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Naoto Kunitomo and Seisho Sato

Discussion Paper 96-E-24

IMES

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Estimation of Asymmetrical Volatility for Asset Prices : The Simultaneous Switching ARIMA Approach *

Naoto Kunitomo[†]

and

Seisho Sato [‡]

June 1996

Abstract

The asymmetrical movement between the downward and upward phases of the sample paths of many financial time series has been commonly noted by economists. Since this feature cannot be described by the Autoregressive Integrated Moving-average (ARIMA) model and the Autoregressive Conditional Heteroskedastic (ARCH) model, we introduce a class of the Simultaneous Switching Autoregressive Integrated Moving-Average (SSARIMA) model with ARCH disturbances. The asymmetrical volatility function of financial time series with daily effects can easily be estimated by this modelling. We also report a simple empirical result on stock price daily indices of the Nikkei-225 and SP-500.

Key Words

Asymmetrical Volatility, Stock Prices, Simultaneous Switching ARIMA Model, Conditional Heteroskedasticity, Daily Effect, Holiday Effect.

*Discussion Paper Version (16 June 1996). The research was conducted while the first author was a visiting research fellow at the Institute of Monetary and Economic Studies, Bank of Japan. We thank Dr. Kunio Okina for some suggestions on interpreting our empirical findings.

[†]Faculty of Economics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, JAPAN.

[‡]Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106, JAPAN.

1. Introduction

In the past decade, several non-linear time series models have been proposed by econometricians. In particular, considerable attention has been paid to the autoregressive conditional heteroskedasticity (ARCH) model, which was originally proposed by Engle (1982). A number of extensions of the standard ARCH model have been proposed and some of them have been used in empirical studies. In addition, some non-parametric and semi-parametric estimation methods for the conditional heteroskedasticities in asset prices and returns have been proposed. The related issues of modelling the conditional heteroskedasticities in asset prices and their volatilities have been discussed by Bollerslev (1986) and Chapter 21 of Hamilton, for instance.

In this paper we shall propose to use a new class of non-linear time series models called the Simultaneous Switching Autoregressive Integrated Moving-average (SSARIMA) models for analyzing stock prices and possibly other asset prices. In particular, we shall use the SSARIMA model with the autoregressive conditional heteroskedastic (ARCH) disturbances to estimate the volatility functions of stock prices. The main reason for using this class of non-linear time series models is because we are convinced that the class of Autoregressive Integrated Moving-average (ARIMA) time series model and the standard ARCH models cannot describe one important aspect in some financial time series, that is, the asymmetrical movement in the upward phase (or regime) and in the downward phase (or regime). It has often been observed and argued that major financial series including stock prices display some kind of asymmetrical movements in the upward and downward phases. This feature of time series can be regarded as one form of time irreversibility discussed in the statistical time series analysis : see Chapter 4 of Tong (1990).

Earlier, we introduced a simple stationary SSAR time series model and discussed its statistical properties in some detail (Kunitomo and Sato (1996a)). Let $\{y_t\}$ be a sequence of scalar time series satisfying

$$(1.1) \quad y_t = \begin{cases} Ay_{t-1} + \sigma_1 v_t & \text{if } y_t \geq y_{t-1} \\ By_{t-1} + \sigma_2 v_t & \text{if } y_t < y_{t-1} \end{cases},$$

where A, B, σ_i ($\sigma_i > 0, i = 1, 2$) are scalar unknown coefficients, and $\{v_t\}$ are a sequence of i.i.d. random variables followed by $N(0, 1)$. By imposing the coherency condition given by

$$(1.2) \quad \frac{1-A}{\sigma_1} = \frac{1-B}{\sigma_2} = \gamma,$$

this time series model has the Markovian representation

$$(1.3) \quad y_t = y_{t-1} + [\sigma_1 I(v_t \geq \gamma y_{t-1}) + \sigma_2 I(v_t < \gamma y_{t-1})][-\gamma y_{t-1} + v_t],$$

where γ is an unknown parameter and $I(\cdot)$ is the indicator function. When $\sigma_1 = \sigma_2 = \sigma$, then the Simultaneous Switching Autoregressive (SSAR) model becomes the standard $AR(1)$ model by re-parametrizing $A = B = 1 - \sigma\gamma$. As we have shown (Kunitomo and Sato (1996a)), even this simplest univariate SSAR model, called SSAR(1), gives us some explanations and descriptions of a very important aspect of the asymmetrical movement of time series in two different phases. Although this characteristic of economic time series has been observed by a number of economists, as yet there has not been a simple but general time series model incorporating this feature directly as far as we know in the econometric literature.

In subsequent work (Kunitomo and Sato (1996b)), we have extended the basic SSAR model that we proposed in Kunitomo and Sato (1996a) into two important directions for econometric applications. First, we allow the disturbance terms in the SSAR model to be auto-correlated and have a finite moving-average structure in addition to the non-linear autoregressive part in the time series models. Second, we considered a class of non-stationary SSAR models, which is useful for applications to major financial time series. It is important to note that it is not possible to describe the kind of asymmetrical patterns in the upward and downward phases by using the standard linear non-stationary time series models including the ARIMA time series model and the standard ARCH model proposed by Engle (1982). This issue has been previously pointed out by Nelson (1991) and Harvey and Shephard (1993) in the context of the volatility function of stock prices. Although there have been some proposed estimation methods for volatility functions by Nelson (1991) and Harvey and Shephard (1993), the time series models often become very complicated once the asymmetrical forms of the volatility functions are introduced. Since the stationary SSARMA and non-stationary SSARIMA models are natural extensions of the corresponding ARMA and ARIMA models in some sense, they can easily be extended to handle the asymmetrical conditional heteroskedasticities.

The main purpose of this paper is to give a new estimation method of the volatility functions of asset prices, which can be asymmetrical in two phases, by the use of the SSARIMA modelling with ARCH disturbances. Unlike other parametric, non-parametric, and semi-parametric methods already available, our formulation is a very simple parametric approach and it provides an easy way to handle daily effects and holiday effects in the volatility functions of asset prices. Since these effects have been observed by some financial economists and practitioners in financial markets, our method for estimating the volatility functions may have very real applications. Furthermore, our estimation method can be justified by its asymptotic properties and hence the model selection procedure within the class of SSARIMA models based on the information criteria can be developed rather straightforwardly.

In Section 2, we shall introduce the univariate SSARIMA model with ARCH disturbances. We also shall investigate some properties of the SSARIMA(p, q) model with a time trend and ARCH(r) disturbances in some detail and give the method of estimating the asymmetrical conditional heteroskedasticities in asset

prices. We shall also discuss the asymptotic properties of the estimation method and develop the model selection procedure. Then in Section 3, we shall apply the SSARIMA model with a time trend for the analysis of the Nikkei 225 spot index at Tokyo and the SP 500 spot index at New York. In Section 4, some concluding remarks on our econometric approach and empirical findings will be given. The proofs of some theoretical results obtained in this paper will be gathered in the Appendix.

2. The SSARIMA Model

There has been growing interest in the last decade among econometricians to investigate financial time series data by using statistical time series analysis. There are several interesting features often observed in financial time series data. First, many financial time series such as stock prices, bond prices, interest rates, foreign exchange rates, and their derivatives are often too volatile to use the stationary time series models used in standard statistical time series analysis. There are cogent arguments in financial economics that there is a martingale measure for financial time series : see Duffie (1992). Second, the distributions of financial prices, yields, and returns are often not well approximated by the Gaussian distribution. It has often been found that the kurtosis calculated from the daily returns for stock prices is much larger than 3, for instance. However, there is little consensus on the class of distributions to be used for describing financial time series among econometricians. Third, the estimated historical volatility functions for many financial time series are often not constant over time. This leads to the argument for that the conditional variances of time series are not constant over time. Fourth, some financial time series exhibit asymmetrical movements in the upward phase and the downward phase. In particular, a number of economists have observed this type of asymmetrical time series movement in stock prices.

The standard linear time series models such as the autoregressive integrated moving average (ARIMA) process can go towards explaining the first and second features, but not to the third and fourth features. The standard autoregressive conditional heteroskedasticity (ARCH) process, which was originally proposed by Engle (1982) and has been sometimes used in recent econometric applications, is consistent with the second and third features, but not with the fourth one. There have thus been several attempts to extend the standard ARCH model.

2.1 A Simple Model of Stock Prices

In this section we first extend the simple econometric model of stock prices that we used in Kunitomo and Sato (1996b). We began by modifying the well-known economic model in financial economics developed by Amihud and Mendelson (1987).

Let the intrinsic value of a security at time t and its observed price be V_t and P_t respectively. We distinguish the intrinsic value of a security and its observed

price. Since the two values V_t and P_t can be different, we can introduce a partial-adjustment model when the intrinsic value V_t at t deviates from the anticipated price V_t^* , which is given by

$$(2.1) \quad P_t - P_{t-1} = \begin{cases} g_1(V_t - V_t^*) & \text{if } V_t - V_t^* \geq 0 \\ g_2(V_t - V_t^*) & \text{if } V_t - V_t^* < 0 \end{cases}.$$

The anticipated price V_t^* at t is calculated by the past realizations of prices and given by

$$(2.2) \quad V_t^* = \sum_{i=0}^{p-1} \beta_i P_{t-i-1}$$

and the adjustment coefficients in the stock price equation satisfy the condition $g_i \geq 0$ ($i = 1, 2$).

Earlier, we dealt with the case when $V_t^* = P_{t-1}$ (Kunitomo and Sato (1996b)), which is included in (2.2) as a special case. The anticipated price V_t^* in (2.2) includes the optimal forecasts of price levels given the past information if P_t could have followed some ARIMA models. Because there are new shocks or news available at t in markets, V_t could be different from V_t^* . In addition, we have allowed the adjustment coefficients g_i ($i = 1, 2$) to take different values. There could be intuitive economic reasons why they can be different. For instance, when $V_t \geq V_t^*$ the current price has been under-evaluated and there is economic pressure mainly from the demand side to force the price up. On the other hand, when $V_t < V_t^*$ the current price has been over-evaluated and there is economic pressure mainly from the supply side to push the price down. Since there are two main forces during the actual price determination process in financial markets, the two coefficients g_i ($i = 1, 2$) could be different. However, instead of discussing the rigorous economic justifications, here we simply point out that this formulation includes many situations as special cases, which are theoretically or practically interesting in financial economics. When $g_1 = g_2$, (2.1) is reduced to the standard linear adjustment model. Further, when $g_1 = g_2 = 1$ and $V_t^* = P_{t-1}$, then $V_t = P_t$ and the intrinsic value of a security is always equal to its observed price. Hence, by using the formulation we have adopted in (2.1) it is possible to examine from the observed time series data if these conditions are reasonable descriptions of reality.

In the recent financial economics, there has been a convention that the logarithm of the intrinsic security value $\{V_t\}$ follows an integrated process $I(1)$ with a drift,

$$(2.3) \quad V_t = V_{t-1} + \sigma e_t + \mu,$$

where μ represents the expected daily return except stock dividends and $\{e_t\}$ are a sequence of random variables generated by the linear stationary stochastic process possessing a MA representation. Let the indicator functions be

$$I_t^{(1)} = I(P_t \geq P_{t-1})$$

and

$$I_t^{(2)} = I(P_t < P_{t-1}),$$

where $I(\omega) = 1$ if the event ω occurs and $I(\omega) = 0$ otherwise. Then we rearrange (2.1) and (2.2). By combining (2.3) with (2.1) and (2.2), we can get the representation of ΔP_t as

$$(2.4) \quad \Delta P_t = g(t) \left[\frac{1}{g(t-1)} - \beta_0 \right] \Delta P_{t-1} - g(t) \sum_{i=1}^{p-1} \beta_i \Delta P_{t-i} + g(t) [\mu + \sigma e_t],$$

where $g(t) = g_1 I_t^{(1)} + g_2 I_t^{(2)}$ and Δ is the difference operator such as $\Delta P_t = P_t - P_{t-1}$. In this representation, $I_t^{(1)} = 1$ if and only if $V_t - V_t^* \geq 0$. But then (2.1) implies that $I_t^{(1)} = 1$ if and only if $\Delta P_t \geq 0$. When $p = 1$ and $\beta_0 = 1$, (2.4) is identical to our SSIAR model in Kunitomo and Sato (1996b).

2.2 The SSARIMA(p, q)-ARCH(r) model

In this section we shall introduce a class of the simultaneous switching autoregressive integrated moving-average (SSARIMA) model. For the specific application to financial time series in this paper we first consider the univariate SSARMA model given by

$$(2.5) \quad y_t = \begin{cases} a_1^* + a_2^* t + \sum_{i=1}^p a_i y_{t-i} + \sigma_1 u_t & (\text{if } y_t \geq y_{t-1}) \\ b_1^* + b_2^* t + \sum_{i=1}^p b_i y_{t-i} + \sigma_2 u_t & (\text{if } y_t < y_{t-1}) \end{cases},$$

where we take $\sigma_i > 0$ ($i = 1, 2$). The disturbance terms $\{u_t\}$ are a sequence of $I(1)$ process satisfying

$$(2.6) \quad \Delta u_t = \sum_{j=0}^q c_j v_{t-j},$$

where $c_0 = 1$ for the normalization factor. The random variables $\{v_t\}$ are martingale differences with $E(v_t | \mathcal{F}_{t-1}) = 0$ and

$$(2.7) \quad E(v_t^2 | \mathcal{F}_{t-1}) = 1 + \sum_{i=1}^r \alpha_i v_{t-i}^2 \quad a.s.,$$

where the unknown coefficients $\{\alpha_i, i = 1, \dots, r\}$ satisfy some restrictions on the positivity of conditional variances $\alpha_i \geq 0$ ($i = 1, \dots, r$). In the above notation the σ -field \mathcal{F}_{t-1} is generated by a set of random variables $\{y_s, v_s; s \leq t-1\}$. Although the conditional heteroskedasticity of $\{v_t\}$ has a simple form as in the original ARCH model and is symmetrical, the resulting conditional heteroskedasticities for $\{y_t\}$ and $\{\Delta y_t\}$ can be asymmetrical.

The univariate non-linear time series model we introduced in (2.5) has the first order multivariate autoregressive form with moving-average disturbances by using the standard state space representation in time series analysis. Let us define $p \times 1$ vectors \mathbf{y}_t and \mathbf{u}_t by

$$(2.8) \quad \mathbf{y}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \quad \mathbf{u}_t = \begin{pmatrix} u_t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$p \times 2$ matrices $\boldsymbol{\mu}_i$ ($i = 1, 2$) by

$$(2.9) \quad \boldsymbol{\mu}_1 = \begin{pmatrix} a_1^* & a_2^* \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\mu}_2 = \begin{pmatrix} b_1^* & b_2^* \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix},$$

and also define $p \times p$ matrices \mathbf{A} and \mathbf{B} by

$$(2.10) \quad \mathbf{A} = \begin{pmatrix} a_1 & \cdots & \cdots & a_p \\ 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 & \cdots & \cdots & b_p \\ 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix}.$$

Then the non-linear time series model we have introduced can be represented as

$$(2.11) \quad \mathbf{y}_t = \begin{cases} \boldsymbol{\mu}_1 \mathbf{z}_t^* + \mathbf{A} \mathbf{y}_{t-1} + \sigma_1 \mathbf{u}_t & \text{if } \mathbf{e}_1' \mathbf{y}_t \geq \mathbf{e}_1' \mathbf{y}_{t-1} \\ \boldsymbol{\mu}_2 \mathbf{z}_t^* + \mathbf{B} \mathbf{y}_{t-1} + \sigma_2 \mathbf{u}_t & \text{if } \mathbf{e}_1' \mathbf{y}_t < \mathbf{e}_1' \mathbf{y}_{t-1} \end{cases},$$

where $\mathbf{e}_1' = (1, 0, \dots, 0)$ and $\mathbf{z}_t^* = (1, t)'$ is the vector of strictly exogenous variables.

The most important feature of this representation is that the time series variables may take quite different values in the two different phases or regimes. This type of statistical time series model could be classified as the threshold model in the recent time series literature. However, since the vector time series and two phases at time t are determined simultaneously, we shall refer this type of time series models to a simultaneous switching autoregressive (SSAR) time series model. The univariate time series model consisting of (2.5) and (2.6) can be called the simultaneous switching autoregressive integrated moving-average (SSARIMA) model. We denote this class of models as SSARIMA($p, 1, q$) or simply SSARIMA(p, q), which is a direct extension of the standard ARIMA(p, q) model in the statistical time series analysis. In the SSARIMA models we need some restrictions on unknown parameters in order to make the stochastic process defined by (2.11) meaningful in a proper statistical sense. This issue has been called the coherency problem. We say the non-linear time series model (2.11) is coherent if and only if the correspondence between $\{\mathbf{y}_t\}$ and $\{\mathbf{u}_t\}$ is one-to-one given the initial condition \mathcal{F}_0 .

The conditions of $\mathbf{e}'_1 \mathbf{y}_t \geq \mathbf{e}'_1 \mathbf{y}_{t-1}$ and $\mathbf{e}'_1 \mathbf{y}_t < \mathbf{e}'_1 \mathbf{y}_{t-1}$ can be rewritten as

$$(2.12) \quad \sigma_1 u_t \geq \mathbf{e}'_1 (\mathbf{I}_p - \mathbf{A}) \mathbf{y}_{t-1} - \mathbf{e}'_1 \boldsymbol{\mu}_1$$

and

$$(2.13) \quad \sigma_2 u_t < \mathbf{e}'_1 (\mathbf{I}_p - \mathbf{B}) \mathbf{y}_{t-1} - \mathbf{e}'_1 \boldsymbol{\mu}_2$$

respectively. Then the set of conditions on coherency for the present case can be summarized by a 1×2 vector $\boldsymbol{\gamma}^{*'} = (\gamma_1^*, \gamma_2^*)$ and a $1 \times p$ vector $\boldsymbol{\gamma}' = (\gamma_1, \dots, \gamma_p)$:

$$(2.14) \quad \begin{aligned} \frac{1}{\sigma_1} [-\mathbf{e}'_1 \boldsymbol{\mu}_1, \mathbf{e}'_1 (\mathbf{I}_p - \mathbf{A})] &= \frac{1}{\sigma_2} [-\mathbf{e}'_1 \boldsymbol{\mu}_2, \mathbf{e}'_1 (\mathbf{I}_p - \mathbf{B})] \\ &= (\boldsymbol{\gamma}^{*'}, \boldsymbol{\gamma}') . \end{aligned}$$

Because the univariate SSARIMA(p, q) has some specific structure in the class of the general multivariate SSARIMA model, it has a simple representation. For instance, we do not need some additional conditions to (2.14) for the coherency of the stochastic process. By using Theorem 2.1 of Kunitomo and Sato (1996a), we have the one-to-one relation between the stochastic processes $\{\mathbf{y}_t\}$ and $\{u_t\}$ under the condition given by (2.14).

In order to obtain a useful representation of the process $\{\Delta \mathbf{y}_t\}$, we use the indicator functions

$$I_t^{(1)} = I(\mathbf{e}'_1 \mathbf{y}_t \geq \mathbf{e}'_1 \mathbf{y}_{t-1})$$

and

$$I_t^{(2)} = I(\mathbf{e}'_1 \mathbf{y}_t < \mathbf{e}'_1 \mathbf{y}_{t-1}) .$$

We can then obtain the representation

$$(2.15) \quad \mathbf{y}_t = \boldsymbol{\mu}(t) \mathbf{z}_t^* + \mathbf{A}(t) \mathbf{y}_{t-1} + \sigma(t) \mathbf{u}_t ,$$

where

$$(2.16) \quad \boldsymbol{\mu}(t) = \sum_{i=1}^2 \boldsymbol{\mu}_i I_t^{(i)} ,$$

$$(2.17) \quad \mathbf{A}(t) = \mathbf{A} I_t^{(1)} + \mathbf{B} I_t^{(2)} ,$$

and

$$(2.18) \quad \sigma(t) = \sum_{i=1}^2 \sigma_i I_t^{(i)} .$$

When $\{u_t\}$ in (2.15) is an $I(1)$ process, the stochastic process $\{\mathbf{y}_t\}$ is a non-ergodic process. Hence it is of interest to investigate the ergodicity conditions for $\{\Delta \mathbf{y}_t\}$. In the present univariate case we can simplify some coefficients by the coherency conditions (2.14), that is, we have the relations

$$(2.19) \quad \boldsymbol{\mu}(t) = -(\gamma_1^*, \gamma_2^*) \sigma(t)$$

and

$$(2.20) \quad \mathbf{e}'_1 - \mathbf{e}'_1 \mathbf{A}(t) = (\gamma_1, \dots, \gamma_p) \sigma(t) .$$

By using these relations, we re-write the disturbance terms $\{u_t\}$ as

$$(2.21) \quad u_t = \frac{1}{\sigma(t)} \Delta y_t + \gamma_1^* + \gamma_2^* t + \sum_{i=1}^p \gamma_i y_{t-i} .$$

Because of (2.6) and (2.21), given the information available at $t - 1$, we have to consider four phases for Δy_t at t depending on $I_t^{(i)}$ and $I_{t-1}^{(i)}$ ($i = 1, 2$). By taking the difference operation on (2.21) and using (2.14) and (2.15), we have the representation as

$$(2.22) \quad \Delta y_t = \sigma(t) \left\{ A_0 + \sum_{i=1}^p A_i(t-i) \sigma(t-i)^{-1} \Delta y_{t-i} + \Delta u_t \right\} ,$$

where $A_0 = -\gamma_2^*$,

$$A_1(t-1) = 1 - \gamma_1 \sigma(t-1) ,$$

and

$$A_i(t-i) = -\gamma_i \sigma(t-i) \quad (i = 2, \dots, p) .$$

It is immediately clear that the time series model defined by (2.4) is a special case of (2.21). Moreover, we have the following characterization result on $\{\Delta y_t\}$, which is a simple extension of Theorem 2.1 of Kunitomo and Sato (1996b).

Theorem 2.1 : *Define the non-linear transformation of $\{\Delta y_t\}$ by*

$$(2.23) \quad T(\Delta y_t) = \sigma(t)^{-1} \Delta y_t .$$

Then the transformed stochastic process $\{T(\Delta y_t)\}$ satisfies

$$(2.24) \quad T(\Delta y_t) = A_0 + \sum_{i=1}^p A_i(t-i) T(\Delta y_{t-1}) + \Delta u_t ,$$

where A_0 and $A_i(t-i)$ ($i = 1, \dots, p$) are defined by (2.22).

The time series model defined by (2.24) can be called the Threshold Autoregressive Moving-average (TARMA) model with time-varying coefficients in the non-linear time series analysis. From this result it can be deduced that the stochastic process for $\{\Delta y_t\}$ is slightly different from the TARMA(p, q) model. Also the stochastic process $\{\Delta y_t\}$ has the representation

$$\Delta y_t = \begin{cases} a_2^* + \sum_{i=1}^p a_i \Delta y_{t-i} + \sigma_1 \Delta u_t & (\text{if } \Delta y_{t-1} \geq 0, \Delta y_t \geq 0) \\ a_2^* + \left(\frac{\sigma_1}{\sigma_2}\right) b_1 \Delta y_{t-1} + \sum_{i=2}^p a_i \Delta y_{t-i} + \sigma_1 \Delta u_t & (\text{if } \Delta y_{t-1} < 0, \Delta y_t \geq 0) \\ b_2^* + \left(\frac{\sigma_2}{\sigma_1}\right) a_1 \Delta y_{t-1} + \sum_{i=2}^p b_i \Delta y_{t-i} + \sigma_2 \Delta u_t & (\text{if } \Delta y_{t-1} \geq 0, \Delta y_t < 0) \\ b_2^* + \sum_{i=1}^p b_i \Delta y_{t-i} + \sigma_2 \Delta u_t & (\text{if } \Delta y_{t-1} < 0, \Delta y_t < 0) \end{cases} \quad (2.25)$$

By this form of representation, we notice that the differenced process $\{\Delta y_t\}$ from the SSARIMA model has not only the simultaneous switching characteristic, but also a characteristic of the threshold type time series model. For the stochastic process $\{\Delta y_t\}$ defined by (2.25), we can present a set of sufficient conditions for its ergodicity. A proof is provided in the Appendix.

Theorem 2.2 : Suppose (i) p and q are finite numbers, (ii) the coherency condition (2.14) holds, (iii) the density function $g(v)$ of $\{v_t\}$ is everywhere positive with respect to the Lebesgue measure, and (iv) $\sup_{t \geq 1} E[|v_t|] < +\infty$. Then the Markov chain defined by (2.25) for $\{\Delta y_t\}$ is ergodic if there exist p positive numbers δ_i ($i = 1, \dots, p$) satisfying (v) $a_1 < \delta_1, b_1 < \delta_1, |a_i| < \delta_i, |b_i| < \delta_i$ ($i = 2, \dots, p$),

$$(2.26) \quad [a_1 - \delta_1][b_1 - (1 - \delta_1 - \frac{\sigma_2}{\sigma_1})] < 0 ,$$

$$(2.27) \quad [b_1 - \delta_1][a_1 - (1 - \delta_1 - \frac{\sigma_1}{\sigma_2})] < 0 ,$$

and (vi)

$$(2.28) \quad \sum_{i=1}^p \delta_i < 1,$$

For the precise definition and discussions on the ergodicity for Markov chains on a general state space, see Tweedie (1975), Liu and Susko (1992), or Meyn and Tweedie (1993). When $p = 1$, we have shown (Kunitomo and Sato (1996b)) that the necessary and sufficient conditions for the ergodicity of $\{\Delta y_t\}$ are

$$(2.29) \quad a_1 < 1, b_1 < 1, a_1 b_1 < 1 .$$

By a simple calculation, the sufficient conditions in the above theorem reduce to these ones when $p = 1$. When $p \geq 2$ and $q = 0$, the obvious sufficient conditions for the ergodicity are given by

$$(2.30) \quad \sum_{i=1}^p |a_i| < 1, \sum_{i=1}^p |b_i| < 1 .$$

But they exclude many interesting non-linear phenomena such as the case when there are some over-reactions and subsequent gradual adjustments to its mean level. It seems that the conditions we have for the ergodicity cover some of these important cases, but are too strong as the necessary conditions when $p \geq 2$. From our limited number of simulations, we have found that our sufficient conditions exclude some important cases even when $p = 2$.

2.3 Estimation and Model Selection

In order to estimate the SSARIMA models, we have proposed and investigated the maximum likelihood (ML) method (Kunitomo and Sato (1996b)). We are also proposing to use the ML method for estimating the SSARIMA models with ARCH disturbances. However, because of the ARCH effects for the disturbance terms, we need to modify the likelihood function for the standard SSARIMA model slightly.

We set the initial conditions such that $v_0 = v_{-1} = \dots = v_{-\max\{q,r\}} = 0$ and $\Delta y_t, 1 \leq t \leq p$ are fixed for simplicity's sake. Then there is an important aspect in the present model that the Jacobian of the transformation from $\{\Delta u_t, p+1 \leq t \leq T\}$ to $\{\Delta y_t, p+1 \leq t \leq T\}$ is given by

$$\prod_{t=p+1}^T \sigma(t)^{-1}.$$

The Jacobian of the transformation from $\{v_t, p+1 \leq t \leq T\}$ to $\{\Delta u_t, p+1 \leq t \leq T\}$ is 1 provided that (2.6) is an invertible MA process.

Under the assumption that the disturbance terms $\{v_t\}$ given \mathcal{F}_{t-1} are conditionally normal random variables, the normalized log-likelihood function for $\{\Delta y_t, p+1 \leq t \leq T\}$ given the initial conditions can be written as

$$\begin{aligned} (2.31) \log L_T(\boldsymbol{\theta}) &= -\frac{(T-1)m}{2T} \log 2\pi \\ &\quad - \frac{1}{2T} \sum_{t=p+1}^T \sum_{i=1}^2 I_t^{(i)} \log[\sigma_i^2 \omega_t(\boldsymbol{\theta})] - \frac{1}{2T} \sum_{t=p+1}^T \omega_t(\boldsymbol{\theta})^{-1} v_t(\boldsymbol{\theta})^2, \end{aligned}$$

where $\{v_t(\boldsymbol{\theta})\}$ are $\{v_t\}$ rewritten from (2.5) and (2.6) as functions of $\{\Delta y_t\}$, $\boldsymbol{\theta}$ is a vector of structural parameters appearing in the original SSARIMA model, and

$$(2.32) \quad \omega_t(\boldsymbol{\theta}) = 1 + \sum_{i=1}^r \alpha_i v_{t-i}^2.$$

The unknown coefficients $\{\alpha_i (i = 1, \dots, r)\}$ satisfy

$$(2.33) \quad \alpha_i \geq 0, \quad \sum_{i=1}^r \alpha_i < 1.$$

In this representation of the SSARIMA model with ARCH disturbances the vector of unknown coefficients consists of

$$\theta' = (\gamma_2^*, \gamma_1, \dots, \gamma_p, c_1, \dots, c_q, \sigma_1, \sigma_2, \alpha_1, \dots, \alpha_r) .$$

The maximum likelihood (ML) estimator can be defined as the maximum of $\log L_T(\theta)$ with respect to the unknown parameters in θ , where the parameter space Θ is restricted by the coherency conditions given by (2.14). The asymptotic properties of the ML estimator in the SSARIMA(1,q) has been established by Kunitomo and Sato (1996b). By modifying those results slightly, we can establish in the present case that the ML estimator is consistent and asymptotically normal in the general SSARIMA(p,q)-ARCH(r) model. A sketchy proof of this is given in the Appendix.

Theorem 2.3 : *For the SSARIMA(p,q) model given by (2.25) with ARCH(r) disturbances, suppose (i) the sufficient conditions for the coherency in (2.14) and the ergodicity in Theorem 2.2 hold, (ii) the disturbance terms $\{v_t\}$ are conditionally normally distributed given \mathcal{F}_{t-1} with $\sup_{t \geq 1} E[v_t^{4+\delta}] < +\infty$ for some $\delta > 0$ and $\alpha_i \geq 0$ ($i = 1, \dots, r$), (iii) the MA order q is a finite number and (2.6) is invertible, and (iv) $\sigma_i > 0$ ($i = 1, 2$). Also suppose (v) the true parameter vector θ_0 is an interior point of a compact set Θ_0 in the parameter space Θ . Then the ML estimators $\hat{\theta}_{ML}$ of unknown parameters in θ are consistent and asymptotically normally distributed as*

$$(2.34) \quad \sqrt{T} (\hat{\theta}_{ML} - \theta) \xrightarrow{d} N [0, I_{\theta}^{-1}] ,$$

provided

$$(2.35) \quad I_{\theta} = \text{plim}_{T \rightarrow \infty} \left[-\frac{\partial^2 \log L_T(\theta)}{\partial \theta \partial \theta'} \right]$$

is a positive definite matrix.

The conditions in the above theorem are the same as in Theorem 2.4 of Kunitomo and Sato (1996b) except (ii). In the present case, we need some sufficient conditions for the non-negativity of conditional variance functions and the existence of unconditional moments, which imply some restrictions on the parameter space. These conditions could be considerably relaxed. A related issue has been discussed by Chapter 21 of Hamilton (1994). The asymptotic information matrix can be consistently estimated by

$$(2.36) \quad \left[-\frac{\partial^2 \log L_T(\theta)}{\partial \theta \partial \theta'} \right] \Big|_{\hat{\theta}_{ML}} .$$

Although we have obtained the consistency and the asymptotic normality of the ML estimator under a set of sufficient conditions, presently we do not know much of the behavior of the ML estimator when the sample size is not very large and the conditional distribution of disturbances are not normal. In this

respect, we have investigated the finite sample properties of the ML estimator in a systematic way for the SSARIMA(1,0) model when $T = 100$ and $T = 500$ (Kunitomo and Sato (1996b)). We produced some evidence for the use of the ML estimation method for the SSARIMA(1,0) model even when the sample size is not very large and the conditional distribution of disturbances are different from normal in certain extent. From those results we expect that similar finite sample properties could hold in the present situation.

It is important to note that the class of the SSARIMA(p, q)-ARCH(r) models we have introduced in this paper includes many statistical time series models as special cases. For instance, it includes the ARIMA($p, 1, q$) models with ARCH(r) disturbances. In order to select an appropriate model within the class of the SSARIMA(p, q)-ARCH(r) models, we propose to use the minimum Akaike's Information Criterion (AIC): see Akaike (1973). Although there have been many proposed criteria for model selection procedures, it is widely known that the minimum AIC is useful and successful in many applications. The value of AIC for the SSARIMA(p, q)-ARCH(r) model in the present case can be defined by

$$(2.37) \quad AIC(p, q, r) = \frac{1}{T} \sum_{i=1}^2 \sum_{t=p+1}^T I_t^{(i)} \log[\hat{\sigma}_i^2 \omega_t(\hat{\boldsymbol{\theta}})] \\ + \frac{1}{T} \sum_{t=p+1}^T \omega_t(\hat{\boldsymbol{\theta}})^{-1} v_t(\hat{\boldsymbol{\theta}})^2 + 2(p + q + r),$$

where $\hat{\sigma}_i$ ($i = 1, 2$) denote the ML estimator of σ_i ($i = 1, 2$) and $\hat{\boldsymbol{\theta}}$ denotes the vector of the ML estimator for $\boldsymbol{\theta}$.

3. An Application to Financial Data

In this section we shall report an empirical result using some stock time series data. In our data analysis we have used the time series data of the Nikkei-225 indices which are the most popular stock price index traded in Japan and the SP-500 from the U.S. The first data sets are the closing daily data of the Nekkei-225 Spot index from January 1985 to December 1994 and the second data sets are the closing daily data of the SP-500 from January 1985 to December 1994. All data were transformed into their logarithms before the estimation of the non-stationary SSAR-ARCH model. It may be of some interests for economists to compare the time series movements of these spot prices, which are representative indices in two major stock markets in the world.

It has often been argued by financial economists and practitioners in major financial markets that there have been significant daily effects in the volatility functions of financial prices. The typical argument for the existence of daily effects is from the observation that the information flows are different from day to day and there are some trading day effects and holiday effects. By taking account of this observation, in actual estimation we have the volatility function

$$(3.1) \quad \omega_t(\boldsymbol{\theta}) = 1 + \sum_{i=1}^r \alpha_i v_{t-i}^2 + \sum_{i=1}^{r^*} \beta_i D_{it},$$

where D_{it} are the dummy variables defined by

$$(3.2) \quad D_{it} = \begin{cases} 1 & \text{if } t \in \mathbf{J}_i \\ 0 & \text{if } t \notin \mathbf{J}_i \end{cases},$$

and the index sets \mathbf{J}_i ($i = 1, \dots, r^*$) denote the calendar days.

We have estimated the SSARMA(p, q)–ARCH(r) model with the dummy variables of the form (3.1). The estimation of structural parameters in the SSARMA(p, q)–ARCH(r) model has been conducted by the ML method under the assumption of the normal disturbances. Since we cannot obtain an explicit formula for the ML estimators of unknown parameters, we have used a numerical non-linear optimization technique with the coherency restrictions on parameters. In the actual estimation we took $q = 0$ because we could not find any significant MA terms in most cases and there are some technical difficulties in MA estimation at present. We have divided the full sample period for Nikkei-225 into 4 sub-periods, each consisting of about 200 data points. This has been done because many different phenomena occurred in the full sample period and the assumption of constant coefficients in the underlying SSARMA model could be unreasonable from a practical point of view ¹. The resulting estimation results are summarized by Table 1 for the Nikkei-225 data and Table 2 for the SP-500 data.

There are several interesting empirical observations from Table 1. First, the spot stock price index sometimes shows sharp asymmetrical movements either in the upward or downward phase. This phenomenon was evident in 1985 and 1987. Actually we already knew that there was a sharp decline in October of 1987. During these sharp downward phases, the estimated values of the downward coefficients are often smaller than the corresponding upward coefficients. This agrees with the fact that the estimated volatility in the downward phase is often larger than the volatility in the upward phase. Second, we have found significant daily effects on the volatility functions in most cases. In our estimation of daily effects we have normalized the volatility coefficients such that the Monday volatility always takes the value of 1. Except for the first data period ², we have found negative daily effects from the Monday volatility, which are statistically significant. This agrees with the notion that the Monday volatility effect is significantly larger than other days because of holiday effects, but also it is less than

¹For instance, many changes in regulations, taxes, and financial institutions have occurred at financial markets in Japan during the full sample period.

²The estimated Monday and Tuesday volatilities in this period are different from our intuition. However, Dr. Kunio Okina has suggested as possible explanation that there was a substantial amount of informational flow from the U.S. financial markets to the Tokyo stock market during the first period. Because there is a time lag in two countries, the Tuesday volatility in Tokyo was quite large.

the holiday length from Friday to Monday times the normal level of volatility. In Japanese financial markets there was some trading on Saturday mornings during 1980s. It would not be possible to find any significant differences in the daily volatility differences because the trading on Saturday was terminated at the end of the 1980s. Also we found that the inclusion of daily effects in the form of (3.1) does not have a significant effect on the estimation of asymmetry and volatility level. In most cases the estimated coefficients of a_i , b_i , and α_i are similar for the model (2.25) with or without daily effects. Third, after around 1990³, there have not been many occasions as was the case previously when the asymmetrical movements of the price indices are evident and the estimated coefficients are not very large. The differences of AIC in $ARIMA(p, 1, 0)$ and $SSARIMA(p, q)$ are not very large in most cases in this period. Fourth, as far as the SP-500 data is concerned, we find some evidence of the asymmetry discussed during the periods before around 1990. This is quite evident for the data including that for 1987. In addition, we often find some asymmetry whenever we use a large quantity of data over more than 5 years before 1990. However, in the data of 1990s it is difficult to find any asymmetry and the estimated coefficients are generally not very large. We only show the estimation results by using the data from January 1990 to June 1992 in Table 2, which seem to be typical of recent SP-500 data. The model chosen by AIC is $ARIMA(0, 1, 0)$ with $ARCH(2)$ and daily effects. It is often possible to find significant ARCH effects as well as daily effects on volatility in the 1990s by comparing the values of AIC.

These empirical problems and findings may have some implications for financial economists. Needless to say, these observations from our empirical results on stock indices are preliminary. But clearly it has not been easy to detect these features of the financial time series data by using the existing methods and the linear time series modelling in particular.

4. Conclusions

In this paper we have focused on one important aspect in many financial economic time series, which has been sometimes ignored in the past econometric studies on the volatilities of major financial time series. Since the asymmetrical pattern in the movements of time series between the upward phase and the downward phase often observed by economists can not be represented properly by the standard ARMA, ARIMA, and ARCH processes, we have proposed to use the class of simultaneous switching autoregressive integrated moving-average (SSARIMA) models with autoregressive conditional heteroskedastic (ARCH) disturbances, which was recently proposed by Kunitomo and Sato (1996b). In this paper we have investigated some properties of the $SSARIMA(p, q)-ARCH(r)$ model and the asymptotic properties of the maximum likelihood estimation method

³There can be economic as well as institutional reasons for this phenomenon. For instance, we have compared the movements of the Nikkei-225 spot index and futures index (Kunitomo and Sato (1996b)). The active trading of the Nikkei-225 futures in Japan was started at the end of 1989.

for estimating its unknown parameters. Unlike other methods already available for estimating the volatilities of financial time series, our method is a very simple but general one for investigating whether they are asymmetrical or not from a medium size set of time series data. The model selection procedure in the class of the SSARIMA(p, q)-ARCH(r) can be straightforwardly developed.

We have also tried to show that there are some reasons why the SSARIMA model with ARCH disturbances is a useful tool to analyze many financial time series in financial markets. The point is that if we permit the intrinsic value of security to be different from the observed price and have a non-linear adjustment process, the result is the SSARIMA model. Then the estimated coefficients in the upward phase and in the downward phase can be different and the resulting volatility of financial time series can be asymmetrical around the normal volatility level.

In this paper we have used the SSARIMA-ARCH modelling to examine the movements of the Nikkei-225 stock index at Tokyo and the SP-500 index at New York from 1985 to 1994. We have found some evidence of the asymmetrical movements of stock indices before around 1990 in particular, which has been consistent with the view of some financial economists and practitioners in the Japanese financial markets. By using our empirical example, at least we have shown that our modelling approach is useful for analysing some financial prices.

Finally, we should mention that our approach can be easily extended to a more general form of conditional heteroskedasticity models such as the generalized ARCH model by Bollerslev (1986). In addition, our formulation does not need a large quantity of data for estimation as some non-parametric and semi-parametric time series methods do. We think there are some advantageous aspects to our modelling compared to other methods.

5. Mathematical Appendix

In this appendix, we gather some mathematical details which we have omitted in the previous sections. However, since most of the theoretical results reported in this paper are rather straightforward extensions of the corresponding ones in Kunitomo and Sato (1996b), we freely refer to the notations and results in that paper. We use the notation $x_t = \Delta y_t$ in the Appendix.

Proof of Theorem 2.2 : We shall use a method similar to the one used in Theorem 2.3 of Kunitomo and Sato (1996b). Define a $(p + q) \times 1$ vector \mathbf{X}_t by

$$(A.1) \quad \mathbf{X}_t = \begin{pmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \\ v_t \\ v_{t-1} \\ \vdots \\ v_{t-q+1} \end{pmatrix}.$$

We then consider the Markovian representation for $\{\mathbf{X}_t\}$. For the sake of simplicity, we set $\gamma_2^* = 0$. The condition $x_t \geq 0$ is equivalent to $v_t \geq \mathbf{a}'_{t-1} \mathbf{X}_{t-1}$, where

$$(A.2) \quad \mathbf{a}'_{t-1} = (\gamma_1 - \frac{1}{\sigma(t-1)}, \gamma_2, \dots, \gamma_p, -c_1, -c_2, \dots, -c_q)$$

and $\{c_j, j \leq j \leq q\}$ are the MA coefficients of $\{\Delta \mathbf{u}_t\}$. From (2.20) we have the representation

$$(A.3) \quad \mathbf{X}_t = \mathbf{H}(\mathbf{X}_{t-1}, v_t),$$

where

$$(A.4) \quad \mathbf{H}(\mathbf{X}_{t-1}, v_t) = \begin{pmatrix} -\sigma(t) \mathbf{a}'_{t-1} \mathbf{X}_{t-1} + \sigma(t) v_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \\ v_t \\ v_{t-1} \\ \vdots \\ v_{t-q+1} \end{pmatrix}.$$

We use the criterion function

$$(A.5) \quad G(\boldsymbol{\xi}) = \sum_{i=1}^{p+q} h(\xi_i),$$

where $h(\xi_i) = |\xi_i|$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{p+q})$.

If we use a sequence of compact sets $K_n = [-n, n] \times \dots \times [-n, n], n = 1, 2, \dots$, then $\inf_{\boldsymbol{\xi} \in K_n^c} G(\boldsymbol{\xi}) \rightarrow +\infty$ as $n \rightarrow +\infty$. For $t \geq q$,

$$(A.6) \quad \begin{aligned} E[G(\mathbf{X}_t) | \mathcal{F}_0] &= E\left[\sum_{j=0}^{p-1} h(x_{t-j}) + \sum_{j=0}^{q-1} h(v_{t-j}) \middle| \mathcal{F}_0\right] \\ &\leq k_1 + \sum_{j=0}^{p-1} E[E[h(x_{t-j}) | \mathcal{F}_{t-j-1}] | \dots | \mathcal{F}_0] \end{aligned}$$

because $E[|v_t|] < +\infty$ and k_1 is a positive constant.

Let

$$(A.7) \quad \begin{aligned} Q_{t|t-i} &= E[h(x_t)|\mathcal{F}_{t-i}] \\ &= E\{h[-\sigma(t)\mathbf{a}'_{t-1}\mathbf{X}_{t-1} + \sigma(t)v_t]|\mathcal{F}_{t-i}\} \end{aligned}$$

for any $i \geq 1$. We first consider the case when $x_{t-1} > 0$. In this case from (2.25) we have two phases at t given the condition $x_{t-1} > 0$ and we have

$$(A.8) \quad \begin{aligned} Q_{t|t-1} &= \sigma_1 \int_{z \geq (-\frac{1}{\sigma_1})x_{t-1} + \gamma' \mathbf{x}_{[t-1]} - \mathbf{c}' \mathbf{v}_{[t-1]}} \left[\frac{1}{\sigma_1} x_{t-1} - \gamma' \mathbf{x}_{[t-1]} + \mathbf{c}' \mathbf{v}_{[t-1]} + z \right] g(z) dz \\ &\quad - \sigma_2 \int_{z < (-\frac{1}{\sigma_1})x_{t-1} + \gamma' \mathbf{x}_{[t-1]} - \mathbf{c}' \mathbf{v}_{[t-1]}} \left[\frac{1}{\sigma_1} x_{t-1} - \gamma' \mathbf{x}_{[t-1]} + \mathbf{c}' \mathbf{v}_{[t-1]} + z \right] g(z) dz, \end{aligned}$$

where $\mathbf{c}' = (c_1, \dots, c_q)$, $\mathbf{x}_{[t-1]} = (x_{t-1}, \dots, x_{t-p})$, and $\mathbf{v}_{[t-1]} = (v_{t-1}, \dots, v_{t-q})$. We note that (2.26) and (2.27) in Theorem 2.2 imply $0 < (1 - \delta_1)/\sigma_1 < \gamma_1 < 1/\sigma_1 + \delta_1/\sigma_2$ and $0 < (1 - \delta_1)/\sigma_2 < \gamma_1 < 1/\sigma_2 + \delta_1/\sigma_1$. Then by using the conditions in Theorem 2.2, we have an inequality

$$(A.9) \quad Q_{t|t-1} \leq k_2(1 + \sum_{i=1}^q |v_{t-i}|) + \sum_{i_1=1}^p \delta_{i_1} |x_{t-i_1}|.$$

When $x_{t-1} > 0$, we have the above inequality by using the conditions in Theorem 2.2. Then we have

$$(A.10) \quad \begin{aligned} Q_{t|t-p-1} &\leq E[k_2(1 + \sum_{j=1}^q |v_{t-j}|) + \sum_{i_1=1}^p \delta_{i_1} k_2(1 + \sum_{j=1}^q |v_{t-i_1-j}|) | \mathcal{F}_{t-p-1}] \\ &\quad + \sum_{i_1=1}^p \delta_{i_1} \sum_{i_2=1}^p \delta_{i_2} E[|x_{t-i_1-i_2}| | \mathcal{F}_{t-p-1}]. \end{aligned}$$

By repeating this procedure and taking $\delta = \sum_{i=1}^p \delta_i$ ($0 < \delta < 1$), we have the relation

$$(A.11) \quad E[G(\mathbf{X}_t) | \mathcal{F}_0] \leq k_3 \sum_{k=0}^{t-1} \delta^k \left(1 + \sum_{j=1}^q \eta_{t-k-j} \right) + p\delta^t G(\mathbf{X}_0),$$

where k_3 is a positive constant and $\eta_k = E[|v_k| I(k > 0) + |v_k| I(k \leq 0)]$. Hence we have established the boundedness condition on the criterion function

$$(A.12) \quad \sup_{t \geq 1} E[G(\mathbf{X}_t) | \mathcal{F}_0] < +\infty.$$

The rest of our proof is the same as the sufficiency part of the proof of Theorem 2.3 in Kunitomo and Sato (1996b). The essential part is to use Theorem 1.1 of Liu and Susko (1992) for the ergodicity of Markov chain with a general state space. We can use their theorem because we have the following Lemma A.1, which is an straightforward extension of Lemma A.1 of Kunitomo and Sato (1996b) and we omit its proof. *Q.E.D.*

Lemma A.1 : Let $\{v_t\}$ in the SSARIMA(p, q) model be independently and identically distributed random variables with the density function $g(v)$, which is everywhere positive in \mathbf{R} . Then given

$$(x_{t-1}, \dots, x_{t-p}, v_{t-1}, \dots, v_{t-q}) = (z_1, \dots, z_p, z_{p+1}, \dots, z_{p+q}),$$

the conditional probability

$$(A.13) \quad \Pr\{x_t \leq x | x_{t-1} = z_1, \dots, x_{t-p} = z_p, v_{t-1} = z_{p+1}, \dots, v_{t-q} = z_{p+q}\}$$

is a continuous function of $\mathbf{z}' = (z_1, \dots, z_{p+q})$.

Next, we shall give Lemma A.2 on the existence of moments for $\{\Delta y_t\}$. Kunitomo and Sato (1996b) have proved Lemma A.2 when $p = 1$ as their Lemma A.2. However, it is straightforward to extend that proof for an arbitrary p and so we omit the details.

Lemma A.2 : In the SSARIMA model given by (2.25), assume (i) the coherency conditions (2.14), (ii) the ergodicity conditions (2.26)-(2.28), and (iii) $\sup_{t \geq 1} E[|v_t|^k] < +\infty$ for some $k \geq 1$. Then

$$(A.14) \quad \sup_{t \geq 1} E[|x_t|^k] < +\infty.$$

Proof of Theorem 2.3 : The method of our proof is similar to the one used in Kunitomo and Sato (1996b). However, some modifications are necessary because there is an ARCH effect in the disturbance terms and p is an arbitrary integer.

Step 1: Let the stochastic process $\{\Delta u_t(\boldsymbol{\theta})\}$ be defined by

$$(A.15) \quad \Delta u_t(\boldsymbol{\theta}) = D(t)^{-1}x_t + \gamma_2^* + [\gamma_1 - D(t-1)^{-1}]x_{t-1} + \sum_{i=2}^p \gamma_i x_{t-i},$$

which is identical to $\{\Delta u_t\}$ in (2.25). We denote the vector of true parameter values of

$$\boldsymbol{\theta}' = (\gamma_2^*, \gamma_1, \dots, \gamma_p, \sigma_1, \sigma_2, c_1, \dots, c_q, \alpha_1, \dots, \alpha_r)$$

as

$$\boldsymbol{\theta}'_0 = (\gamma_2^{*(0)}, \gamma_1^{(0)}, \dots, \gamma_p^{(0)}, \sigma_1^{(0)}, \sigma_2^{(0)}, c_1^{(0)}, \dots, c_q^{(0)}, \alpha_1^{(0)}, \dots, \alpha_r^{(0)}).$$

Also we denote $\boldsymbol{\alpha}' = (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\alpha}'_0 = (\alpha_1^{(0)}, \dots, \alpha_r^{(0)})$. It is then possible to prepare the following lemma, which is similar to Lemma 3.1 and Lemma 3.2 of Weiss (1986) and we give a sketchy proof at the end of Step 1.

Lemma A.3 : (i) Define a $(1 + p + q) \times 1$ vector of parameters

$$\mathbf{m}' = (\gamma_2^*, \gamma_1, \dots, \gamma_p, c_1, \dots, c_q)$$

and assume $E[v_t^2] < +\infty$. Then

$$(A.16) \quad E\left[\frac{\partial v_t(\boldsymbol{\theta})}{\partial \mathbf{m}} \frac{\partial v_t(\boldsymbol{\theta})}{\partial \mathbf{m}'}\right] > 0 \text{ a.s..}$$

(ii) Assume $E[v_t^4] < +\infty$. Then

$$(A.17) \quad E\left[\frac{\partial \omega_t}{\partial \boldsymbol{\alpha}} \frac{\partial \omega_t}{\partial \boldsymbol{\alpha}'}\right] > 0 \text{ a.s..}$$

Under the assumptions we have made for the ergodicity, we can follow the argument used in Theorem 3.1 of Weiss (1986). The criterion function in the present case

$$(A.18) \quad R_T(\boldsymbol{\theta}) = -\frac{1}{2} \frac{1}{T} \sum_{t=2}^T \sum_{i=1}^2 I_t^{(i)} \log \omega_t(\boldsymbol{\alpha}) - \frac{1}{2} \frac{1}{T} \sum_{t=2}^T v_t(\boldsymbol{\theta})^2 \omega_t(\boldsymbol{\alpha})^{-1}$$

converges in probability to

$$(A.19) \quad R(\boldsymbol{\theta}) = -\frac{1}{2} E[\log \omega_t(\boldsymbol{\alpha})] - \frac{1}{2} E[v_t(\boldsymbol{\theta})^2 \omega_t(\boldsymbol{\alpha})^{-1}].$$

Then by expanding $v_t(\boldsymbol{\theta})$ around $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$ and using the second half of Lemma A.2, we have

$$(A.20) \quad \begin{aligned} E[v_t(\boldsymbol{\alpha})] &= E[v_t(\boldsymbol{\alpha}^{(0)})] + (\boldsymbol{\alpha} - \boldsymbol{\alpha}^{(0)})' E\left[\frac{\partial \omega_t}{\partial \boldsymbol{\alpha}} \frac{\partial \omega_t}{\partial \boldsymbol{\alpha}'}\right] (\boldsymbol{\alpha} - \boldsymbol{\alpha}^{(0)}) \\ &\geq E[v_t(\boldsymbol{\alpha}^{(0)})^2]. \end{aligned}$$

under the assumptions we have made. The equality holds only at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$. Also we have an inequality

$$(A.21) \quad -\frac{1}{2} E[\log \omega_t(\boldsymbol{\alpha})] - \frac{1}{2} E[v_t(\boldsymbol{\theta})^2 \omega_t(\boldsymbol{\alpha})^{-1}] \leq -\frac{1}{2} E[\log \omega_t(\boldsymbol{\alpha}^{(0)})] - \frac{1}{2},$$

while the equality holds only at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$.

Sketch Proof of Lemma A.3 : Since the MA error is invertible, we have

$$(A.22) \quad v_t(\boldsymbol{\theta}) = v_t(\boldsymbol{\theta}_0) + \frac{\partial v_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \Big|_{\boldsymbol{\theta}^*} (\boldsymbol{\theta} - \boldsymbol{\theta}_0),$$

where $\boldsymbol{\theta}^*$ is a point between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$ in Θ and $\frac{\partial v_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} \Big|_{\boldsymbol{\theta}^*}$ is in \mathcal{F}_{t-1} . Also from (A.15), we have a representation

$$(A.23) \quad x_t = D(t) v_t(\boldsymbol{\theta}_0) + D(t) x_{t-1}^*,$$

where $x_t = \Delta y_t$ and x_{t-1}^* is in \mathcal{F}_{t-1} . Without loss of generality we consider the simple case of $p = q = 1$. Suppose we have a 1×3 vector $\lambda' = (\lambda_1, \lambda_2, \lambda_3)$ such that

$$(A.24) \quad \lambda' \begin{bmatrix} v_t \\ 1 \\ x_t \end{bmatrix} = 0 \text{ a.s..}$$

Then we can show that $\lambda_i \neq 0$ ($i = 1, 2, 3$). Since

$$(A.25) \quad x_t = \left(-\frac{\lambda_2}{\lambda_3}\right) + \left(-\frac{\lambda_1}{\lambda_3}\right)v_t \text{ a.s. ,}$$

we have a contradiction with (A.23) and (A.24). Thus we obtain (A.17). We can use the same argument in order to establish (A.18).

Step 2: Next, we consider the maximization problem of the criterion function $R(\theta)$ with respect to $(\gamma_2^*, \gamma_1, \dots, \gamma_p, c_1, \dots, c_q)$. By using the first half of Lemma 3.2 and similar arguments to Weiss (1986), $R(\theta)$ is uniquely maximized at

$$(\gamma_2^*, \gamma_1, \dots, \gamma_p, c_1, \dots, c_q) = (\gamma_2^{*(0)}, \gamma_1^{(0)}, \dots, \gamma_p^{(0)}, c_1^{(0)}, \dots, c_q^{(0)})$$

under the assumptions we have made. The rest of our argument with respect to σ_i ($i = 1, 2$) is the same as the one in Kunitomo and Sato (1996b). Hence we can prove that the probability limit of

$$(A.26) \quad Q_T(\theta) = -\frac{1}{2} \frac{1}{T} \sum_{t=2}^T \sum_{i=1}^2 I_t^{(i)} \log[D_i \omega_t(\alpha) D_i'] - \frac{1}{2} \frac{1}{T} \sum_{t=2}^T v_t(\theta)^2 \omega_t(\alpha)^{-1} .$$

is uniquely maximized at $\theta = \theta_0$. Also the probability convergence is uniform in $\theta_0 \subset \Theta$ with respect to θ . Then by applying Theorem 4.1.1 of Amemiya (1985), we obtain the consistency of $\hat{\theta}_{ML}$.

Step 3 : In order to prove the asymptotic normality of the ML estimator in the SSARIMA model under the assumptions we have made, the most important step is to use the martingale property of the partial derivatives of the log-likelihood function summarized in the following Lemma A.4.

Lemma A.4 : *Let θ be a vector of unknown parameters in the SSARMA model with ARCH(r) disturbances. Then we have*

$$(A.27) \quad E \left[\frac{\partial \log L_t(\theta)}{\partial \theta} | \mathcal{F}_{t-1} \right] = \frac{\partial \log L_{t-1}(\theta)}{\partial \theta},$$

where \mathcal{F}_{t-1} is the σ -field generated by $\{y_s, s \leq t-1; v_s, s \leq t-1\}$.

Sketch Proof of Lemma A.4 : For the parameters in θ except α_j ($j = 1, \dots, r$), we can obtain the result by slightly modifying the derivation of Lemma

A.4 of Kunitomo and Sato (1996b). For the ARCH parameters α_j ($j = 1, \dots, r$), we have

$$(A.28) \quad \frac{\partial \log L_T(\boldsymbol{\theta})}{\partial \alpha_j} = -\frac{1}{T} \sum_{t=1}^T \left(\frac{1}{2}\right) \omega_t^{-1} \frac{\partial \omega_t}{\partial \alpha_j} [v_t^2(\boldsymbol{\theta}) \omega_t^{-1} - 1].$$

It is straightforward to show that

$$T \times \frac{\partial \log L_T(\boldsymbol{\theta})}{\partial \alpha_j}$$

is a martingale.

Step 4 : The last step is to use the central limit theorem for martingales. (See Hall and Heyde (1980), or Anderson and Kunitomo (1992), for instance.) We can utilize the existence of unconditional moments $\sup_{t \geq 1} E[v_t^{4+\delta}] < \infty$ to show the conditional Lindeberg type condition. For instance, define a $1 \times q$ vector $\mathbf{Z}_t = (z_{jt}; j = 1, \dots, q)$ by

$$(A.29) \quad z_{jt} = \omega_t^{-1} v_{t-j}^2 [v_t^2 \omega_t^{-1} - 1].$$

Then we have

$$(A.30) \quad \frac{1}{T} \sum_{t=1}^T E[\|\mathbf{Z}_t\|^2 | \mathcal{F}_{t-1}]$$

converges to $E[\|\mathbf{Z}_t\|^2]$ in probability under the assumptions we have made. Also we have an inequality

$$(A.31) \quad \frac{1}{T} \sum_{t=1}^T E[\|\mathbf{Z}_t\|^2 I(\|\mathbf{Z}_t\| > \sqrt{T}c) | \mathcal{F}_{t-1}] \leq \frac{1}{T} \left(\frac{1}{c\sqrt{T}}\right)^\delta \sum_{t=1}^T E[\|\mathbf{Z}_t\|^{2+\delta} | \mathcal{F}_{t-1}]$$

for any positive constant c and some positive constant δ . The last term converges to zero in probability because its expectation is bounded by

$$(A.32) \quad \frac{1}{c^\delta} \left(\frac{1}{T}\right)^{\delta/2} \sup_{t \geq 1} E[\|\mathbf{Z}_t\|^{2+\delta}].$$

As the rest of the proof is similar to the arguments used in Sato and Kunitomo (1996), we omit the details. *Q.E.D.*

References

- [1] Akaike, H. (1973), "Information Theory and an Extension of the Maximum Likelihood Principle," in Petrov, B.N. and Csaki, F. eds., *Second International Symposium on Information Theory*, Akademia Kiado, Budapest, 267-281.
- [2] Amemiya, T. (1985), *Advanced Econometrics*, Basil-Blackwell.
- [3] Amihud, Y. and Mendelson, H. (1987), "Trading Mechanisms and Stock Returns : An Empirical Investigation", *Journal of Finance*, Vol.XLII, 3, 533-555.
- [4] Anderson, T.W. and Kunitomo, N. (1992), "Asymptotic Distributions of Regression and Autoregression Coefficients with Martingale Differences", *Journal of Multivariate Analysis*, 40, 221-243.
- [5] Bollerslev, T. (1986), "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, 307-327.
- [6] Brockwell, P.J. and Davis, R.A. (1991), *Time Series: Theory and Method*, 2nd Edition, Springer-Verlag.
- [7] Duffie, D. (1992), *Dynamic Asset Pricing Theory*, Princeton University Press.
- [8] Engle, R. (1982), "Autoregressive Conditional Heteroskedasticity with Estimates of the Variances of U.K. Inflation," *Econometrica*, 50, 987-1008.
- [9] Hall, P. and Heyde, C. (1980), *Martingale Limit Theory and its Applications*, Academic Press.
- [10] Hamilton, J. (1994), *Time Series Analysis*, Princeton University Press.
- [11] Harvey, A.C. and Shephard, N. (1993), "Estimation of an Asymmetric Stochastic Volatility Model for Asset Returns," Unpublished manuscript.
- [12] Kunitomo, N. and Sato, S. (1994), "A Non-linearity in Economic Time Series and Disequilibrium Econometric Models", in Takeuchi, K. and Takemura, A. (ed.), *Theory and Application of Mathematical Statistics*, University of Tokyo Press (in Japanese).
- [13] Kunitomo, N. and Sato, S. (1996a), "Asymmetry in Economic Time Series and Simultaneous Switching Autoregressive Model", *Structural Change and Economic Dynamics*, 7, 1-34.
- [14] Kunitomo, N. and Sato, S. (1996b), "Stationary and Non-stationary Simultaneous Switching Autoregressive Models with an Application to Financial Time Series," Discussion Paper No. 96-F-11, Faculty of Economics, University of Tokyo.

- [15] Liu, J. and Susko, E. (1992), "On Strict Stationarity and Ergodicity of a Non-linear ARMA Model", *Journal of Applied Probability*, **29**, 363-373.
- [16] Meyn, S.P. and Tweedie, R.L. (1993), *Markov Chain and Stochastic Stability*, Springer-Verlag.
- [17] Nelson, D.B. (1991), "Conditional Heteroskedasticity in Asset Returns: A New Approach," *Econometrica*, **59**, 2, 347-370.
- [18] Sato, S. and Kunitomo, N. (1996), "Some Properties of the Maximum Likelihood Estimator in Simultaneous Switching Autoregressive Model", *Journal of Time Series Analysis*, **17**, 287-307.
- [19] Tong, H. (1990), *Non-linear Time Series*, Oxford University Press, Oxford.
- [20] Tweedie, R.L. (1975), "Sufficient Conditions for Ergodicity and Recurrence of Markov Chains on a General State Space," *Stochastic Processes and their Applications*, **3**, 385-403.
- [21] Weiss, A.A. (1986), "Asymptotic Theory for ARCH Models: Estimation and Testing," *Econometric Theory*, **2**, 107-131.

Table 1: Nikkei 225 Index (Jan. 1, 1985 – Feb. 20, 1987)

(i) Modelling with daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4174.02	-4239.82	-4297.53	-4298.33
SSARIMA(2)	-4178.76	-4239.16	-4297.63	-4297.67
SSARIMA(3)	-4176.76	-4237.32	-4295.67	-4295.83
ARIMA(0)	-4143.68	-4213.05	-4268.96	-4270.93
ARIMA(1)	-4166.34	-4234.70	-4286.93	-4287.64
ARIMA(2)	-4170.94	-4234.17	-4286.98	-4287.19
ARIMA(3)	-4169.05	-4232.41	-4285.00	-4285.52

Selected Model: SSARIMA(1,0) – ARCH(3) AIC: -4298.33

	1	2	3	σ		α_1	α_2	α_3
a (up)	0.2676	–	–	0.0030	ARCH coef.	0.19	0.37	0.07
b (down)	0.0752	–	–	0.0038				

	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.
Daily effects	1	3.20	1.97	1.81	2.26	0.60

(ii) Modelling without daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4163.98	-4227.60	-4284.09	-4282.21
SSARIMA(2)	-4165.67	-4227.98	-4282.41	-4280.46
SSARIMA(3)	-4163.97	-4226.54	-4280.61	-4278.66
ARIMA(0)	-4135.46	-4207.97	-4255.02	-4253.02
ARIMA(1)	-4154.55	-4223.05	-4271.77	-4269.77
ARIMA(2)	-4155.80	-4224.08	-4270.08	-4268.08
ARIMA(3)	-4154.64	-4222.85	-4268.51	-4266.51

Selected Model: SSARIMA(1,0) – ARCH(2) AIC: -4284.09

	1	2	3	σ		α_1	α_2	α_3
a (up)	0.2719	–	–	0.0043	ARCH coef.	0.23	0.34	–
b (down)	0.0693	–	–	0.0056				

Table 1 (cont.): Nikkei 225 Index (Feb. 21, 1987 – Dec. 31, 1989)

(i) Modelling with daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4828.46	-5114.55	-5167.47	-5167.29
SSARIMA(2)	-4836.28	-5129.81	-5166.09	-5165.70
SSARIMA(3)	-4834.52	-5129.29	-5164.11	-5163.80
ARIMA(0)	-4741.86	-5096.03	-5151.75	-5154.42
ARIMA(1)	-4740.41	-5098.90	-5163.12	-5163.86
ARIMA(2)	-4762.00	-5102.56	-5161.77	-5162.18
ARIMA(3)	-4761.40	-5122.66	-5159.77	-5160.23

Selected Model: SSARIMA(1,0) – ARCH(2) AIC: -5167.47

	1	2	3	σ		α_1	α_2	α_3
a (up)	0.2083	–	–	0.0058	ARCH coef.	0.70	0.16	–
b (down)	0.0776	–	–	0.0068				

	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.
Daily effects	1	0.68	0.63	0.56	0.58	0.87

(ii) Modelling without daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4782.42	-5111.23	-5169.37	-5168.64
SSARIMA(2)	-4787.27	-5122.35	-5167.88	-5167.01
SSARIMA(3)	-4785.35	-5121.75	-5165.89	-5165.07
ARIMA(0)	-4658.77	-5071.68	-5149.73	-5152.83
ARIMA(1)	-4658.20	-5086.33	-5161.78	-5162.11
ARIMA(2)	-4666.46	-5091.56	-5160.47	-5160.42
ARIMA(3)	-4664.74	-5093.18	-5158.50	-5158.43

Selected Model: SSARIMA(1,0) – ARCH(2) AIC: -5169.37

	1	2	3	σ		α_1	α_2	α_3
a (up)	0.2177	–	–	0.0048	ARCH coef.	0.70	0.17	–
b (down)	0.0624	–	–	0.0057				

Table 1 (cont.): Nikkei 225 Index (Jan. 1, 1990 – Jun 11, 1992)

(i) Modelling with daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-3117.69	-3174.44	-3203.24	-3210.30
SSARIMA(2)	-3135.06	-3193.48	-3222.36	-3225.73
SSARIMA(3)	-3133.11	-3191.66	-3220.53	-3223.77
ARIMA(0)	-3115.75	-3171.58	-3201.35	-3209.02
ARIMA(1)	-3115.24	-3174.18	-3201.64	-3208.62
ARIMA(2)	-3132.40	-3192.57	-3222.68	-3225.92
ARIMA(3)	-3130.46	-3190.85	-3220.73	-3223.93

Selected Model: ARIMA(2,0) – ARCH(3) AIC: -3225.92

	1	2	3	σ	ARCH coef.	α_1	α_2	α_3
a (up)	0.0866	-0.2126	–	0.0153		0.21	0.28	0.09
b (down)	0.0866	-0.2126	–	0.0153				
	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.		
Daily effects	1	0.42	0.66	0.34	0.50	–		

(ii) Modelling without daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-3112.69	-3166.62	-3191.60	-3200.92
SSARIMA(2)	-3129.99	-3185.10	-3212.33	-3217.27
SSARIMA(3)	-3128.13	-3183.23	-3210.41	-3215.28
ARIMA(0)	-3110.39	-3162.32	-3189.49	-3199.39
ARIMA(1)	-3110.52	-3166.22	-3191.03	-3200.28
ARIMA(2)	-3127.91	-3184.49	-3213.30	-3218.28
ARIMA(3)	-3126.04	-3182.67	-3211.33	-3216.28

Selected Model: ARIMA(2,0) – ARCH(3) AIC: -3218.28

	1	2	3	σ	ARCH coef.	α_1	α_2	α_3
a (up)	0.1097	-0.2114	–	0.0113		0.24	0.26	0.12
b (down)	0.1097	-0.2114	–	0.0113				

Table 1 (cont.): Nikkei 225 Index (Jun 12, 1992 – Dec. 15, 1994)

(i) Modelling with daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-3541.45	-3565.11	-3591.16	-3604.67
SSARIMA(2)	-3539.49	-3563.46	-3598.44	-3605.82
SSARIMA(3)	-3539.28	-3562.02	-3597.07	-3603.82
ARIMA(0)	-3544.81	-3566.08	-3592.55	-3603.28
ARIMA(1)	-3543.32	-3565.25	-3591.39	-3604.31
ARIMA(2)	-3541.35	-3563.49	-3598.73	-3605.75
ARIMA(3)	-3541.10	-3561.97	-3597.33	-3603.77

Selected Model: SSARIMA(2,0) – ARCH(3)					AIC: <u>-3605.82</u>			
	1	2	3	σ		α_1	α_2	α_3
a (up)	-0.0920	0.1228	–	0.0138	ARCH coef.	0.10	0.28	0.12
b (down)	-0.0005	0.1125	–	0.0127				
	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.		
Daily effects	1	0.62	0.47	0.45	0.31	–		

(ii) Modelling without daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-3532.68	-3554.39	-3571.48	-3591.93
SSARIMA(2)	-3530.71	-3552.51	-3577.91	-3592.61
SSARIMA(3)	-3529.69	-3551.01	-3577.14	-3590.61
ARIMA(0)	-3536.31	-3554.74	-3572.29	-3589.20
ARIMA(1)	-3534.36	-3553.42	-3570.62	-3591.18
ARIMA(2)	-3532.38	-3551.49	-3577.72	-3591.97
ARIMA(3)	-3531.29	-3549.89	-3577.05	-3589.98

Selected Model: SSARIMA(2,0) – ARCH(3)					AIC: -3592.61			
	1	2	3	σ		α_1	α_2	α_3
a (up)	-0.1192	0.0950	–	0.0105	ARCH coef.	0.12	0.19	0.18
b (down)	-0.0153	0.0861	–	0.0095				

Table 2 : S & P 500 Index (Jan. 1, 1990 – Jun 11, 1992)

(i) Modelling with daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4034.48	-4033.51	-4043.41	-4043.50
SSARIMA(2)	-4033.53	-4032.35	-4042.65	-4042.56
SSARIMA(3)	-4031.83	-4030.65	-4040.89	-4041.11
ARIMA(0)	-4036.02	-4035.33	-4046.02	-4045.88
ARIMA(1)	-4036.10	-4035.28	-4045.17	-4045.38
ARIMA(2)	-4035.08	-4034.05	-4044.32	-4044.39
ARIMA(3)	-4033.42	-4032.37	-4042.60	-4042.99

Selected Model: ARIMA(0,0) – ARCH(2) AIC: -4046.02

	1	2	3	σ	ARCH coef.	α_1	α_2	α_3
a (up)	–	–	–	0.0094		0.03	0.11	–
b (down)	–	–	–	0.0094				

	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.
Daily effects	1	0.80	0.49	0.75	0.91	–

(ii) Modelling without daily effects

	AIC			
	ARCH(0)	ARCH(1)	ARCH(2)	ARCH(3)
SSARIMA(1)	-4033.19	-4033.18	-4039.60	-4040.22
SSARIMA(2)	-4032.07	-4031.72	-4038.50	-4038.95
SSARIMA(3)	-4030.49	-4030.01	-4036.78	-4037.53
ARIMA(0)	-4033.46	-4034.17	-4041.31	-4041.56
ARIMA(1)	-4034.32	-4034.62	-4040.84	-4041.74
ARIMA(2)	-4033.09	-4033.07	-4039.61	-4040.36
ARIMA(3)	-4031.58	-4031.39	-4037.94	-4039.04

Selected Model: ARIMA(1,0) – ARCH(3) AIC: -4041.74

	1	2	3	σ	ARCH coef.	α_1	α_2	α_3
a (up)	0.0640	–	–	0.0081		0.03	0.09	0.08
b (down)	0.0640	–	–	0.0081				