external assets of other countries. Because of this, the following three equilibrium equations for the supply and demand of assets hold.

$$E_1^3 (Y_1^1 + Y_1^2 + Y_1^3) = 0$$

$$E_2^3 (Y_2^1 + Y_2^2 + Y_2^3) = 0 \qquad (8)$$

$$E_3^3 (Y_3^1 + Y_3^2 + Y_3^3) = 0$$

From these equilibrium equations for the supply of and demand for assets denominated in each currency and the balance sheet restrictions for each country, the equilibrium equations for supply and demand in the foreign exchange market can be derived. The foreign exchange market in a narrow sense usually refers to the interbank market which handles foreign exchange transactions between banks. In the wider sense, it is understood as a market which includes interbank transactions and the bank transactions with customers. However, here, in order to understand the supply of and demand for foreign exchange in a comprehensive fashion, the foreign exchange market is considered more broadly to be a market for transactions of financial assets denominated in foreign courrencies (including forward exchange positions) in each country.

The foreign exchange market defined in this way can be thought of as a transactions market which does not include the elements lying along the diagonal drawn from the upper left hand corner to the lower right hand corner of Table 3. This is because these diagonal elements (Y_1^1, Y_2^2, Y_3^3) are net foreign assets denominated in own country currency held by each country's investors and hence bear no exchange risk. Other elements in the columns (Y_i^j, $i \neq j$) are net foreign assets denominated in foreign currencies held by each country and consequently do bear exchange risk.

Investors in each country adjust the level of assets and liabilities denominated in

15. These equations combine the balance sheet constraints for the private sector given by (1) in Appendix1 with the identity that own currency denominated debt of the government sector equals the cumulative government deficit.

each currency according to the expected rate of return. The supply and demand by investors in each country for assets and liabilities denominated in each currency are not equal ex ante, but supply and demand is adjusted according to changes in spot exchange rates under Assumption (A-5) that interest rates are given. Hence, the supply and demand equations (8) hold ex post.

Thus, using the balance sheet equations (7), we can eliminate the diagonal terms from equations (8), and derive the equilibrium conditions for supply and demand in the foreign exchange market. For instance, transforming the first equation of equations (7),

$$-E_1^3 Y_1^1 = E_2^3 Y_2^1 + E_3^3 Y_3^1 - B_1 \qquad (9)$$

is obtained. The right hand side of this equation is the Japanese cumulative current account deficit ($-B_1$) plus the foreign exchange position of Japanese investors (the first two terms). These terms are all considered as the stock supply of yen assets. Also, the first equation of the equilibrium equations (8) can be rewritten in the following fashion

$$-E_1^3 Y_1^1 = E_1^3 Y_1^2 + E_1^3 Y_1^3 \qquad (10)$$

The right hand side of this equation is the exchange position in yen denominated assets for the U.S. and West Germany and can be viewed as a stock demand for yen assets. Using equation (9) and eliminating $E_1^3 Y_1^1$ from equation (10), the equilibrium condition for the supply of and demand for yen in the foreign exchange market can be obtained

$$\underbrace{E_2^3 Y_2^1 + E_3^3 Y_3^1 - B_1}_{\text{Supply of yen}} = \underbrace{E_1^3 Y_1^2 + E_1^3 Y_1^3}_{\text{Demand for yen}} \qquad (11)$$

This equation can be simplified by ignoring the influence on the supply of and demand for exchange arising from capital gains and losses caused by exchange rate fluctuations.[16] In other words, replacing the spot exchange rates E_i^3 (i = 1,2,3) by the equilibrium exchange rates $\bar{F}_i^3$ (i = 1,2,3) determined by purchasing power parity, the following equation is obtained

$$X_2^1 + X_3^1 - B_1 = X_1^2 + X_1^3$$
$$[\text{ where } X_i^j = \bar{F}_i^3 Y_i^j, \ i, \ j = 1, 2, 3 \ ] \qquad (12)$$

This is the equilibium equation for the supply of and demand for yen. Equations

16. Concerning this point, see Appendix1, Section2.

for the supply and demand equilibria for other currencies can be derived in a similar fashion. It will be noted here that the equation for the equilibrium of supply and demand in the foreign exchange market in the general case of an n currency world can be written as[17]

$$(\underline{X}-\underline{X}^{t})\,\underline{\eta}+\underline{B}=\underline{0} \tag{13}$$

B. Exchange Rate Equations

Deriving the demand functions for foreign currency denominated assets for investors in the other n-1 countries (which correspond to equation (6) for the nth-country), substituting them into the equilibrium condition (13) for supply and demand in foreign exchange markets, and solving for the spot exchange rates gives the following equation for exchange rate determination.

$$e_k^n = \bar{f}_k^n + (r_k - r_n) + \frac{1}{c}\sum_{j=1}^{n-1} M_{kj}^n B_j \tag{14}$$

e_k^n : The price of the kth currency in terms of the nth-currency (in logarithm).
$\bar{f}_k^n$: The purchasing power parity exchange rate for e_k^n at the end of the period (in logarithm).
r_k : Nominal interest rate on assets denominated in the kth-currency.
r_n : Nominal interest rate on assets denominated in the nth-currency.
c : Coefficient showing the degree of risk aversion of the world's investors as a whole.
M_{kj}^n : The covariance between the exchange rates of the jth-currency.and the kth-currency at the end of the period, with both currencies denominated in terms of the nth-currency (If k = j, this is the variance of the kth-currency).
B_j : Amount of net foreign assets of the jth-country (largely equal to the cumulative current account balance).

17. Rewriting equation (13) in matrix notation, the following is obtained.

$$\left[\begin{pmatrix} 0 & X_1^2 & X_1^3 & \cdots & X_1^n \\ X_2^1 & 0 & X_2^3 & \cdots & X_2^n \\ X_3^1 & X_3^2 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & X_{n-1}^n \\ X_n^1 & X_n^2 & X_n^3 & \cdots\; X_n^{n-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & X_2^1 & X_3^1 & \cdots & X_n^1 \\ X_1^2 & 0 & X_3^2 & \cdots & X_n^2 \\ X_1^3 & X_2^3 & 0 & \ddots & \vdots \\ \vdots & & & \ddots & X_n^{n-1} \\ X_1^n & X_2^n & X_3^n & \cdots\; X_{n-1}^n & 0 \end{pmatrix}\right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This equation has been derived under the assumption (A-3) that there is no inflation in any country and is therefore not general. Now, assume that there is inflation in each country, but that future inflation rates are foreseen with certainty. Transforming equation (14), the following final equation for the determination of exchange rates is obtained. (This is equation (38′) in Appendix 1, Section 2).

$$e_k^n = g_k^n + [(r_k - \pi_k) - (r_n - \pi_n)] + \frac{1}{c}\sum_{j=1}^{n-1} M_{kj}^n B_j \qquad (15)$$

$$k = 1, 2, \cdots, n-1$$

g_k^n: The present purchasing power parity rate of the kth-currency in terms of the nth-currency (in logarithm).

π_k: Expected inflation rate in the kth-country.

π_n: Expected inflation rate in the nth-country.

As can be understood from this equation for exchange rate determination, the exchange rate for the kth-currency in terms of the nth-currency e_k^n is determined by three factors. The first is the purchasing power parity rate g_k^n at present, the second is the real interest differential between the kth-country and the nth-country $[(r_k - \pi_k) - (r_n - \pi_n)]$, and the third is the weighted total the amount of net foreign assets of countries other than the nth-country, with the weights being elements M_{kj}^n of the variance-covariance matrix (The term $\sum_{j=1}^{n-1} M_{kj}^n B_j$ shows the risk premium)[18].

C. Inflation in the Multi-Currency Model

Equation (15) is obtained from the theory of exchange rate determination developed in this paper. This is an extension to the n country case of an earlier exchange rate determination equation derived by the author using a two country real exchange risk model (Fukao (2) p. 17). As the relationship between the expected future spot rate and the forward rate as well as the relationship between exchange rates and real interest differentials for each country have already been explained in this previous paper, these topics will not be addressed here. Concerning the effects of intervention, see Part V. for a detailed discussion.

The biggest difference between the two country model and the n country model lies in the risk premium term (third term, equation (15)). This term, which depends on the balance of indebtedness of each country, depends not only on the own country balance of indebtedness, but on third countries' balance of indebtedness. The extent to which a third country's net foreign asset balance affects the exchange rate between the base country and the own country depends upon the covariance between the own

18. Concerning the correspondence of this term with the risk premium, see Fukao (5).

country's exchange rate and the third country's exchange rate at the end of the period (both exchange rates are rates against the currency of the base country). For example, using the U.S. as the base country, the extent to which changes in the West German balance of indebtedness affects the yen-dollar exchange rate depends upon the magnitude of the expected future covariance between the yen-dollar rate and the mark-dollar rate. The magnitude of this covariance, in turn, depends upon the correlation between the exchange rates of each currency with the base currency. The strength of this third country effect is thus determined by the substitutability and complementarity between currencies discussed in Part II.

In order to analyze this third country effect, consider the simple case of a world consisting only of three countries. Furthermore, in order to focus only on the influence of balance of indebtedness, assume that the real interest rates of each country are equal. With these assumptions, equation (15) for exchange rate determination becomes

$$e_1^3 = g_1^3 + \frac{1}{c}\left[M_{11}^3 B_1 + M_{12}^3 B_2 \right] \tag{16}$$

$$e_2^3 = g_2^3 + \frac{1}{c}\left[M_{21}^3 B_1 + M_{22}^3 B_2 \right] \tag{17}$$

These equations state that the exchange rates of currencies 1 and 2 against the third currency are influenced by the respective balance of indebtedness B_1 and B_2. Here, M_{11}^3, and M_{22}^3 are the expected exchange rate variances at the end of the period and are always positive. Because of this, an increase in the balance of indebtedness of country 1 ($dB_1 > 0$) always increases the exchange rate of the first currency measured in terms of the third currency ($de_1^3 > 0$). Similarly, an increase in the balance of indebtedness of the second country ($dB_2 > 0$) always increases the exchange rate of the second currency measured in terms of the third currency ($de_2^3 > 0$). On the other hand, if the correlation between currency 1 and 2 is expected to be positive from the viewpoint of currency 3, currencies 1 and 2 are substitutes and M_{12}^3 ($= M_{21}^3$) is positive. In the case where the correlation is negative and therefore the two are complements, then M_{12}^3 ($= M_{21}^3$) is negative.

Consider first the case where currencies 1 and 2 are substitutes. This corresponds to the case in Part I where currencies 1,2 and 3 are the yen, the mark, and the U.S. dollar respectively. From the viewpoint of the U.S. dollar, the yen and the mark have had a positive correlation in the past (Table 2a). If this relationship is expected to continue in the future, M_{12}^3 is positive. Accordingly, if B_2 increases through a surplus in the West German current account, the mark rises against the U.S. dollar ($de_2^3 > 0$). At the same time, the yen rises against the dollar ($de_1^3 > 0$). In this case, because $M_{12}^3 = M_{21}^3$, a surplus in the Japanese current account ($dB_1 > 0$) causes the yen to rise ($de_1^3 > 0$) and at the same time also brings about a rise in the mark ($de_2^3 >

0). This third country effect is stronger the larger is the covariance M^3_{12}. If the yen and the mark are expected to move perfectly parallel with one another, then $M^3_{11} = M^3_{12}$, and fluctuations in the Japanese and West German foreign asset balances affect the yen dollar exchange rate in exactly the same way.[19] This third country effect is caused in the following way. Even if yen and mark denominated assets are not perfect substitutes, disequilibrium in the West German current account causes a change in the supply and demand of assets denominated in marks and influences the supply and demand of yen denominated assets because there is some degree of substitutability.

Next, consider the case where currencies one and two are complements. Let currencies one, two, and three be the yen, the Canadian dollar, and the U.S. dollar, respectively. As can be seen from Table 2a, the yen and the Canadian dollar have a weak negative correlation. If this relationship is expected to continue in the future, these two currencies can be considered as complements. In this case, $M^3_{12} < 0$. If the Canadian current account shows a surplus ($dB_2 > 0$), the Canadian dollar rises against the U.S. dollar ($de^3_2 > 0$). On the other hand, the yen- U.S. dollar exchange rate falls. This is because external Canadian dollar debt decreases when the Canadian current account turns to a surplus. Since a portion of this debt was hedged by yen holding, demand for yen denominated assets decreases and the yen rate falls. In the case of complementary currencies, $M^3_{21} = M^3_{12} < 0$. Hence, a surplus in the Japanese current account ($dB_1 > 0$) causes not only the yen to rise against the dollar ($de^3_1 > 0$) but also causes the currency of the third country, Canada, to fall against the dollar ($de^3_2 < 0$).

The above analysis was conducted as if net foreign asset holdings of each country could all move independently. However, in reality, the total of the three countries' net foreign asset balances always equals zero.

$$B_1 + B_2 + B_3 \equiv 0$$

Consequently, in the above analysis of the three country case, it was tacitly assumed that changes in one country's net foreign assets were all accommodated by changes in the net foreign asset balance of the U.S. (B_3). If Japan has a current account surplus, the effect of this surplus on the yen-U.S. dollar exchange rate differs according to the

19. This case is equivalent to the yen and the mark having a fixed rate between them and floating jointly against the dollar. That $M^3_{11} = M^3_{12}$ is proved in the following fashion. Let $\tilde{F}^3_1$ and $\tilde{F}^3_2$ be random variables for the yen-dollar and the mark-dollar exchange rates. If the yen and the mark are expected to move perfectly parallel against the dollar, then $\tilde{F}^3_1 = \alpha \tilde{F}^3_2$, Where α is a fixed positive constant. Taking logarithms, $\tilde{f}^3_1 = \ell n\, \alpha + \tilde{f}^3_2$, where small letters indicate logarithmic values. Subtracting the expected value of these ($\bar{f}^3_1$, $\ell n\, \alpha + \bar{f}^3_2$) from each side, $\tilde{f}^3_1 - \bar{f}^3_1 = \tilde{f}^3_2 - \bar{f}^3_2$
Multiply both sides of the above by ($\tilde{f}^3_1 - \bar{f}^3_1$) and taking expectations, the following is obtained.

$$M^3_{11} = M^3_{12}$$

deficit countries which correspond to the Japanese surplus. In the previous example of Japan, the U.S. and West Germany, if the deficit country corresponding to Japan's current account surplus is West Germany, then the effect of the strengthening of the yen by this surplus ($M^3_{11}\ dB_1 > 0$) is partially negated by weakening of the yen caused by the West German deficit ($M^3_{12}\ dB_2 < 0$). The upward pressure on the yen-U.S. dollar exchange rate exerted by the Japanese surplus is therefore small. However, if the deficit country corresponding to Japan's surplus is the U.S., then the effect of strengthening the yen in this case is stronger than the earlier case.[20]

V. The Effect of Intervention Policies and the Choice of Currency Used for Intervention

In the paper to date, it has been assumed that the monetary authorities of each country do not intervene in the exchange market (Part III, Section A). Here, the balance sheets of the monetary authorities will be explicitly introduced in the theoretical model, and the mechanism by which intervention changes the balance of each currency denominated asset held by the private sector and affects the exchange rate will be considered. Below, the relationship between the exchange rate and sterilized intervention will be analyzed. Sterilized intervention here refers to the case in which changes in high powered money brought about by intervention are immediately offset by own currency denominated bond operations.

This sterilization behavior can be recorded on the monetary authorities' balance sheets in the form of changes in assets and liabilities denominated in each currency. In the case of n currencies, the balance sheet of a particular monetary authority appears as follows:

$$\sum_{i=1}^{n} Z_i = 0 \tag{18}$$

Z_i : Assets denominated in the currency of the ith-country held by the monetary authority (If it is a debt, $Z_i < 0$).

Because real assets and the capital account of monetary authorities are ignored, the right hand side of equation (18) is zero. Incorporating the balance sheet of

20. This discussion of current account surplus and deficit concerns multilateral balances between all countries and does not concern bilateral balances. For example, in a world consisting of Japan, the U.S. and West Germany, even if each country's muntilateral current balance is in equilibrium, it is possible for Japan to have a surplus with the U.S., the U.S. with West Germany, and West Germany with Japan. In this case, bilateral balances are in disequilibrium, but in the model presented in this paper, these bilateral imbalances do not cause exchange rates to change.

monetary authorities in the international balance of indebetedness table for the three country case given in Table 3, Table 4 is obtained. As can be clearly seen, the form of this table is exactly the same for whatever country's monetary authorities are considered. Consequently, reference will be made henceforth to simplify the monetary authorities rather than the monetary authorities of a particular country. In the general case, the consolidated balance sheets of several countries' monetary authorities can be introduced as the Z_i terms when considering simultaneous actions of monetary authorities in several countries.

Table 4 International Balance of Indebtedness When Intervention Occurs

(in U.S. dollars)

	Japan (1)	West Germany (2)	the U.S. (3)	Monetary Authorities	Total
Yen (1)	X_1^1	X_1^2	X_1^3	Z_1	0
Mark (2)	X_2^1	X_2^2	X_2^3	Z_2	0
Dollar (3)	X_3^1	X_3^2	X_3^3	Z_3	0
Total	B_1	B_2	B_3	0	0

X_n^j : Quantity of net assets denominated in the ith-currency held by the private sector of the jth-country, converted into dollars using the purchaing power parity exchange rate.

Z_i: Quantity of net assets denominated in the ith-currency held by monetary authorities, converted into dollars using the purchasing power parity rates.

From Table 4, the equations for equilibrium of supply of and demand for assets denominated in each country's currency are given as follows.

$$\begin{cases} X_1^1 + X_1^2 + X_1^3 + Z_1 = 0 \\ X_2^1 + X_2^2 + X_2^3 + Z_2 = 0 \\ X_3^1 + X_3^2 + X_3^3 + Z_3 = 0 \end{cases}$$

This can be rewritten for the n country case as

$$\sum_{j=1}^{n} X_i^j + Z_i = 0 \quad i = 1, 2, \cdots\cdots, n \tag{19}$$

Also, the following balance sheet constraints hold

$$\begin{cases} X_1^1 + X_2^1 + X_3^1 = B_1 \\ X_1^2 + X_2^2 + X_3^2 = B_2 \\ X_1^3 + X_2^3 + X_3^3 = B_3 \end{cases}$$

$$Z_1 + Z_2 + Z_3 = 0$$

or, for the general case of n countries.

$$\sum_{i=1}^{n} X_i^j = B_j \qquad j = 1, 2, \cdots\cdots, n \tag{20}$$

$$\sum_{i=1}^{n} Z_i = 0 \tag{21}$$

From these equations, just as in Part IV, Section A, the equations for equilibrium of supply and demand in the foreign exchange market in the matrix notation can be obtained after eliminating the diagonal elements (X_1^1, X_2^2, X_3^3, or for the general case, X_i^i ,i = 1,2,....,n) (see footnote 17 for reference)

$$(\underline{X} - \underline{X}^t)\,\underline{\eta} + (\underline{B} + \underline{Z}) = \underline{0} \tag{22}$$

$$\underline{Z} = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix}$$: A vector that expresses the balance sheet of monetary authorities.

This equilibrium equation is a generalization of equation (13) for the case where no intervention occurs. The only difference between the two is that the final term $\underline{B}$ in equation (13) has now become $(\underline{B}+\underline{Z})$. Using equation (22), the equation for determining exchange rates can be derived as in the pervious section. This is given by

$$e_k^n = g_k^n + [(r_k - \pi_k) - (r_n - \pi_n)] + \frac{1}{c} \sum_{j=1}^{n-1} M_{kj}^n (B_j + Z_j) \tag{23}$$

Comparing the two equations (15) and (23), the third term which shows the risk premium has been modified to include the balance sheet term Z_j of the monetary authoroities for the case where intervention occurs. This indicates that in the framework of this model, intervention is effective in changing exchange rates.

Intervention by monetary authorities, under the balance sheet constraints given by equation (18), is conducted by changing their holdings of assets and liabilities (Z_j)

denominated in each currency. As can be understood from equation (23), the influence of intervention on exchange rates is the same as the influence of changes in the balances of indebtedness B_j. Since this influence was explained in the last section, only a brief explanation will be presented here.

In order to support the exchange rate of the ith-currency against the base currency (nth-currency), intervention in which the ith currency is bought by selling the jth currency can be expressed as follows (j can be equal to n).

$$dZ_i = -dZ_j > 0 \tag{24}$$

The influence of changes in Z_i and Z_j from this intervention on the exchange rate of a particular currency expressed in terms of the nth currency can be analyzed using equation (23). In other words, totally differentiating (23) and setting the variables other than de^n_k, dZ_i, dZ_j equal to zero,

$$de^n_k = \begin{cases} \frac{1}{c}\left[M^n_{ki}\, dZ_i + M^n_{kj}\, dZ_j\right] & j \neq n \\ \frac{1}{c} M^n_{ki}\, dZ_i & j = n \end{cases}$$

is obtained. Now, using equation (24) to replaced Z_j by dZ_i the following equation is derived.

$$de^n_k = \begin{cases} \frac{1}{c}\left[M^n_{ki} - M^n_{kj}\right] dZ_i & j \neq n \\ \frac{1}{c} M^n_{ki}\, dZ_i & j = n \end{cases} \tag{25}$$

This equation shows the effect on the exchange rate of the kth-currency denominated in the nth-currency when monetary authorities practice intervention by selling the jth-currency and buying the ith-currency. In other words, changes in the exchange rate of the kth-currency arising as a result of intervention de^n_k (k = 1,2,..,n-1) depend on the degree of risk aversion of investors as a whole who participate in the foreign exchange market (c: the greater is c, the smaller is the degree of risk aversion). In addition, this change depends on the substitutability and complementarity between the kth-currency and the two currencies i and j which were used for intervention (M^n_{ki}, M^n_{kj}). Finally, this change depends also on the amount of intervention (dZ_i).

From equation (25), several important implications can be derived. In order to easily understand these, let the ith country be Japan, and the base country be the U.S. (the nth country). In order to support the yen against the dollar, consider the case in which yen are bought in exchange for U.S. dollars, Canadian dollars, or marks. Further assume that from the point of view of the U.S. dollar, the yen and the mark are substitute currencies, and the yen and the Canadian dollar are complementary currencies.[21]

1) When buying yen in order to support the yen rate, the effect on the yen-U.S. dollar exchange rate differs according to the currency which is sold. If the sold currency is a substitute for the yen from the point of view of the U.S. dollar like the mark, the effect on the yen-U.S. dollar exchange rate is smaller than the case where U.S. dollars are sold. On the other hand, if the sold currency is a complement to the yen from the point of view of the U.S. dollar like the Canadian dollar, the effect on the yen-U.S. dollar exchange rate is greater than in the case of selling U.S. dollars.

2) When intervening by selling U.S. dollars and buying yen, not only the yen-U.S. dollar exchange rate is affected, but the exchange rates of third countries such as West Germany is affected as well. For instance, selling U.S. dollars and buying yen raises the mark against the U.S. dollar since the mark is a substitute currency for the yen from the point of view of the U.S. dollar. This operation also lowers the Canadian dollar against the U.S. dollar because the Canadian dollar is a complementary currency to the yen from the point of view of the U.S. dollar.

3) Cooperative intervention by selling U.S. dollars and buying both the yen and marks is more effective than independent intervention because of the existence of this third country effect. Hence, comparing the case in which the Bank of Japan independently buys yen and sells dollars in order to prevent a weakening of the yen with the case in which the Bundesbank cooperates by selling dollars and buying marks, the effect of strengthening the yen is greater in this latter case.

4) The following two cases when intervention is not effective can also be imagined. The first case is when intervention occurs using only two currencies which are extremely good substitutes. In other words, when supporting the yen by buying yen and selling dollars or marks, if the yen and the mark, or the yen and the dollar are very good substitutes, this intervention resutls in simply exchanging two similar currencies and is therefore ineffective.

The second case occurs when investors are risk neutral. In this case, $b_n = 0$ in the utility function of Assumption (A-2) in Part III. Hence, investors do not consider the variances which express the risk of the portfolio held by each, and only attempt to maximize the expected return. Because of this, if the return on assets denominated in two different currencies differs only a little, investors attempt to shift all assets entirely to the one with higher yeild. Consequently the asset market is in equilibrium only when the expected return on all assets is the same. In this situation, for risk neutral

21. The assumption that the yen and the mark are substitute assets is fairly realistic. However, concerning the complementarity of the yen and the Canadian dollar, the negative correlation between the two is low, and after the Second Oil Crisis when the yen and the mark were falling against the U.S. dollar, the Canadian dollar was also falling (Chart 1). Consequently, care is necessary in interpreting the realism of the analysis which follows.

investors, all currencies are perfect substitutes. The coefficient c in equation (23) for exchange rate determination is therefore infinite, and the third term of equation (23) is zero. Exchange rates in this situation depend only on domestic and foreign real interest rates and present purchasing power parity rates. Exchange rates are not influenced by intervention or the amount of net foreign assets held. (This latter amount is approximately equal to the cumulative current account balance). Consequently, in the case where investors are risk neutral, not only is intervention in any currency ineffective, but current account imbalances of each country also do not influence exchange rates.[22]

The case in which the coefficient c is infinite is, of course, extreme. However, if many investors participating in the foreign exchange market are not particularly risk averse and a large foreign currency exchange position can be accommodated without changing the exchange rate much, then the market as a whole can be considered to be largely risk neutral. In this case, intervention cannot be expected to influence exchange risk and change exchange rates.

In the analysis of this section, the effect of intervention on the expectations of investors has not been examined. However, in the foreign exchange market, in the case where many factors such as the policy stance of governments, inflation rates, interest rates, and foreign and domestic price levels are not sufficiently assimilated, intervention can convey inforamtion concerning these factors. Consequently, this information effect of intervention can influence the exchange rate. This factor can be important in determining the short term effects of intervention, but because it is difficult to analyze theoretically the analysis in this paper has assumed this effect away. Also, in cases where large-scale intervetion for a medium and long run occurs which considerably changes the correlation between exchange rates, it is possible that the substitutability-complementarity relationship (M_{ij}^{n}) between currencies may change. This relationship may also be changed by oil shocks, international political instability, and internations finanacial instability, because the pattern of exchange rate fluctuation that people expect in the future will change, thus changing the substitutability-complementarity relationship between currencies. As a result of these factors, it is very difficult to predict the effects of intervention.

VI. Conclusion

In this paper, exchange rate determination in a multi-country, multi-currency model has been analyzed theoretically. Important implications are summarized below.

22. This case corresponds to Dornbusch's overshooting model (Fukao (3), pp15-29).

1) The yen-U.S. dollar exchange rate is affected not only by the current account balances of the U.S. and Japan, but also by the current account balances of third countries such as the EC countries. As a result, if a third country or currency area whose currency is a very good substitute for the yen has a large current account deficit, the yen will depreciate against the dollar even if the Japanese current account itself is in surplus.

2) Sterilized intervention policies change the supply of assets denominated in different currencies. Intervention can also change the expectations of investors. Through these two routes, intervention affects exchange rates. This paper has examined the effect of intervention on exchange rates through changes in the supply of and demand for assets, but the necessity of considering the substitutability and complementarity with third country currencies was also pointed out.

Concerning substitutability and complementarity, two important implications were examined. The first involves the effects of cooperative intervention. Suppose the Bank of Japan buys yen and sells dollars to prevent the yen from depreciating and at the same time the Bundesbank intervenes by selling dollars and buying marks. Even though the Bundesbank does not use yen as the intervention currency, the effect of protecting the yen from depreciating against the dollar is greater in this case of cooperative intervention than when the Bank of Japan intervenes alone. This greater effect is caused by the yen and the mark being good substitutes for one another.

The second important point is that even when intervening independently, the effectiveness of intervention greatly differs according to the choice of the currency sold. For instance, in the case of Japan, intervention has traditionally involved U.S. dollars and yen. However, if intervention is extended to include other currencies such as the mark, the substitute-complement relationship is extremely important. In other words, according to the conclusions of this paper, if the Bank of Japan buys yen and sells marks in order to support the yen-U.S. dollar exchange rate, the effect on this exchange rate is much less than when dollars are sold in exchange for yen because the yen and the mark are good substitutes. On the other hand, if Canadian dollars are sold and yen are purchased, because the yen and the Canadian dollar are complementary currencies, the effect on the yen-U.S. dollar exchange rate is greater than if yen are bought and U.S. dollars are sold.

The substitutability and complementarity amongst currencies, which is an important factor in the analysis of the supply of and demand for exchange amongst multiple currencies, depends on expectations of the correlations between the exchange rates of each country. However, concerning what factors fundamentally determine these correlations themselves, the analysis in this paper is insufficient. This remains an important topic for future research.

APPENDIX 1

Mathematical Derivation of Exchange Rate Equation

As much as possible, mathematical exposition has been avoided in the main body of this paper. Only the results of the theory of exchange rate determination have been closely explained. Here, mathematical proofs corresponding to the important resutls will be provided.

(Section 1.) Demand Functions for Assets Denominated in Each Currency.

Here, the demand functions which were given in equation (6), Part III, Section B for foreign currency denominated assets will be derived. Based upon Assumptions (A-1) through (A-6) investors in the nth country invest in riskless, own currency assets and assets denominated in n-1 foreign currencies which bear real exchange risk. The slection of the optimal portfolio satisfies the maximization of the utility function given by equation (1) subject to the initial balance sheet constraints at time t = 0. The initial balance sheet constraint for investors in the nth country is given by

$$W_o^n = \sum_{k=1}^{n-1} E_k^n Y_k^n + D^n = (\underline{E}^n)^t \, \underline{Y}^n + D^n \qquad (1')$$ [23]

W_o^n : Initial wealth of investors (t = 0).

E_k^n ; The sport exchange rate at time t =0 of the kth-currency in terms of the nth-currency.

D^n : Amount of assets denominated in the nth-currency.

Y_k^n : Amount of assets denominated in the kth-currency.

$$\underline{E}^n = \begin{pmatrix} E_1^n \\ E_2^n \\ \vdots \\ E_{n-1}^n \end{pmatrix} \quad : \text{ Vector of spot exchage rates.}$$

23. This balance sheet constraint ignores real assets held by investors in the nth country. Because of this, W_o^n is the quantity of financial assets held by the nth-country's investors at the beginning of the period. This is equal to the total of the cumulative deficit of the government of the nth-country (where all of the deficit is assumed to be financed by own currency denominated bonds) and the net quantity of foreign assets held by the nth country. (See Fukao (4), Table 2).

$$\underline{Y}^n = \begin{bmatrix} Y_1^n \\ Y_2^n \\ \vdots \\ Y_{n-1}^n \end{bmatrix}$$: Vector of the portfolio of foreign currency denominated assets held by investors in the nth-country.

(To distinguish vectors and matrices from scalers in this paper, the former are underlined. Transpose matrices are denoted by the superscript t).

While fulfilling equation (1′), investors in the nth country choose their portfolio $Y_1^n, Y_2^n, \ldots, Y_{n-1}^n, D^n$, to maximize the utility function given by equation (1).

$$u = a_n \mu_w - b_n \sigma_w^2 \qquad (1)$$

(An explanation of the notation is given in Assumption (A-2) of Part III, Section A.) If Y_i^n is positive, this is an asset and if negative, a liability. Portfolio management of each currency denominated asset or liability can be freely conducted at market interest rates.

At time t = 0 when the portfolio is determined, the wealth $\widetilde{W}^n$ of nth-country investors at the end of the period is a random variable because exchange rates at the end of the period are not known with certainty. (In this paper, random variables are indicated by a tilda). $\widetilde{W}^n$ can be written as follows:

$$\widetilde{W}^n = \sum_{k=1}^{n-1} Y_k^n (1 + R_k) \widetilde{F}_k^n + D^n (1 + R_n) \qquad (2')$$

R_k : The nominal interest rate in the kth county.

$\widetilde{F}_k^n$: Exchange rate of the kth-currency measured in the nth-currency at the end of the period.

R_n : The nominal interest rate in the nth-country (home country).

According to Assumption (A-4), the expected valued of the logarithm $\widetilde{f}_k^n$ ($\equiv \ell_n \widetilde{F}_k^n$) of the exchange rate $\widetilde{F}_k^n$ at the end of the period equals the logarithm $\bar{f}_k^n$ ($\equiv \ell_n \bar{F}_k^n$) of the purchasing power parity rate $\bar{F}_k^n$ at the end of the Period (see Appendix B). That is, if ε denotes the mathematical expectation operator,

$$\varepsilon[\widetilde{f}_k^n] = \bar{f}_k^n \quad k = 1, 2, \cdots, n-1 \qquad (3')$$

holds.[24] Also, it is assumed that investors' expectations as to future exchange risk is given by the following variance-covariance matrix.

24. Here it is not assumed that the expected value of the exchange rate equals the purchasing power parity rate but rather that the expected value of the logarithm of the exchange rate equals the logarithm of the purchasing power parity rate.

$$\varepsilon\left[(\tilde{f}_j^n - \bar{f}_j^n)(\tilde{f}_k^n - \bar{f}_k^n)\right] = M_{jk}^n \tag{4'}$$

M_{jk}^n : If $j \neq k$, this is the covariance between $\tilde{f}_j^n$ and $\tilde{f}_k^n$.
If $j = k$, this is the variance of $\tilde{f}_j^n$.

Moverover, in order to simplify the below equations, domestic and foreign interest rates are defined as the logarithm of one plus the interest rate.

$$r_k = \ell_n(1+R_k) \quad k=1,2,\cdots, n \tag{5'}$$

Using the new variables above and rewriting equation (2′) which shows the end of the period value of assets, the following is obtained where exp(x) represent the exponential function e^x.

$$\tilde{W}^n = \sum_{k=1}^{n-1} Y_k^n \exp(r_k + \tilde{f}_k^n - \bar{f}_k^n + \bar{f}_k^n) + D^n \exp(r_n)$$

$$= \sum_k Y_k^n \exp(\bar{f}_k^n) \exp(r_k + \tilde{f}_k^n - \bar{f}_k^n) + D^n \exp(r_n)$$

$$\cong \sum_k Y_k^n \bar{F}_k^n (1 + r_k + \tilde{f}_k^n - \bar{f}_k^n) + D^n (1 + r_n)^{25}$$

Consequently, the expected value μ_w and the variance σ_w^2 of the end of the period assets $\tilde{W}^n$ is given by the following:

$$\mu_w = \varepsilon(\tilde{W}^n)$$

$$= \sum_{k=1}^{n-1} Y_k^n \bar{F}_k^n (1 + r_k) + D^n (1 + r_n) \tag{6'}$$

$$= (\underline{Y}^n)^t \underline{F}^n (\underline{\eta} + \underline{r}^n) + D^n (1 + r_n)$$

$$\underline{F}^n = \begin{pmatrix} \bar{F}_1^n & 0 & \cdots & 0 \\ 0 & \bar{F}_2^n & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 \cdots 0 & & \bar{F}_{n-1}^n \end{pmatrix}$$: Diagonal matrix of purchasing power parity rates of each currency against the currency of the nth-country.

25. In transforming this equation, the following approximation was used for small x.

$$e^x = \exp x \cong 1 + x$$

$$\underline{\eta} = \overset{(n-1)\times 1}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}$$: Vector of one's of length n - 1.

$$\underline{r}^n = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \end{pmatrix}$$: Vector of the interest rates of countries other than the nth-country.

$$\begin{aligned} \sigma_w^2 &= Var(\widetilde{W}^n) \\ &= \varepsilon(\widetilde{W}^n - \mu_w)^2 \\ &= \varepsilon\left(\sum_{k=1}^{n-1} \bar{Y}_k^n \bar{F}_k^n (\tilde{f}_k^n - \bar{f}_k^n)\right)^2 \\ &= \sum_j \sum_k (Y_j^n \bar{F}_j^n) M_{jk}^n (Y_k^n \bar{F}_k^n) \\ &= (\underline{Y}^n)^t \underline{F}^n \underline{M}^n \underline{F}^n \underline{Y}^n \end{aligned} \tag{7'}$$

$$\underline{M}^n = \begin{pmatrix} M_{11}^n & M_{12}^n & \cdots & M_{1n-1}^n \\ M_{21}^n & M_{22}^n & & M_{2n-1}^n \\ \vdots & & \ddots & \vdots \\ M_{n-1,1}^n & \cdots & & M_{n-1,n-1}^n \end{pmatrix}$$: Matrix of the variances and covariances between exchange rates at the end of the period, det $M^n \neq 0$.[26]

Inserting expression for μ_w and σ_w^2 derived above into equation (1) gives

$$\begin{aligned} u = {} & a_n[(\underline{Y}^n)^t \underline{F}^n(\underline{\eta} + \underline{r}^n) + D^n(1+r_n)] \\ & - b_n[(\underline{Y}^n)^t \underline{F}^n \underline{M}^n \underline{F}^n \underline{Y}^n] \end{aligned} \tag{8'}$$

26. The assumption that det $\underline{M}^n \neq 0$ implies that all currencies float against one another and are not pegged against a single currency or a basket of currencies. Concerning the properties of this variance-covariance matrix, see Lindgren (10, p464).

The investors' optimal portfolio ($\underline{Y}^n$, D^n) is obtained by maximizing (8′) subject to the constraints given by equation (1′).

This constrained maximization problem gives the following necessary conditions in addition to the conditions provided by equation (1′)

$$a_n \underline{F}^n(\underline{\eta}+\underline{r}^n)-2b_n\underline{F}^n\underline{M}^n\underline{F}^n\underline{Y}^n-\lambda_n\underline{E}^n=\underline{0} \quad (9')$$

$$a_n(1+r_n)-\lambda_n=0 \quad (10')$$

where λ_n is the Lagrangian multiplier.

From equation (9′) and (10′), the following is obtained for the demand by nth-country investors for foreign currency denominated assets.

$$\underline{F}^n\underline{Y}^n=c_n[\underline{M}^n]^{-1}[(\underline{\eta}+\underline{r}^n)-(1+r_n)(\underline{F}^n)^{-1}\underline{E}^n] \quad (11')$$

$c_n \equiv \frac{a_n}{2b_n}$: Parameter of the utility function.

The right hand side of equation (11′) can be simplified by noting that the kth-element of the vector given by the longest parenthetical terms is

$$1+r_k-(1+r_n)\exp(e_k^n-\bar{f}_k^n)$$

$$\cong(1+r_k)-(1+r_n)(1+e_k^n-\bar{f}_k^n)$$

$$\cong(r_k-r_n)+(\bar{f}_k^n-e_k^n)$$

e_k^n : Logarithm of the spot exchange rate $e_k^n=\ell n\, E_k^n$

Hence, the following holds.

$$\begin{aligned}&(\underline{\eta}+\underline{r}^n)-(1+r_n)(\underline{F}^n)^{-1}\underline{E}^n\\&=(\underline{r}^n-r_n\underline{\eta})+(\underline{f}^n-\underline{e}^n)\end{aligned} \quad (12')$$

$$\underline{f}^n=\begin{pmatrix}\bar{f}_1^n\\ \bar{f}_2^n\\ \vdots\\ \bar{f}_{n-1}^n\end{pmatrix}$$

: Vector of the logarithm of the purchasing power parity exchange rates at the end of the period.

$$\underline{e}^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{n-1}^n \end{bmatrix}$$: Vector of the logarithm of the spot exchange rates e_k^n.

Inserting this into equation (11′), the demand function for foreign currency denominate assets for investors in the nth-country is given by

$$\underline{X}^n = c_n [\underline{M}^n]^{-1} [(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)] \qquad (13')$$

This equation is the same as equation (6) of Part III. $\underline{X}^n \equiv \underline{F}^n Y^n$ is the demand for foreign currency denominated assets for investors in the nth-country converted into nth currency units using purchasing power parity rates.

The investors' utility function parameter c_n in this demand function for foreign currency denominated assets is related to the coefficient which shows the degree of risk aversion of investors.[27] Also, $\underline{M}^n$ expresses as a variance-covariance matrix the degree of risk from future exchange rate fluctuations and the correlation between exchange rates. Lastly, $[(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)]$ is a vector which shows the expected rate of return differentials between foreign and domestic assests. In other words, writing the kth-element of this as $(r_k + \bar{f}_k^n - e_k^n) - r_n$, it can be seen that the first term shows the interest rate r_k on assets denominated in the kth-currency plus the expected rate of increase in the exchange rate of the kth-currency against the nth-currency $(\bar{f}_k^n - e_k^n)$. Hence, this first term shows the expected rate of return of assets denominated in the kth-currency. The second term gives the yeild on assets denominated in the nth-currency. The difference between these two terms is the expected yeild differential between foreign and domestic assets for investors in the nth country.

Properties of the demand function for foreign currency denominated assets given by equation (13′) have already been explained in detail in Part III, Section B, and further elaboration is not necessary. The demand for own currency denominated assets by investors in the nth-country.can be obtained from equation (13′) and the balance sheet constraints given by equation (1′). That is,

27. If as is usually assumed, the degree of risk aversion decreases when investors' net assets increase, then c_n becomes larger when the quantity of assets of the nth country increases, and the demand for foreign currency denominated assets increases. Here, however, analysis is conducted under the assumption that c_n is constant.

$$D^n = W_o^n - \sum_{k=1}^{n-1} E_k^n Y_k^n = W_o^n - \sum_{k=1}^{n-1} E_k^n \left(\frac{X_k^n}{\bar{F}_k^n}\right) \qquad (14')$$

holds.[28] X_k^n in the above equation is given by equation (13′).

(Section 2.) Equation for Exchange Rate Determination.

In this section, equation (15) for exchange rate determination which is introduced in Part IV of the text will be derived. First, after deriving the equation for equilibrium of supply and demand in the foreign exchange market in matrix notation, the exchange rate determination equation will be derived for the special case in which only investors in the nth-country hold foreign currency denominated assets and liabilities. Then, an exchange rate determination equation which has the same shapes as this equation will be derived for the general case in which investors in all countries hold foreign currency denominated assets and liabilities.

(Supply and Demand in the Foreign Exchange Market)

The equation for the equilibrium of supply of and demand for yen assets in the foreign exchange market was derived in equation (11), Part IV, Section A. Here, that equation will be generalized to the n country case by utlizing matrix notation. First, the following matirces are defined.

$$\underline{Y} = \begin{bmatrix} 0 & Y_1^2 & Y_1^3 & \cdots\cdots & Y_1^n \\ Y_2^1 & 0 & Y_2^3 & \cdots\cdots & Y_2^n \\ Y_3^1 & Y_3^2 & 0 & \cdots\cdots & Y_3^n \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & Y_{n-1}^n \\ Y_n^1 & Y_n^2 & \cdots\cdots & Y_n^{n-1} & 0 \end{bmatrix} \quad n \times n$$

28. D^n in equation (14’) shows the total amount held by investors in the nth-country of own currency denominated assets. Hence, this differs from the net quantity of foreign assets denominated in own currency. For instance, in Table 3, thinking of the U.S. as the nth-country, the net quantity of foreign assets denominated in own currency Y_3^3 differs from the amount held by the U.S. private sector of dollar denominated assets D^3. This difference arises because the U.S. private sector can hold domestic assets such as dollar denominated U.S. government bonds.

$$\underline{E} = \begin{bmatrix} E_1^n & 0 & 0 & \cdots & 0 \\ 0 & E_2^n & 0 & \cdots & 0 \\ 0 & 0 & E_3^n & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & 0 & E_n^n \end{bmatrix} \quad n \times n$$

$$\underset{n\times 1}{\underline{B} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}} \qquad \underset{n\times 1}{\underline{\eta} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}$$ [29]

Using these matrices, conditions for the equilibrium of supply and demand in the foreign exchange market can be written as

$$\underbrace{\underline{Y}^t \underline{E}^t \underline{\eta} - \underline{B}}_{\text{supply}} = \underbrace{\underline{E}\,\underline{Y}\,\underline{\eta}}_{\text{demand}} \tag{15'}$$

$$(\underline{E}\,\underline{Y} - \underline{Y}^t \underline{E}^t)\,\underline{\eta} + \underline{B} = \underline{0} \tag{16'}$$

These equilibrium equations consist of n equations corresponding to n currencies. If equilibrium holds for the supply of and demand for n-1 currencies, then it must also hold for the nth currency. Hence, these n equation are not independent (Walras' Law). This can be shown by premultiplying the left hand side of equation (16′) by η^t. In other words,

$$\eta^t(\underline{E}\,\underline{Y} - \underline{Y}^t\underline{E}^t)\,\underline{\eta} + \underline{\eta}^t\underline{B} = \underline{\eta}^t\underline{E}\,\underline{Y}\,\underline{\eta} - \underline{\eta}^t\underline{Y}^t\,\underline{E}^t\eta + \underline{\eta}^t\underline{B}$$

$$= \underline{\eta}^t\underline{E}\,\underline{Y}\,\underline{\eta} - \underline{\eta}^t\,\underline{E}\,\underline{Y}\,\underline{\eta} + \underline{\eta}^t\,\underline{B} = \underline{\eta}^t\underline{B}$$

29. In equation (6'), $\underline{\eta}$ was a vector of ones of length n-1. Here, however, it is a vector of ones of length n.

Here, $\underline{\eta}^t\underline{B}$ equals zero because it is the sum of balance of indebtedness of all countries. As a result, these n equations are not independent.

A precise expression for the conditions for the equilibrium of supply and demand in the foreign exchange market was derived above. This will, however, be transformed for the purpose of the following analysis. First, consider the conditions for a hypothetical long run equilibrium in the world ecomony. That is, the exchange rates of each country E^n_j equal the exchange rates determined by purchasing power parity $\bar{F}^n_j$. Furthermore, assume that the real interest rates in each country are equal. (In Assumption (A-3) of this paper, inflation rates were assumed to be zero, and consequently the above assumption means that nominal interest rates are equal). In such a world, as can be understood from equation (13′) for the demand functions for foreign currency denominated assets for the nth country investors, demands by each country for foreign currency denominated assets are zero. Consequently, in this long run equilibrium, equations (17′) and (18′) hold.

$$\underline{Y} = \underline{0} \tag{17′}$$

$$\underline{E} = \underline{F} \tag{18′}$$

$$\underline{F} = \begin{pmatrix} \bar{F}^n_1 & 0 & \cdots & 0 \\ 0 & \bar{F}^n_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 0 & 0 & \cdots\ 0 & \bar{F}^n_n \end{pmatrix}$$: Diagonal matrix of purchasing power parrity rates

Starting from this long run equilibrium and totally differentiating (16′), the next equation is obtained

$$(\underline{dE}\,\underline{Y} + \underline{E}\,\underline{dY} - \underline{dY}^t\underline{E}^t - \underline{Y}^t\,\underline{dE}^t)\underline{\eta} + \underline{dB} = \underline{0} \tag{19′}$$

Here, $\underline{dE}$, $\underline{dY}$, and $\underline{dB}$ are the total differentials of $\underline{E}$, $\underline{Y}$, and $\underline{B}$. Substituting (17′) and (18′) into (19′), equation (20′) is obtained.

$$(\underline{F}\,\underline{dY} - \underline{dY}^t\underline{F}^t)\,\underline{\eta} + \underline{dB} = \underline{0} \tag{20′}$$

Consequently, around this long run equilibrium, the following conditions for the equilibrium of supply and demand in the foreign exchange market can be used for small $\underline{Y}$'s and $\underline{B}$'s.

$$(\underline{F}\,\underline{Y} - \underline{Y}^t\,\underline{F}^t)\,\underline{\eta} + \underline{B} = \underline{0} \tag{21′}$$

Defining the new matrix $\underline{X}$ by the following

$$\underline{X} \equiv \underline{F}\,\underline{Y} \tag{22'}$$

the next equation is obtained.

$$(\underline{X} - \underline{X}^{t})\,\underline{\eta} + \underline{B} = \underline{0} \tag{23'}$$

Hence, this equation shows that if deviations from this long run equilibrium are small, the purchasing power parity rates $\bar{F}^{n}_{j}$ can be used to aggregate the demand functions Y^{j}_{i} for foreign currency denominated assets. Because analysis can be greatly simplified by using the conditions for equilibrium in the foreign exchange market expressed in the form of equation (23′), this equation will be used below.

(The Equation for Exchange Rate Determination in the Case Where only One Country Bears Exchange Risk)

Here, for simplification, consider the case in which only investors in the nth-country hold foreign currency denominated assets and liabilities. The other n-1 countries hold no foreign countries hold no foreign currency denominated assets or debt, Using this extremely strong assumption, the equation for exchange rate determination will be derived. However, as will be shown later, the exchange rate determination equation for the general case in which investors of all n countries hold foreign assets and liabilities is of exactly the same shape as that for this special case. Hence, the results of the analysis for this special case are sufficiently general.

When only the nth country bears exchange risk, the international balance of indebtedness can be written as is shown in Table 1′ for the four countries' case. Using the results of the analysis obtained in equation (23′), X^{j}_{i} is used in place of Y^{j}_{n} because the foreign currency denominated asset position is evaluated by using purchasing power parity rates.

The matrix $\underline{X}$ corresponding to this table is written as follows, replacing the diagonal terms with zeros (see the definition of $\underline{Y}$ for equations (15′) and (22′) for reference).

$$\underline{X} = \begin{pmatrix} 0 & 0 & 0 & X^{4}_{1} \\ 0 & 0 & 0 & X^{4}_{2} \\ 0 & 0 & 0 & X^{4}_{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 1′ Internation Balance of Indebtedness when only the U.S. holds Foreign Currency Denominated Assects and Liabilities

(in U.S. dollars)

	Japan (1)	West Germany (2)	Canada (3)	the U.S. (4)	Total
Yen (1)	X_1^1	0	0	X_1^4	0
Mark (2)	0	X_2^2	0	X_2^4	0
Canadian dollar (3)	0	0	X_3^3	X_3^4	0
U.S. dollar (4)	0	0	0	0	0
Total	B_1	B_2	B_3	B_4	0

X_i^j : Net quantity of assets denominated in the ith-currency held by the jth-country. converted into dollars using purchasing power parity rates.
$(X_i^j = F_i^4\ Y_i^j \quad i, j = 1 \sim 4)$

Substituting this into the equation for the market clearning conditions in the foreign exchange marekt given by equation (23′), equation (24′) is obtained:

$$(\underline{X} - \underline{X}^t)\,\underline{\eta} + \underline{B} = \begin{pmatrix} X_1^4 + B_1 \\ X_2^4 + B_2 \\ X_3^4 + B_3 \\ -X_1^4 - X_2^4 - X_3^4 + B_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (24')$$

In the case of n countries, the following holds for the market clearing conditions.

$$\begin{pmatrix} X_1^n + B_1 \\ X_2^n + B_2 \\ \vdots \\ X_{n-1}^n + B_{n-1} \\ -\sum_{i=1}^{n-1} X_i^n + B_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \qquad (25')$$

These equations are not independent because of Walras' Law, and hence any n-1 of these may be selected as the market clearing conditions. Choosing the first n-1 equations, the market clearing conditions for the foreign exchange market become

$$\underline{X}^n + \underline{B}^n = \underline{0} \qquad (26')$$

$$(n-1)\times 1$$

$$\underline{B}^n = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \end{bmatrix}$$: Vector of the net quantity of assets held by each country other than the nth-country.

The equation for exchange rate determination is given by substituting (13′) into equation (26′) and solving for the spot exchange rate. In other words, equation (27′) is solved for e^n giving equation (28′).

$$c_n [M^n]^{-1} [(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)] + \underline{B}^n = \underline{0} \qquad (27')$$

$$\underline{e}^n = \underline{f}^n + \underline{r}^n - r_n \underline{\eta} + \frac{1}{c_n} \underline{M}^n \underline{B}^n \qquad (28')$$

Where, $c_n = \frac{a_n}{2b_n}$ is the coefficient showing the degree of risk aversion of investors in the nth-country as a whole. Looking at only the kth-element of the above vector equation, equation (29′) is obtained for the determination of the exchange rate between the kth-currency and the nth-currency.

$$e_k^n = \bar{f}_k^n + (r_k - r_n) + \frac{1}{c_n} \sum_{j=1}^{n-1} M_{kj}^n B_j \qquad (29')$$

As can be seen from this equation, the exchange rate of the kth currency in terms of the nth-currency is determined by three factors. The first is the interest rate differential between the kth-country and the nth-country $(r_k - r_n)$, the second is the expected value $\bar{f}_k^n$ of the spot rate at the end of the period, and the third is the risk premium which is shown as the last term on the right hand side of the above equation.[30] This risk premium is a weighted total of the net foreign assets excluding those of the nth-country. where the weights are the terms M_{kj}^n of the variance-covariance matrix. Equation (29′) was derived under the assumption that only investors in the nth country hold foreign currency assets and liabilities. However, even when this assumption is removed, an equation of exactly the same shape holds, as will be shown below.

30. Concerning the fact that this term is the risk premium, see Fukao (2).

(The Equation for Exchange Rate Determination in the Case Where All Countries Bear Exchange Risk)

The privious analysis is expanded below to the general case where investors in all n countries hold foreign currency denominated assets and liabilities. First, the demand function for foreign curency denoinated assets for investors in the jth-country is obtainedin the same way as was equation (13′) for investors in the nth-country. This demand function is given by

$$\begin{aligned} \underline{X}^j &= \overline{F}^n_j \, \underline{F}^j \, \underline{Y}^j \\ &= \overline{F}^n_j c_j [\underline{M}^j]^{-1} [(\underline{r}^j - r_j \underline{\eta}) + (\underline{f}^j - \underline{e}^j)] \end{aligned} \tag{30'}$$

$$\underline{X}^j = \begin{bmatrix} X^j_1 \\ X^j_2 \\ \vdots \\ X^j_{j-1} \\ X^j_{j+1} \\ \vdots \\ X^j_n \end{bmatrix}$$: Vector of foreign currency denominated assets demanded by investors in the jth-country, This has been converted into the nth-currency using the purchasing power parity rate $\overline{F}^n_j$.

c_j : A parameter of the utility function for investors in the jth-country. This corresponds to c_n.

$$\underline{E}_j = \begin{bmatrix} E^j_1 \\ E^j_2 \\ \vdots \\ E^j_{j-1} \\ E^j_{j+1} \\ \vdots \\ E^j_n \end{bmatrix}$$: vector of spot exchange rates. E^j_k is the exchange rate of the kth-currency in terms of the jth-currentcy, and consequently, this is a vector of exchange rates for n-1 currencies from the point of view of the jth-country. $\underline{e}^j$、$\widetilde{\underline{f}}^j$、$\underline{f}^j$、$\underline{F}^j$ are defined similary to $\underline{E}^j$. All are vectors or matrices corresponding to the previous variable for the nth-currency.

$$\underline{r}^j = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{j-1} \\ r_{j+1} \\ \vdots \\ r_n \end{bmatrix}$$

: Vector of interest rates for all countries except country j expressed in logarithms. That is, R_k is the interest rate of the kth-country and

$$r_k = \ell n\,(1+R_k)$$

$$\underline{Y}^j = \begin{bmatrix} Y^j_1 \\ Y^j_2 \\ \vdots \\ Y^j_{j-1} \\ Y^j_{j+1} \\ \vdots \\ Y^j_n \end{bmatrix}$$

: The demand function for foreign currency denominated assets expressed in the corresponding currency. Thus, Y^j_k is the amount of assets denominated in the kth-currency held by investors of the jth-country.

Substituting this $\underline{X}^j$ in the $\underline{X}$ matrix in equation (23′) which gives the market clearing conditions and solving for the spot exchange rate vector $\underline{e}^j$, the eqution for exchange rate determination should be obtained. However, the respective X^j's (j= 1,2,..n) incorporate own country variables and vectors ($\underline{M}^j$, $\underline{r}^j$,$\underline{f}^j$, $\underline{e}^j$), and solving for spot exchange rates in the manner outlined above is difficult because the variables are numerous. Consequently, this problem will be solved below by converting equation (30′), which is written from the point of view of the jth-country. into an equation from the point of view of the nth-country.

As was shown in Part II, exchange rate fluctuations between the yen, the mark, the Canadian dollar, and the U.S. dollar differ greatly according to the perspective of which country is adopted (Charts 1-4). Also, depending on which point of view is adopted, the vector of exchange rates and the variance-covariance matrix which shows the correlation amongst currencies also differ. Here, we consider the transformation of the vector $\underline{e}^j$ and the matrix $\underline{M}^j$ which are written from the point of view of the jth-country into the vector $\underline{e}^n$ and the matrix $\underline{M}^n$ written from the perspective of the base country.

Concerning this matrix and vector transformation, the following lemmas hold.

Lemma 1 (Solnik)[31]

31. This lemma is proved in Solnik (13).

(i) $\underline{e}^{j} = \underline{H}_{j}\, \underline{e}^{n}$

(ii) $\underline{\tilde{f}}^{j} = \underline{H}_{j}\, \underline{\tilde{f}}^{n}$

(iii) $(\underline{r}^{j} - r_{j}\, \underline{\eta}) = \underline{H}_{j}\, (\underline{r}^{n} - r_{n}\, \underline{\eta})$

where

$$\underline{H}_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 & -1 & 0 & & \vdots \\ \vdots & & & \ddots & 1 & -1 & 0 & & \vdots \\ \vdots & & & & 0 & -1 & 1 & \ddots & \vdots \\ \vdots & & & & 0 & -1 & 0 & \ddots & 0 \\ \vdots & & & & & \vdots & & & 1 \\ 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \end{bmatrix} \quad (n-1)\times(n-1)$$

$\uparrow$ j^{th} column

Proof

(i) From the definition of exchange rates,

$$E_k^j = E_k^n \,/\, E_j^n$$

By taking natural logarithms,

$$e_k^j = e_k^n - e_j^n$$

and noting $e_n^n = \ell n\, E_n^n = \ell n\, 1 = 0$, we get

$$\underline{e}^j = \begin{bmatrix} e_1^j \\ e_2^j \\ \vdots \\ e_{j-1}^j \\ e_{j+1}^j \\ \vdots \\ e_{n-1}^j \\ e_n^j \end{bmatrix} = \begin{bmatrix} e_1^n - e_j^n \\ e_2^n - e_j^n \\ \vdots \\ e_{j-1}^n - e_j^n \\ e_{j+1}^n - e_j^n \\ \vdots \\ e_{n-1}^n - e_j^n \\ - e_j^n \end{bmatrix} = \underline{H}_j\, \underline{e}^n$$

(ii) This equation can be proved similarly.

(iii) By conducting the matrix multiplication of the righthand side of the equation, we get the lefthand side.

Corollary 1

(i) $\underline{f}^{j} = \underline{H}_{j}\ \underline{f}^{n}$

(ii) $\underline{M}^{j} = \underline{H}_{j}\ \underline{M}^{n}\ \underline{H}_{j}^{t}$

Proof

(i) This is obvious.

(ii) $$\begin{aligned}\underline{M}^{j} &= \varepsilon\ (\widetilde{\underline{f}^{j}} - \underline{f}^{j})\ (\widetilde{\underline{f}^{j}} - \underline{f}^{j})^{t} \\ &= \varepsilon\ [\underline{H}_{j}(\widetilde{\underline{f}^{n}} - \underline{f}^{n})\ (\widetilde{\underline{f}^{n}} - \underline{f}^{n})^{t}\ \underline{H}_{j}^{t}] \\ &= \underline{H}_{j}\ \underline{M}^{n}\ \underline{H}_{j}^{t}\end{aligned}$$

Lemma 2

$$\underline{H}_{j} = \underline{H}_{1}\ \underline{A}_{j}$$

where

$$\underline{A}_{j} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & & 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 & & 0 \\ 0 & 0 & 1 & & & \vdots & & & \vdots \\ \vdots & & & \ddots & 0 & 0 & 0 & & \vdots \\ \vdots & & & & 1 & 0 & 0 & & \vdots \\ \vdots & & & & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & & & & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots 0 & 1 \end{pmatrix} \quad (n-1)\times(n-1)$$

↑ j^{th} column

Proof

From the definition of $\underline{H}_{j}$ in lemma 1,

$$\underline{H}_{1} = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & & 0 \\ -1 & 0 & 0 & 1 & & \vdots \\ \vdots & & & & \ddots & \\ -1 & 0 & & & 1 & 0 \\ -1 & 0 & & & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad (n-1)\times(n-1)$$

when we post-multiply $\underline{H}_1$ by $\underline{A}_j$, $\underline{A}_j$ shifts the columns of $\underline{H}_1$ as follows:

Original column # 1, 2, 3, 4, ..., j−1, j, j+1, ..., n−1

new column # 1, 2, 3, ..., j−2, j−1, j, j+1, ..., n−1

The newly creatd matrix is obviously equal to $\underline{H}_j$.

Lemma 3

$$\underline{H}_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 1 & 0 & 0 & & 0 & 0 & -1 \\ 0 & 1 & 0 & & 0 & 0 & -1 \\ 0 & 0 & 1 & \ddots & & & \vdots \\ \vdots & & & \ddots & 0 & 0 & -1 \\ \vdots & & & & 1 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix} (n-1)\times(n-1)$$

Proof

By multiplying $\underline{H}_1$ by the above matrix $\underline{H}_1^{-1}$, we get an identity matrix.

Lemma 4

$$\underline{H}_j^{-1} = \underline{A}_j^t \underline{H}_1^{-1}$$

Proof

From lemma 2, we get

$$\underline{H}_j^{-1} = [\underline{H}_1 \underline{A}_j]^{-1} = \underline{A}_j^{-1} \underline{H}_1^{-1}$$

Here, we can find that $\underline{A}_j^{-1} = \underline{A}_j^t$ by multiplying $\underline{A}_j$ by $\underline{A}_j^t$ as follows

$$\underline{A}_j\underline{A}_j^t = \begin{pmatrix} 0\,0\,0 & \cdots & 0\,1\,0 & \cdots & 0 \\ 1\,0\,0 & & 0\,0\,0 & & 0 \\ 0\,1\,0 & & \vdots\,\vdots\,\vdots & & \vdots \\ 0\,0\,1 & \ddots & & & \vdots \\ \vdots & \ddots & 0\,0\,0 & & \vdots \\ \vdots & & 1\,0\,0 & & \vdots \\ \vdots & & 0\,0\,1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & 0\,0 & \cdots\,0 & 1 \end{pmatrix} \begin{pmatrix} 0\,1\,0 & \cdots & & & 0 \\ 0\,0\,1 & \ddots & & & 0 \\ \vdots\,\vdots & & \ddots & & \vdots \\ 0\,0 & & 0\,1\,0 & \cdots & 0 \\ 1\,0 & \cdots & 0\,0\,0 & \cdots & 0 \\ 0\,0 & & 0\,0\,1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0\,0 & \cdots & & 0 & 1 \end{pmatrix} \leftarrow j^{th} \text{ row}$$

↑ j^{th} column

$$= \underline{I}$$

Therefore we get

$$\underline{H}_j^{-1} = \underline{A}_j^t \underline{H}_1^{-1}$$

These four lemmas are necessary mathematical preparations for the transformation of exchange rate vectors. Using these lemmas, transform equation (39′) from the perspective of the jth country to the perspective of the nth-country. From Lemma 1 Corollary1, equation (30′) becomes

$$\underline{X}^j = \bar{F}_j^n c_j \; [\underline{H}_j \underline{M}^n \underline{H}_j^t]^{-1} \cdot [\underline{H}_j(\underline{r}^n - r_n \underline{\eta}) + \underline{H}_j(\underline{f}^n - \underline{e}^n)]$$

$$= \bar{F}_j^n c_j [\underline{H}_j^t]^{-1} [\underline{M}^n]^{-1} \underline{H}_j^{-1} \underline{H}_j \cdot [(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)]$$

$$= \bar{F}_j^n c_j [\underline{H}_j^t]^{-1} [\underline{M}^n]^{-1} [(\underline{r}^n - r_n \underline{\eta}) + (\underline{f}^n - \underline{e}^n)]$$

Now, simplifying by using equation (13′), the following is obtained

$$\underline{X}^j = \frac{\bar{F}_j^n c_j}{c_n} [\underline{H}_j^t]^{-1} \cdot \underline{X}^n = \frac{\bar{F}_j^n c_j}{c_n} [H_j^{-1}]^t \cdot \underline{X}^n$$

Using Lemma 4, this becomes

$$\underline{X}^j = \frac{\bar{F}_j^n c_j}{c_n} [\underline{A}_j^t \; \underline{H}_1^{-1}]^t \cdot \underline{X}^n$$

$$= \frac{\bar{F}_j^n c_j [\underline{H}_1^{-1}]^t \underline{A}_j \underline{X}^n}{c_n}$$

$$= \frac{\bar{F}_j^n c_j}{c_n} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots \; 0 & 1 \\ -1 & -1 & -1 & -1 & \cdots \; -1 & -1 \end{pmatrix} \begin{pmatrix} X_j^n \\ X_1^n \\ X_2^n \\ \vdots \\ X_{j-1}^n \\ X_{j+1}^n \\ \vdots \\ X_{n-1}^n \end{pmatrix}$$

$$
= \frac{\bar{F}^n_j c_j}{c_n} \begin{pmatrix} X^n_1 \\ X^n_2 \\ \vdots \\ X^n_{j-1} \\ X^n_{j+1} \\ \vdots \\ X^n_{n-1} \\ -\sum_{k=1}^{n-1} X^n_k \end{pmatrix}
$$

Consequently $\underline{X}^j$ can be expressed in terms of $\underline{X}^n$ as follows

$$
\underline{X}^j = \begin{pmatrix} X^j_1 \\ X^j_2 \\ \vdots \\ X^j_{j-1} \\ X^j_{j+1} \\ \vdots \\ X^j_{n-1} \\ X^j_n \end{pmatrix} = \frac{\bar{F}^n_j c_j}{c_n} \begin{pmatrix} X^n_1 \\ X^n_2 \\ \vdots \\ X^n_{j-1} \\ X^n_{j+1} \\ \vdots \\ X^n_{n-1} \\ -\sum_{k=1}^{n-1} X^n_k \end{pmatrix} \tag{31'}
$$

This equation expresses the relationship between the demand function $\underline{X}^j$ for foreign currency denominated assets for investors in the jth-country and the demand function $\underline{X}^n$ for investors in the nth country. Making use of this, the matrix $\underline{X}$ in equation (23′) which gives the market clearing conditions can be constructed with elements of the vector $\underline{x}^n$ as follows:

$$
\underline{X} = \frac{1}{c_n} \begin{pmatrix} 0 & X^n_1 & X^n_1 \cdots X^n_1 & X^n_1 \\ X^n_2 & 0 & X^n_2 \cdots X^n_2 & X^n_2 \\ X^n_3 & X^n_3 & 0 \;\cdots X^n_3 & X^n_3 \\ \vdots & & \ddots \;\; \vdots & \vdots \\ X^n_{n-1} & X^n_{n-1} & 0 & X^n_{n-1} \\ -\sum X^n_k & -\sum X^n_k & \cdots -\sum X^n_k & 0 \end{pmatrix} \begin{pmatrix} \bar{F}^n_1 c_1 & 0 & 0 \cdots & 0 \\ 0 & \bar{F}^n_2 c_2 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 \cdots & 0 & \bar{F}^n_n c_n \end{pmatrix}
$$

All summations are over k from 1 to n-1, and $\bar{F}^n_n$ is unity. Substituting the above equation for $\underline{X}$ in the foreign exchange market clearing conditions given as equation (23′) and rearranging terms, the following is obtained

$$\frac{c}{c_n}\begin{pmatrix} X^n_1 \\ X^n_2 \\ \vdots \\ X^n_{n-1} \\ -\sum_{j=1}^{n-1} X^n_j \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n-1} \\ B_n \end{pmatrix} = \underline{0} \qquad (32')$$

Where $c \equiv \sum_{j=1}^{n} \bar{F}^n_j c_j > 0$

Applying Walras' Law and selecting the first n-1 equation of (32′), the next equation holds

$$\frac{c}{c_n}\underline{X}^n + \underline{B}^n = \underline{0} \qquad (33')$$

Now substituting into this equation the demand function for investors of the nth-country (equation (13′)), the following is obtained

$$c\,[\underline{M}^n]^{-1}[(\underline{r}^n - r_n\underline{\eta}) + (\underline{f}^n - \underline{e}^n)] + \underline{B}^n = \underline{0}\ .$$

Finally, solving this for the spot exchange rate vector $\underline{e}^n$, the next equation for exchange rate determination is obtained

$$\underline{e}^n = \underline{f}^n + \underline{r}^n - r_n\underline{\eta} + \frac{1}{c}\underline{M}^n\underline{B}^n \qquad (34')$$

This equation is in vector notation. Looking at the kth-element gives the following which was presented as equation (14) in Part IV, Section A.

$$e^n_k = \bar{f}^n_k + (r_k - r_n) + \frac{1}{c}\sum_{j=1}^{n-1} M^n_{kj}\,B_j \qquad (35')$$

The equation for exchange rate determination in the general case where all n countries hold foreign currency denominated assets is of exactly the same shape as the equation for exchange rate determination in the special case where only one country bears exchange risk. The only differences is that the coefficient of $\underline{M}^n\underline{B}^n$ has become $\frac{1}{c}$.

(Section 3.) Exchange Rate Determination when Inflation is Present

Analysis in Appendix 1 to this point has been conducted under the Assumption (A-3) that there is no inflation in any country. Because of this, purchasing power parity rates at the beginning of the period and the end of the period in equation (35′) for exchange rate determination are the same ($\bar{F}_k^n$ or the log of this $\bar{f}_k^n$).However, introducting perfectly anticipated inflation, the purchasing power parity rates at the beginning and the end of the period (which equals the expected spot rate) differ according to the foreign and domestic inflation differentials. That is, the following equation holds.[32]

$$\bar{F}_k^n = \frac{1+\Pi_n}{1+\Pi_k} G_k^n \tag{36'}$$

Π_n : Expected rate of inflation in the nth-country,
Π_k : Expected rate of inflation in the kth-country.
G_k^n : Purchasing power parity rate at the beginning of the period.
$\bar{F}_k^n$: Purchasing power parity rate at the end of the period. (This equals the expected spot rate.)

Taking the logarithm of both sides of equation (36′), equation (37′) is obtained

$$\bar{f}_k^n = \pi_n - \pi_k + g_k^n \tag{37'}$$

Substituting this into equation (35′), the next equation for exchange rate determination in an inflationary world is obtained

$$e_k^n = g_k^n + [(r_k - \pi_k) - (r_n - \pi_n)] + \frac{1}{c} \sum_{j=1}^{n-1} M_{kj}^n B_j . \tag{38'}$$

The above equation differes from equation (35′) in that the expected spot rate $\bar{f}_k^n$ in equation (35′) has been replaced by the beginning of the period purchasing power parity rate g_k^n. Also, the nominal interest rate differential in equation (35′) has

32. When real economic factors change between the beginning and the end of the period, caused by such factors as different rates of increase in labor productivity in trade and non-trade sectors, equation (36') will not necessarily hold. In other words, the analysis in this paper abstracts from the influence of other factors besides foreign and domestic inflation differentials on purchasing power parity rates.

been transformed into the real interest rate differential $[(r_k - \pi_k) - (r_n - \pi_n)]$.[33] Equation (38′) is the final version of the exchange rate determination equation and was presented previously as equation (15) in the main body of this paper.

APPENDIX 2

Justifications of Mean-Variance Utility Function

Hal R. Varian has suggested the following justifications of the mean-variance utility function.

1) If the utility is quadratic,

$$f(\tilde{W}) = c_1 + c_2\tilde{W} + c_3\tilde{W}^2 \quad \text{where } c_2 > 0,\ c_3 < 0 .$$

Then

$$u = \varepsilon[f(\tilde{W})] = c_1 + c_2\bar{W} + c_3(\bar{W}^2 + \sigma_w^2)$$

$$du = (c_2 + 2\bar{W}c_3)\,d\bar{W} + c_3\,d\sigma_w^2$$

Thus, we have the following correspondence with equation (1) disregarding the constant term:

$$a_n \longleftrightarrow c_2 + 2\bar{W}c_3$$

$$-b_n \longleftrightarrow c_3$$

2) By taking the second-order Taylor expansion around W_0 which is close to the mean $\bar{W}$ of $\tilde{W}$, we get

$$f(\tilde{W}) \cong f(W_0) + f'(W_0)(\tilde{W} - W_0) + \frac{1}{2}f''(W_0)(\tilde{W} - W_0)^2$$

$$= f(W_0) + f'(W_0)[(\tilde{W} - \bar{W}) + (\bar{W} - W_0)] + \frac{1}{2}f''(W_0)[(\tilde{W} - \bar{W}) + (\bar{W} - W_0)]^2$$

33. When inflation exists, precisely speaking, it is necessary to use G_j^n in place of $\bar{F}_j^n$ as the conversion rate in equation (23') for the market clearing conditions for the foreign exchange market. As a result, B_j in the third term on the right hand side of equation (38') must be changed somewhat. This factor will, however, be ignored here.

and taking the expectation

$$u = \varepsilon [f(\widetilde{W})]$$
$$= f(W_o) + f'(W_o)(\bar{W} - W_o) + \frac{1}{2} f''(\sigma_w^2 + (\bar{W} - W_o)^2) .$$

Since W_0 is closed to $\bar{W}$, we can disregard the second order term $(\bar{W} - W_o)^2$ and we have

$$u = f(W_o) + f'(W_o)(\bar{W} - W_o) + \frac{1}{2} f''(W_o) \sigma_w^2$$

By taking the total derivative, we get

$$du = f'(W_o) d\bar{W} + \frac{1}{2} f''(W_o) d\sigma_w^2$$

Thus we have the following correspondence:

$$a_n \longleftrightarrow f'(W_o)$$
$$-b_n \longleftrightarrow \frac{1}{2} f''(W_o)$$

Note that since the ratio $2\frac{b_n}{a_n} = -\frac{f''}{f'}$ is the measure of absolute risk aversion, c_n defined in equation (11′) is the reciprocal of this measure. If investors' utility functions have the property that their measure of absolute risk aversion is declining, as usually assumed, c_n will increase as their wealth increases.

3) If the investors have constant-absolute-risk-aversion utility functions and the distribution of the terminal wealth is normal, we have

$$f(\widetilde{W}) = -\exp(-a\widetilde{W})$$
$$\widetilde{W} \sim N(\bar{W}, \sigma_w^2)$$

Then,

$$u = \varepsilon [f(\widetilde{W})] = -\frac{1}{\sqrt{2\pi}\,\sigma_w} \int \exp\left(-aW - \frac{(W-\bar{W})^2}{2\sigma_w^2}\right) dW$$

Since

$$-aW - \frac{(W-\bar{W})^2}{2\sigma_w^2} = -\frac{W^2 + 2(a\sigma_w^2 - \bar{W})W + \bar{W}^2}{2\sigma_w^2}$$

$$= -\frac{[W + (a\sigma_w^2 - \bar{W})]^2 - (a^2\sigma_w^4 - 2\bar{W}a\sigma_w^2)}{2\sigma_w^2}$$

$$= -\frac{[W + (a\sigma_w^2 - \bar{W})]^2}{2\sigma_w^2} - a(\bar{W} - \frac{a}{2}\sigma_w^2) ,$$

we get

$$u=-\left[\frac{1}{\sqrt{2\pi}\,\sigma_w}\int \exp\left(-\frac{W+(a\sigma_w^2-\bar{W})^2}{2\sigma_w^2}\right)dW\right]\cdot \exp\left(-a\left(\bar{W}-\frac{a}{2}\sigma_w^2\right)\right)$$

Since the inside of the brackets is a normal distribution function, it is equal to one. Therefore,

$$u=-\exp\left(-a\left(\bar{W}-\frac{a}{2}\sigma_w^2\right)\right)$$

Taking a monotonic transformation of u, we get

$$u=\bar{W}-\frac{a}{2}\sigma_w^2 .$$

Because $a=-\frac{f''}{f'}$, it is the measure of absolute risk aversion.

APPENDIX 3

The Definition of Purchasing Power Parity in an N-Country, N-Currency World

The definition of purchasing power parity rates will be given for the n-1 independent currencies in an n-country, n-currency world. First, a definition of long run equilibrium rates is presented

Definition

Long run equilibrium real exchange rates are those real rates which bring about equilibrium in the long run in the current accounts of all n-countries.

From this definition, unique equilibrium real exchange rates are determined for the n-1 independent currencies. This is proved in the following fashion. First, variables are defined as follows.

$T_{ij}(\underline{S}^n)$: Amount of goods and services exported by the ith country to the jth-country denominated in the nth-country's currency. This is a function of the terms of trade vector which is of dimension n-1.

$$\underline{S}^n=\begin{bmatrix} S_1^n \\ S_2^n \\ \vdots \\ S_{n-1}^n \end{bmatrix}$$

: A vector consisting of elements S_k^n of the terms of trade between the nth-country and the kth-country.

$$\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$: Current account balance of each country denominated in the currency of the nth-country.

$$\underline{T}(\underline{S}^n) = \begin{bmatrix} 0 & T_{12} & T_{13} & \cdots & T_{1n} \\ T_{21} & 0 & T_{23} & \cdots & T_{2n} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & T_{n-1n} \\ T_{n1} & T_{n2} & & T_{n\,n-1} & 0 \end{bmatrix}$$: Trade flow matrix.

From these definitions, the following identity holds.

$$(\underline{T}(\underline{S}^n) - \underline{T}(\underline{S}^n)^t)\,\underline{\eta} = \underline{c}$$

These n equations fulfill the following equation.

$$\underline{\eta}^t(\underline{T}(\underline{S}^n) - \underline{T}(\underline{S}^n)^t)\,\underline{\eta} = 0 = \underline{\eta}^t\,\underline{c}$$

Consequently, the equilibrium conditions for the current account balance of each country is given by

$$(\underline{T}(\underline{S}^n) - \underline{T}(\underline{S}^n)^t)\,\underline{\eta} = \underline{0}$$

Solving n-1 of these equation for $\underline{S}^n$, the long run equilibrium real exchange rates are obtained. The nominal exchange rates corresponding to these long run rates are the purchasing power parity rates.

If changes in real econmic factors are small, changes in these purchasing power parity rates largely reflect foreign and domestic inflation differentials, and the "purchasing power parity relatinship" holds. However, when large changes in the real economy occur, purchasing power parity rates also change reflecting these real changes. Changes in the real economic sturcture can arise through such factors as different rates of increase in productivity by sector or the discovery of new raw material deposits.

REFERENCES

[1] Mitsuhiro Fukao. "Kokusai Shūshi to Kawase Jukyū—Kokusai Shikin Junkan Kanjō ni Yoru Bunseki." (The Ballance of Payments and Foreign Exchange). *Kenkyu Shiryō*, 57, Ken 1-1, Nihon Ginkō Kin'yū Kenkyū Kyoku, Feb, 1982.

[2] __________, "The Risk Premium in the Foreign Exchange Market." The Bank of Japan, *Discussion Paper Series* No. 12. May 1982.

[3] __________, "Hendō Sōba Seidoka ni okeru Kawase Sōba Kettei Riron no Hatten. " (The Development of Exchange Rate Determination Under the Flexible Exchange Rate System.) *Kin'yū Kenkyū* No.2, March 1983.

[4] Mitsuhiro Fukao and Takashi Okubo. "International Linkage of Interest Rates: The Case of Japan and the United States." the Bank of Japan, *Discussion Paper Series* No.13, June 1982.

[5] Mitsuhiro Fukao. *The Risk Premium in the Foreign Exchange Market.* Unpublished doctoral dissertation, Univ. of Michigan, 1981.

[6] Genburg, Hans. "Effects of Central Bank Intervention in the Foreign Exchange Market." *IMF Staff Papers* 28, No.3, September 1981.

[7] Hickman, Bert G. "Exchange Rates in Project Link." Proc. of the International Workshop on Exchange Rates in Multicountry Econometric Models, Univ. of Leuven, Belgium, November 26-28, 1981.

[8] Hooper, P., R.D.Haas and S.A.Symansky. "Revision of Exchange Rate Determination in the MCM: A Progress Report." Proc. of the International Workshop on Exchange Rates in Multicountry Models, Univ. of Leuven, Belgium, November 26-28, 1981.

[9] Jensen, Michael C. "Capital Markets: Theory and Evidence." *Bell Journal of Economics and Management Science* 3, 1972, pp.357-98.

[10] Lindgren, Bernard W. *Statistical Theory,* 3rd ed. New York: Macmillan, 1976.

[11] Porter, Michael G. "A Theoretical and Empirical Framework for Analyzing the Term Structure of Exchange Rate Expectations." *IMF Staff Papers* 18, November 1971, pp.613-45.

[12] Ross, Stephen A. "The Current Status of the Capital Asset Princing Model (CAPM)." *The Journal of Finance* 33, June 1978, pp.885-901.

[13] Solnik, Bruno H. *European Capital Markets: Towards a General Theory of International Investment.* Lexington Books, 1973.

[14] Varian, Hal R. *Microeconomic Analysis.* New York: Norton, 1978.