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## ABSTRACT

This note explores technical issues in estimating a smooth yield curve of forward or spot interest rates from a finite number of observed market rates. After surveying possible methods for estimating a yield curve for Japanese yen, a new estimation method is proposed to derive the implied forward rate curve from yield to maturity data for Japanese government coupon bonds, which aims at investigating the information content of yield curves for monetary policy. In order to realize the smooth interpolation of discrete data and the conversion of yields to maturity into spot or forward rates, we make use of third and fifth order spline functions. Although the algorism of this method still includes some approximations, the estimation error is reduced sufficiently without too heavy a computational burden. Finally, we compare the results obtained by our estimation method with those from commonly used linear interpolation.

KEY WORDS: Yield curve; Spline functions; Japanese government bonds; Implied forward rate

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## 1. Introduction

- Deriving a continuous smooth yield curve from a finite number of observed market rates is the nucleus for a wide range of financial analyses such as asset pricing and market surveillance. There are a number of estimation methods, each of which differs in degree of accuracy and convenience of calculation. For example, one of the simplest approaches for practical purposes is to replace spot rate data with "yield to maturity" data (hereinafter YTM) of government bonds and to linearly interpolate these discrete observations. However, this method does not always guarantee the accuracy necessary for specific analytical purposes.

- This note explores technical issues in the estimation of Japanese yen yield curves. After surveying possible methods, a new estimation method is proposed to derive the implied forward rate curve from yield to maturity data for Japanese government coupon bonds, which aims at investigating the information content of yield curves for monetary policy[1]. In order to realize the smooth interpolation of discrete data and the conversion of yields to maturity into spot or forward rates, we make use of third and fifth-order spline functions[2]. Although the algorism of this method still includes some approximations, the estimation error is reduced sufficiently without too heavy a computational burden. Finally, we compare the results obtained by our estimation method with those from commonly used linear interpolation.

## 2. Background to the Analysis

- The yield curve of implied forward rates is an important indicator for monetary policy since it corresponds to expected nominal interest rates in the future under the assumption of the risk neutrality[3] of market participants. The

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[1] A number of central banks in Europe and North America have also conducted research on this topic. For example, see Deacon and Derry [1994], Breedon [1995], Cooper and Steeley [1996] and Fisher, Nychka and Zervos [1995].

[2] For general features of spline functions, see, for instance, de Boor [1978].

[3] There are many theoretical and empirical researches on the information content or the risk premium of term structure. For an introductory overview, see, for example,

yield curve of spot rates, or zero rates, is also meaningful in the same context.

-- The expected nominal interest rate can be interpreted as the combination of the expected inflation rate and the expected real rate of economic growth, or the expected real rate of return on capital. This note, however, does not go into the methodology of separating out these components.

● In general, the use of government bond market prices is desirable for our purpose mainly for the following two reasons. First, it is easier to analyze risk free interest rates than rates incorporating credit risk premium. Second, interest rates with a wide range of maturities, from short term to long term, are necessary to examine the whole term structure of interest rates.

However, the Japanese government bond (hereinafter JGB) market has some problems as described below.

-- The trading volume of JGBs with short maturities, typically less than three years, tends to be extremely small in the secondary market. Hence, price information is not necessarily reliable for this range. To redress this problem, one can estimate a yield curve from interbank rates, such as LIBOR, interest futures or swap rates, and compare it with the JGB curve. By estimating the potential difference between these two curves arising from such factors as credit risk premium and different tax treatment, we can evaluate the reliability of JGB yield curves.

-- It has been pointed out that the JGB curve is distorted[4] reflecting various institutional factors. Although such distortion is tending to diminish, it is indisputable that some anomalies still remain. For example, transactions in 10-year JGBs tend to be concentrated on the benchmark bond trading at each point in time, which leads to a higher liquidity premium for the current benchmark bond. Moreover, the existence of deliverable JGBs[5] for the

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Shiller and McCulloch [1990].

[4] For more detailed analysis on JGB prices, see, for instance, Kikugawa and Singleton [1994].

[5] JGBs with 7- to 11-year maturity and 17- to 21-year maturity are deliverable for the settlement of 10-year and 20-year JGB futures respectively. Refer also to the Appendix 2 to this note.

settlement of JGB futures transactions affects the shape of yield curves. Since this effect is quite substantial for the cheapest bond, the so-called "squeeze" is often observed in the corresponding maturity of the yield curve as shown later in this note.

- Against this background, it is considered desirable to estimate two yield curves, one from JGB prices and the other from interbank rates, as tools for market analysis. With regard to the yield curve derived from interbank rates, two major technical problems arise: (1) the conversion of interest rates -- for example, from swap rates to spot rates -- and (2) the interpolation of a small number of discrete market rates. In estimating JGB yield curves, the following three points should be addressed. First, individual bond characteristics which contaminate the information content of interest rates should be eliminated as far as possible. Second, the smoothness of yield curves should be preserved. Third, spot rates (or discount factors) should be obtained by converting bond prices (or YTM), since only the coupon bond market (mainly 10-year JGBs) is liquid and there is no strip bond market in Japan.

The analytical focus of this note is on the procedure to derive yield curves -- both spot and forward -- from JGB market data.

Theoretically, there are a number of approaches which can be adopted to estimate JGB yield curves. Each has its own strengths and weaknesses in terms of convenience of calculation and theoretical accuracy. Three estimation method examples:

(1) After smoothing YTM data, calculate discount factors and spot rates of hypothetical par bonds, and interpolate using the fifth-order spline function[6].

[6] The spline function is a smooth piecewise polynomial function composed of different polynomials for each divided section. An m-th order spline function, s(x), with n grid points of $q_1$, $q_2$, , $q_n$, can be described by using truncated power functions as follows:

$$s(x) = p(x) + \sum_{i=1}^{n} a_i (x - q_i)_+^m ,$$

where p(x) is an m-th order polynomial and $a_i$ a constant. The truncated power function above is defined as:

$$(x-q)_+^m = 0 \qquad \text{when } x<q,$$
$$= (x-q)^m \qquad \text{when } x \geq q.$$

(2) Break price data down into cash flows to derive discount factors, smooth the discount factors by the least squares method, and then interpolate by using the fifth-order spline function.

(3) Minimize the difference between actual YTMs/prices and theoretical YTMs/prices (the latter is derived from forward curves smoothed by the spline function).

Our experimental method relies on method (1), which is relatively easy to handle. Details of the steps to be followed and calculation examples are shown in the next section.

-- We can basically categorize approaches to derive JGB yield curves into two approaches in terms of the way the effects of individual bond characteristics are eliminated:

(a) methods where contamination is removed by smoothing the curve; and

(b) methods where contamination is removed by constructing a bond pricing model with explanatory variables for the characteristics of individual bonds and determining the parameters of the model by regression analysis.

Each approach has its advantages and disadvantages: approach (a) works better for the purpose of easily surveying market expectations, our current goal; while approach (b) works better for an exact analysis of individual bonds, such as the detection of mispricing in the bond market. Approach (a) can be further divided into two in terms of smoothing method. One is by making use of spline functions, examples of which include the three methods (1) to (3) explained above[7]. The other is by using non-linear regression analysis. Typical examples are the so-called Nelson-Siegel method[8] (Nelson and Siegel

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[7] There are a variety of methods to estimate yield curves with spline functions. The pioneering works are represented by McCulloch [1975], using polynomial spline functions, Vasicek and Fong [1982], using exponential spline functions, and Shea [1985]. More recent works include Adams and Van Deventer [1994] and Fisher, Nychka and Zervos [1995], both of which belong to category (3) above.

[8] The term structure of the instantaneous forward rate, f(T), is assumed to be of the following shape:

$$f(T) = \beta_0 + \beta_1 \exp(-\frac{T}{\tau_1}) + \beta_2 \frac{T}{\tau_1} \exp(-\frac{T}{\tau_1}),$$

[1987]) and its extended version[9] (Svensson [1994]). An example of approach (b) is seen in a paper prepared by J. P. Morgan [1992, 1995].

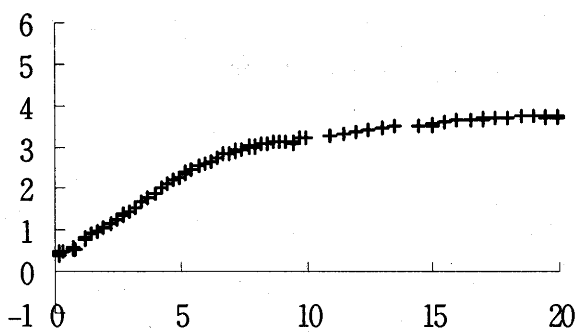
## 3. Details of Estimation Procedures

- This section explains procedures for the yield curve estimation approach we investigated.

- Steps and calculation examples using JGB data of March 29, 1996 are as follows:

(1) Calculate YTMs (Fig. 1) of all 10-year and 20-year coupon JGBs on the basis of closing prices on the Tokyo Stock Exchange. YTMs are semiannually compounded and actual/365-based. The number of bonds traded is currently about 120.

Fig. 1

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where $\beta_0$, $\beta_1$, $\beta_2$ and $\tau_1$ are parameters to be estimated by minimizing either (the sum of squared) price errors or (the sum of squared) yield errors.

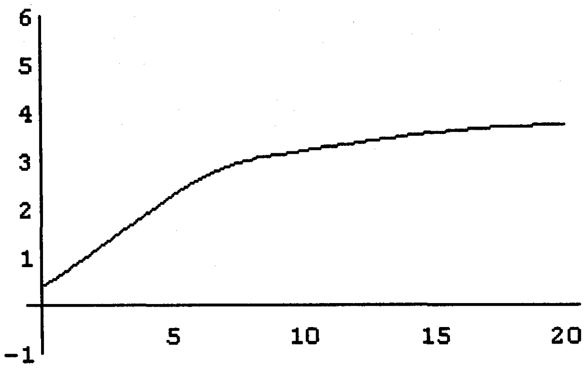
[9] The term structure of the instantaneous forward rate, f(T), is assumed to be of the following shape:

$$f(T) = \beta_0 + \beta_1 \exp(-\frac{T}{\tau_1}) + \beta_2 \frac{T}{\tau_1}\exp(-\frac{T}{\tau_1}) + \beta_3 \frac{T}{\tau_2}\exp(-\frac{T}{\tau_2}),$$

where $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$, $\tau_1$ and $\tau_2$ are parameters. The fourth term of the right-hand side of the above model is newly added to the original Nelson-Siegel model to obtain a more flexible shape.

(2) Apply a cubic natural spline function to YTM data to interpolate smoothly[10] (Fig. 2).

Fig. 2




(3) Create hypothetical par JGBs maturing every six months and plot their YTM data (Fig. 3). Convert the YTMs to 40 semiannual discount factors (for 20 years) by decomposing JGB cash flows. Add the one-day discount factor based on the overnight unsecured call rate. Convert the discount factors to the spot rates, which are continuously compounded and actual/365-based (Fig. 4). Apply a cubic natural spline function to the spot rate data to interpolate smoothly (Fig. 5). We thus obtain the yield curve of spot rates.

[10]Reasons for the adoption of the cubic natural spline function for YTMs here, and for spot rates in (3), are as follows. It is mathematically proved that of all the possible spline functions the cubic natural spline function gives the least value to the target function $\sigma$, defined below, for the given data.

$$\sigma = \sum_{i=1}^{n} w_i \{f(x_i) - y_i\}^2 + g\int_a^b \{f''(x)\}^2 dx$$

In this equation, f(x) is a spline function applied to the given data. The integral interval [a, b] corresponds to the definition interval of f(x). The parameters $w_i$ and g represent the weight in fitting the i-th data and the tradeoff between fit and smoothness of the curve. Note that smoothness is defined as the integral of the squared second-order differential of the curve. In the process of (2) above, the value of $w_i$ is the issuance volume of the i-th bond. Parameter g should be properly set according to the goal of the analysis. The minimization of $\sigma$ assures the achievement of optimized tradeoff between a good fit and smoothness.

Fig. 3

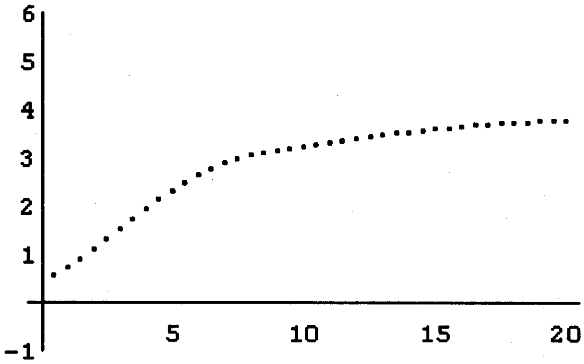



Fig. 4

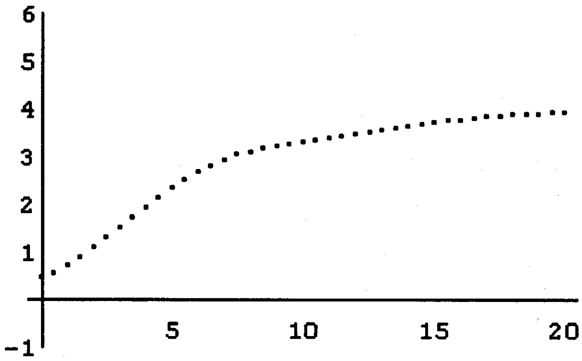
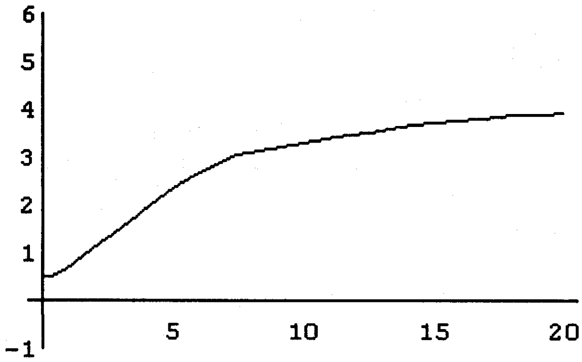



Fig. 5




(4) Plot the values of the spot rate multiplied by time to maturity corresponding to each data in Fig. 4 (Fig. 6). Apply a fifth-order natural spline function[11] to

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[11]Reasons for the adoption of the fifth-order spline function are as follows. It is mathematically proved that of all the possible spline functions the fifth-order natural spline function gives the least value to the target function $\sigma$, defined below, for the

interpolate (Fig. 7).

Fig. 6

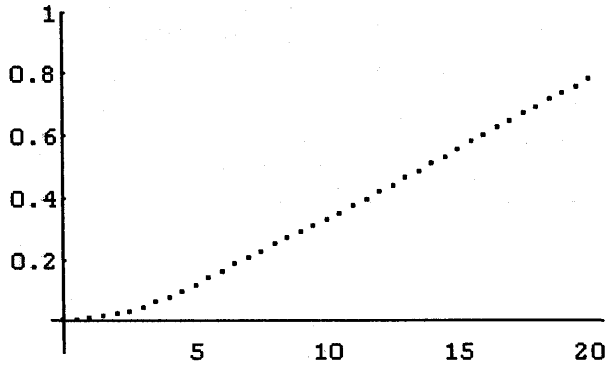
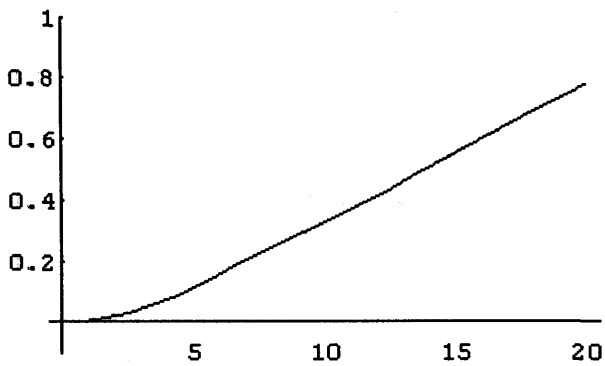



Fig. 7




(5) Derive the yield curve of the instantaneous forward rates, which is continuously compounded and actual/365-based, by differentiating the spline function in Fig. 7 with regard to time to maturity (Fig. 8).

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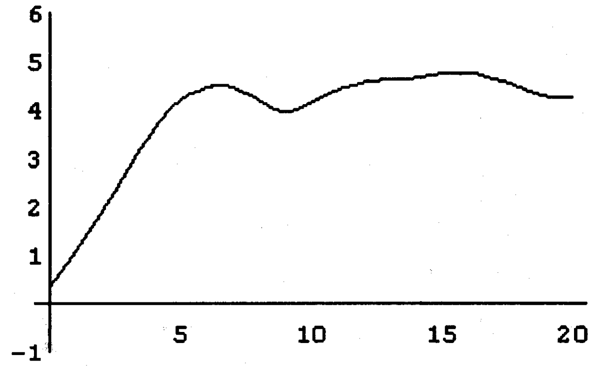
given data.

$$\sigma = \sum_{i=1}^{n} w_i \{f(x_i) - y_i\}^2 + g\int_a^b \{f'''(x)\}^2 dx$$

The first order derivative of f(x), $f'(x)$, is the instantaneous forward rate curve by definition when f(x) is applied to the data of the spot rate multiplied by the time to maturity. Thus, the minimization of $\sigma$ assures the achievement of optimized tradeoff between a good fit regarding f(x) and smoothness of the forward curve $f'(x)$.

Fig. 8




(6) Derive the yield curves of the finite-term forward rates by integrating[12] the instantaneous forward curve in Fig. 8 with regard to time to maturity. For example, the 3-month forward curve (Fig. 9) and the 1-year forward curve (Fig. 10) are shown.

Fig. 9

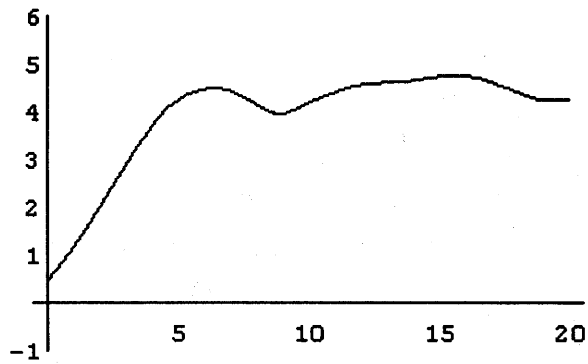
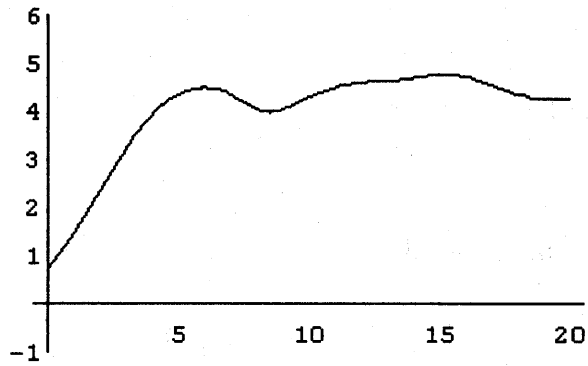



Fig. 10




[12] T-year forward rate at time t, $f_T(t)$, is defined as follows:

$$f_T(t) \equiv \frac{1}{T}\int_t^{t+T} f_i(u)du,$$

where $f_i(t)$ denotes the instantaneous forward rate at time t.

● The advantages and disadvantages of the above procedures can be summarized as follows:

Advantages:

- The smoothest spline curve under a given constraint, whether it is a spot rate curve or a forward rate curve, can be derived.
- One can strike a right balance in a flexible manner between smoothness and fit by choosing parameter g in a target function in accordance with the purpose of the analysis and market environment.
- The shape of the curve is constructed flexibly by setting as many as 40 grid points for a time horizon of 20 years.
- Problems such as divergence and localization can be easily compared with the non-linear optimization method.

Disadvantages:

- The smoothed data are YTMs rather than discount factors or spot rates. YTMs are not an appropriate measure of interest rates for specific periods since they include mixed information combining different discount factors and the mixture of information depends on the characteristics of each bond.
  -- However, if we try to smooth the data which cannot be observed directly in the JGB market, such as spot rates or discount factors, it is often the case that some compromises are required somewhere in the estimation process due to the complexity of calculation.
- All the prices of hypothetical bonds with semiannual maturities are assumed to be at par in the process of converting YTMs to discount factors. The result would be slightly different[13] if non-par bonds are assumed.

## 4. Examples of Estimated Yield Curves

● We show in the Appendix 1 the estimated yield curves of both 3-month forward rates and spot rates as of the end of September from 1988 to 1995. For each yield curve we employed two estimation methods: one is the "new approach"

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[13] Although the difference depends on the shape of the yield curve in general, it is almost negligible under normal market conditions.

explained in the previous section and the other is the "simplified approach," the variations of which are often used for practical analyses.

In the new approach, we chose parameter g for the spline applied to the YTM data by trial and error. More concretely, we fixed parameter $w_i$ to be equal to the issuance volume of the i-th JGB, and then tried to arrive at the most appropriate value of g, which gave rise to the best yield curve for our goal in that the curve represents the interpretable squeeze but almost no noise. Meanwhile, parameter g was set at zero for the splines applied to the spot rate data and the data of the spot rate multiplied by the time to maturity so that the spline penetrates all the given data.

In the simplified approach, we take YTMs as a proxy of spot rates, interpolate linearly these semiannual data, and derive the forward rate curve from the interpolated spot rate curve. This method is not free from two kinds of estimation error. First, instead of smoothing, linear interpolation of spot rate data results in the saw tooth shape of the forward rate curve. Second, approximation of spot rates to YTMs leads to non-local estimation errors.

- Characteristics of estimation results are summarized as follows:

  (1) When the spot rate curve is steep (i. e. 1992 to 1995), the estimation error of the simplified approach tends to be large. When the spot rate curve is flat (1989, for example), the error is almost negligible.

  (2) The estimation error of the simplified approach in the forward rate curve is larger than that in the spot rate curve. During the eight years, the maximum error is about 12% for the forward rate curve and about 6% for the spot rate curve, both realized in the long end of the 1992 yield curve.

  (3) In the estimated 3-month forward rate curve, the squeeze is observed around 7 to 10 years resulting from the existence of the deliverable condition for settlement of JGB futures and the existence of a high liquidity premium for the benchmark bond as described before.

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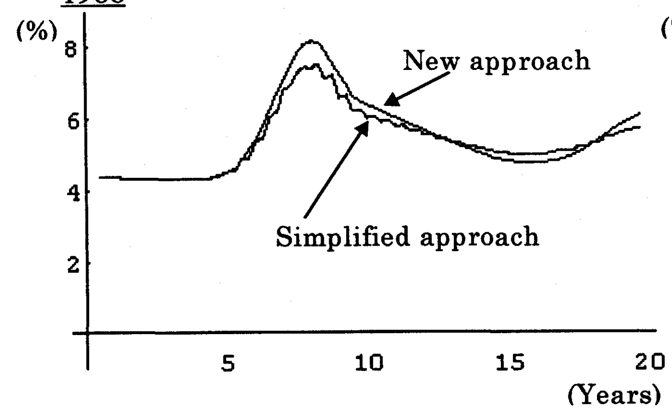
# Appendix 1

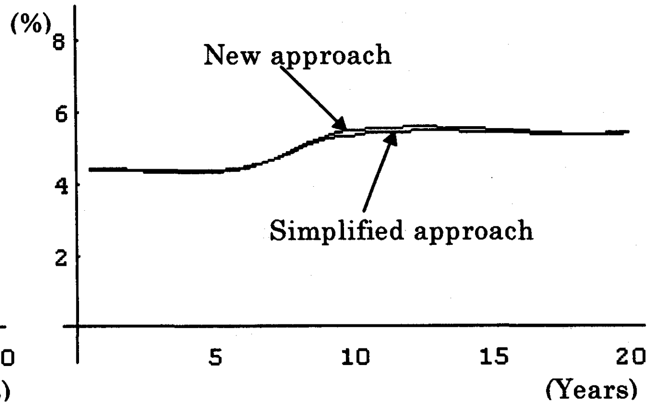
## Estimation Results of Yield Curves

-- Based on JGB data as of end of September each year --

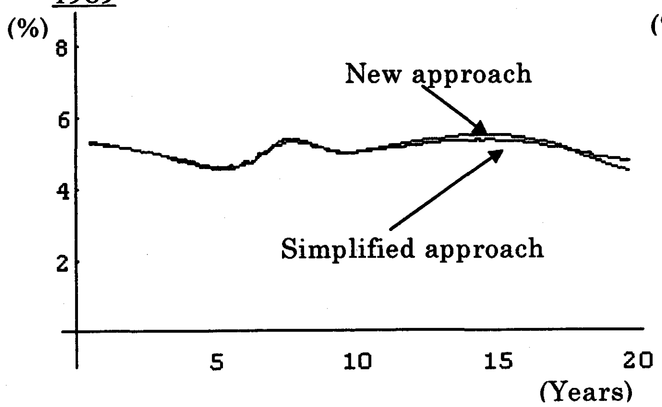
***3-Month Forward Rate Curve*** | ***Spot Rate Curve***

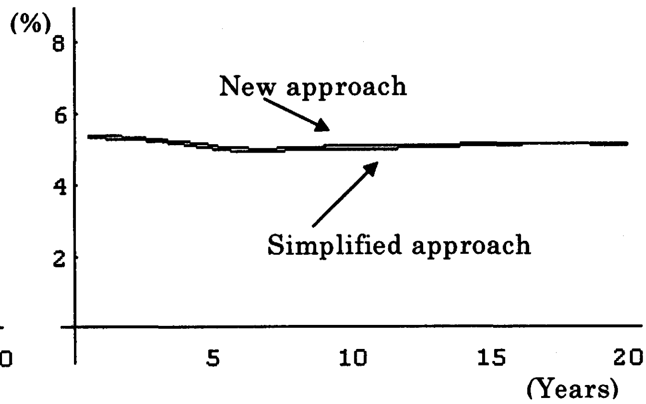
1988



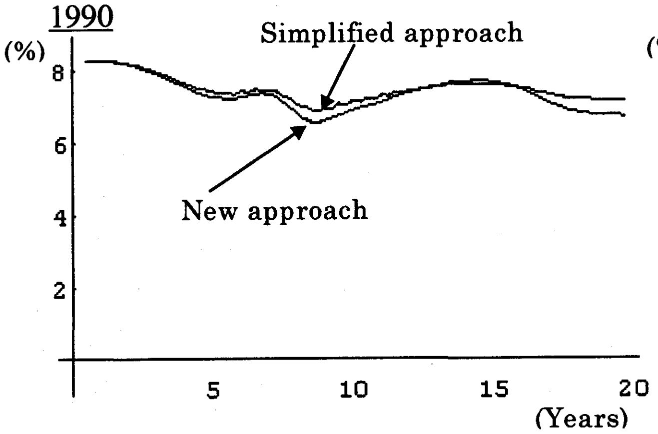



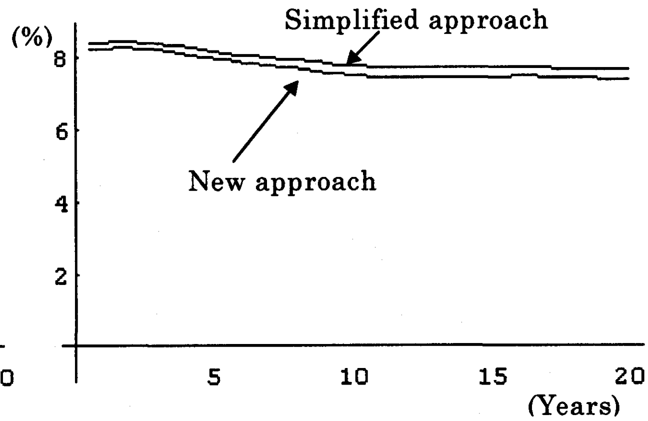

1989



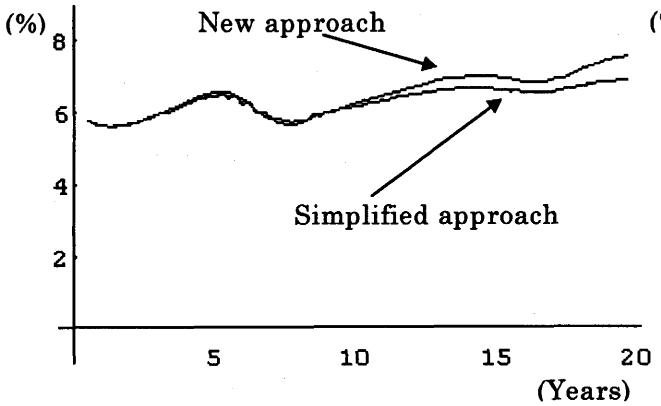

1990

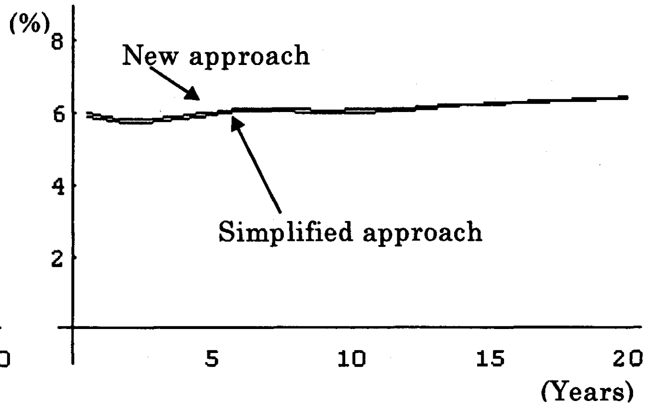


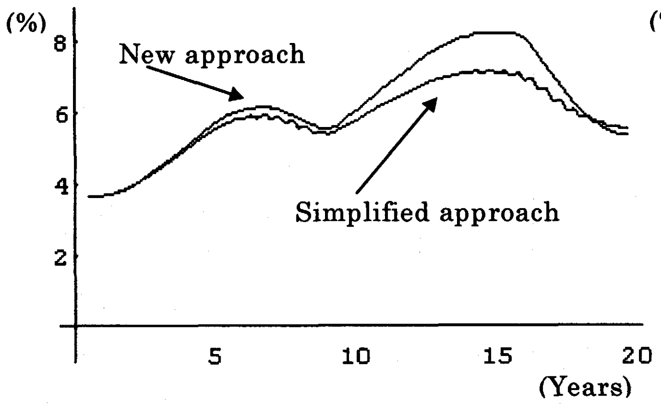
***3-Month Forward Rate Curve*** ***Spot Rate Curve***

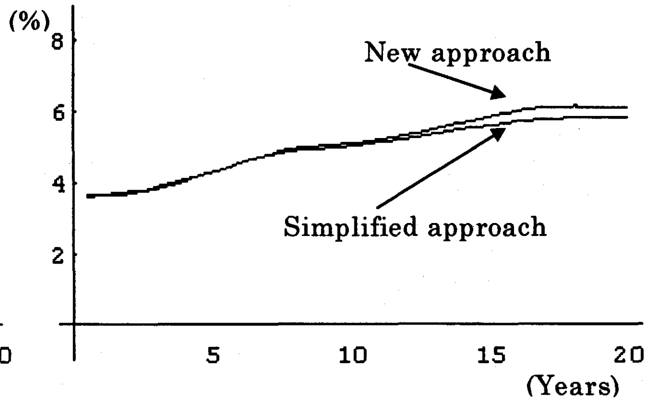
1991



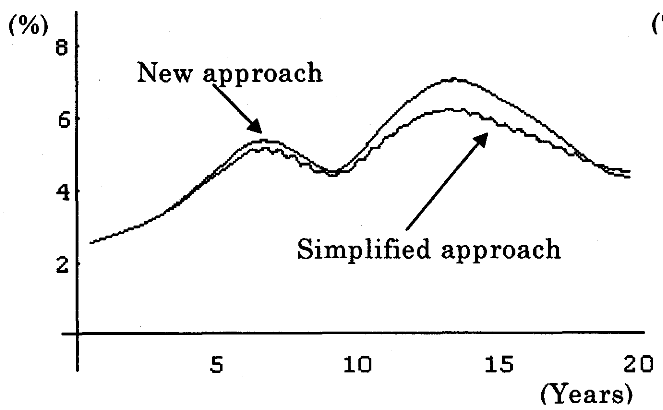



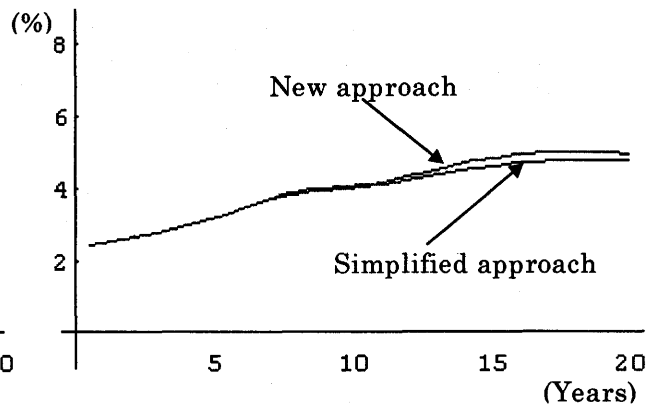

1992



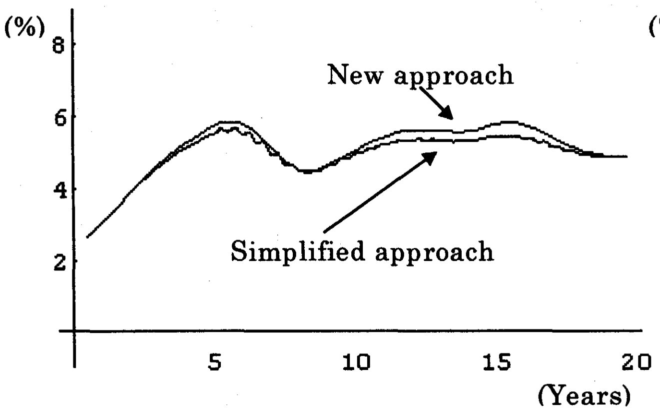

1993

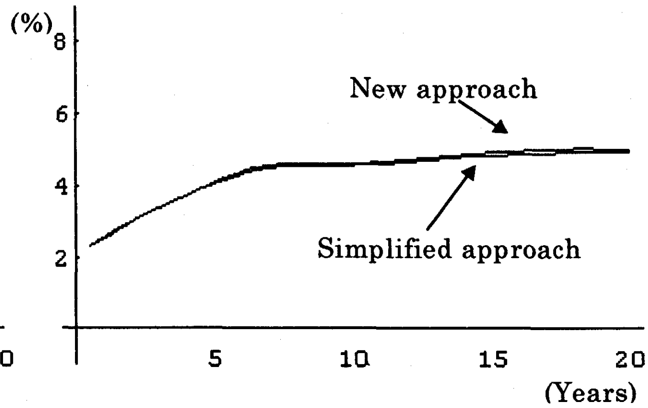


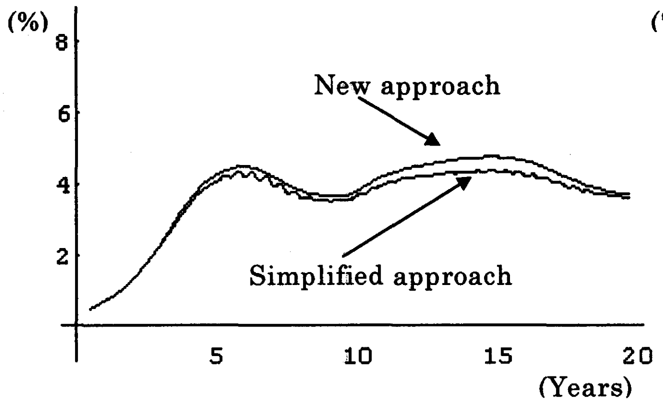
***3-Month Forward Rate Curve*** ***Spot Rate Curve***

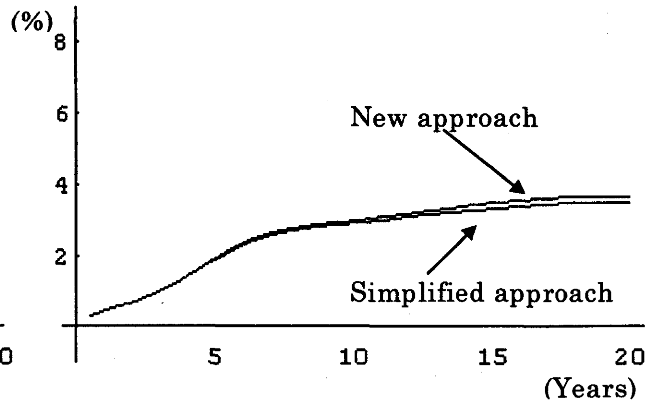
1994




1995



# Appendix 2

## Notes on the Institutional Features of Japanese Government Securities Markets

### 1. Types of Japanese Government Securities

(1) 20-year coupon bonds

The longest Japanese government bonds (JGBs) currently issued. They are issued every other month by public auction. Coupon payments are semiannual.

(2) 10-year coupon bonds

The most common JGBs. These still account for a large portion of JGB issues although there is a drive to use a wider mix of maturity. They are issued every month by public auction and syndicate underwriting. Coupon payments are semiannual.

(3) 5-year discount bonds

These are geared to individual investors and are issued through underwriting syndicates.

(4) Medium-term coupon bonds (2, 4, and 6 years)

Widely held by individual investors, either directly or in *chuki kokusai* fund (medium-term bond funds) accounts.

(5) Treasury bills

These are short-term (currently, 3 or 6 months) rollovers issued at public auctions.

(6) Financing bills

Financing bills (short-term government securities) are issued for terms of less than a year (currently issued for 60-day terms, in principle) in order to cover temporary fund shortages in treasury and special accounts.

## 2. The Secondary Market for Japanese Government Securities

JGBs have a large secondary market. They are mainly traded over the counter. Brokers/dealers, i.e., securities houses and banks, are at the core of the market to make the market and to match trades, while major investors are banks, insurance companies, and investment trusts as well as some foreign central banks and other foreign institutions.

Among the above government securities, 10-year coupon bonds, and secondly 20-year coupon bonds, have dominant transaction volumes. In particular, there is a so-called benchmark bond among 10-year coupon bonds, on which a large portion of JGB transactions is concentrated.

## 3. Taxation

The coupon payments of JGBs are subject to a withholding tax of 20% in total. Returns on bills and discount bonds are subject to income tax. In addition, securities transaction taxes are imposed on the price of securities sold.

## 4. Bond Futures Market

A bond futures market was established on the Tokyo Stock Exchange in 1985. This market has grown over the years and is currently one of the largest in the world.

Trading in the futures market comprises trading in so-called "long-term standard" government bonds, which are actually fictitious bonds with a coupon rate of 6% and a remaining maturity of 10 or 20 years. Settlement dates are the 20th March, June, September, and December. There are two types of settlement for these transactions -- cover settlement, under which uncovered positions are settled through reverse transactions, and delivery settlement. Under delivery settlement, the seller must deliver an amount of government bonds equivalent to the contract amount to the purchaser and the purchaser must pay the seller in funds. In order to ensure an identical price for the standard bond and actual government bonds, a prescribed ratio of exchange, a conversion factor, is used. In addition, the precise bond used for the delivery may be selected from deliverable bonds at the discretion of the seller.