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Shinji Nishida

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ABSTRACT

Systemic risk management is a crucial issue for regulators. An extensive global information network has been widely set up, and the shock of defaults spreads instantaneously all over the world. This paper describes how a banking system is affected by the shock of defaults according to the mixture of well- or under-capitalized banks. The author has developed a two-dimensional model to simulate the "Domino Effect of Defaults" as the energy transfer of systemic shock through the banking system using the physical process of advection and diffusion. Finite difference approximations are used in the computation. The experimental results show; 1) if all banks are well-capitalized, the banking system is quite stable, 2) if all banks are under-capitalized, the banking system is very unstable, and 3) if both well- and under-capitalized banks are present, the stability of the banking system is conditional. The author examines the rationality of "Too Big to Fail," "Prompt Regulatory Actions" and "Least Cost Resolution" policies based on the simulation results.

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The "Domino Effect of Defaults" and its Implications for Regulatory Actions

- Development of a Quantitative Simulation Model for Systemic Shock Waves -

DRAFT

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Abstract

Systemic risk management is a crucial issue for regulators. An extensive global information network has been widely set up, and the shock of defaults spreads instantaneously all over the world. This paper describes how a banking system is affected by the shock of defaults according to the mixture of well- or under-capitalized banks. The author has developed a two-dimensional model to simulate the "Domino Effect of Defaults" as the energy transfer of systemic shock through the banking system using the physical process of advection and diffusion. Finite difference approximations are used in the computation. The experimental results show; 1) if all banks are well-capitalized, the banking system is quite stable, 2) if all banks are under-capitalized, the banking system is very unstable, and 3) if both well- and under-capitalized banks are present, the stability of the banking system is conditional. The author examines the rationality of "Too Big to Fail", "Prompt Regulatory Actions" and "Least Cost Resolution" policies based on the simulation results.

1. Introduction

Since the 80's, financial business has become deregulated rapidly, and the efficiency of banking business has improved dramatically. The deregulation enhanced the competition of banking business, and depositors and borrowers enjoy high and low interest rates' environment respectively. On the bankers' side, the competition forwarded by the deregulation makes it clear that who are the winners and who are the losers. Because the losers generate turbulence, such as defaults, on the banking system, regulators should be aware that this negative impact is the trade-off of the benefits of the deregulation.

In the 80's, the US regulators faced serious problems and experienced a chaotic situation, mainly caused by the defaults of S&Ls. The author points out that this is one of the negative impact of the deregulation that drove S&Ls into a corner, taking unmanageably large risk because of the competition against the banks. Based on this experienced, the US regulators designed a resolution scheme, and succeeded to legislate FDICIA in 1991. In other countries, regulators are now facing similar problems, although the seriousness might be different, and it will require to construct a resolution scheme, like FDICIA.

In terms of resolution scheme, regulators need to know a specific bank's default impact on the banking system. For example, the resolution cost by Pay-off can be estimated only if the banking system would not be affected by the default. So, if the Pay-off causes the "Domino Effect of Default", regulators should not take the action. As a consequence, regulators need to evaluate the magnitude of damage intensity of defaults on the banking system quantitatively. To achieve this goal, the total exposure and its sensitivity of each banks including off-balance contracts must be captured.

In this paper, the author introduces a simple quantitative model which emulates the "Domino Effect of Defaults" using physical process of advection and diffusion. Based on the simulation results, the author evaluates the rationality of "Too Big to Fail", "Prompt Regulatory Action", and "Least Cost Resolution" policies. The author believes that further progress of this type of research, together with the further disclosure of the banks, will rationally show that what type of regulatory action is a better scheme stabilizing the banking system and minimizing the resolution cost for a specific case of bank default.

2. The Systemic Shock Wave Model

In this paper, the author assumes the energy transfer of the systemic shock wave through the banking system can be described using the physical process of advection and diffusion as a model. The governing equations of the two-dimensional systemic shock wave model can be written as follows.

The momentum equations :

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial \xi}{\partial x}+g\frac{u\left(u^2+v^2\right)^{1/2}}{C^2H}=0 \qquad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial \xi}{\partial y}+g\frac{v\left(u^2+v^2\right)^{1/2}}{C^2H}=0 \qquad (2)$$

The continuity equation :

$$\frac{\partial \xi}{\partial t}+\frac{\partial (Hu)}{\partial x}+\frac{\partial (Hv)}{\partial y}=0 \qquad (3)$$

where u: velocity component in the x-direction
v: velocity component in the y-direction
ξ: energy surface elevation of systemic shock wave with respect to the datum plane
H: energy height of systemic shock wave
g : gravitational acceleration
C: coefficient of roughness
h : height between the datum plane and the bottom
t : time

These equations are basically used for the study of circulation and transport phenomena in estuaries, bays and harbors. The model solves depth average momentum and continuity equations in terms of velocity and water surface elevation.

Figure 1 shows an imaginary two-dimensional model of a banking system. As shown, this is a simplified grid system in the x-y plane. Figure 2 represents a schematic view of a systemic shock wave and a bank. The systemic shock wave is driven by the default of banks . If the shock wave exceeds the threshold of the bank, the total energy of the bank will generate a new shock wave and it will be released. Figure 3 explains the mechanism of the systemic shock wave transfer.

In this model, the distance between banks, the height of the threshold and the total energy of the bank are the critical parameters. The distance between banks represents the net exposure between two banks. If the distance is small, the net exposure is high. Because of the dumping of the energy, the effect of the systemic shock wave on the banks gradually decreases. The threshold and the total energy represent the capital and the total exposure of the banks respectively. If the threshold is high, the bank is well-capitalized. And if the total energy is high, the bank is large-scale, so the magnitude of the released energy is large.

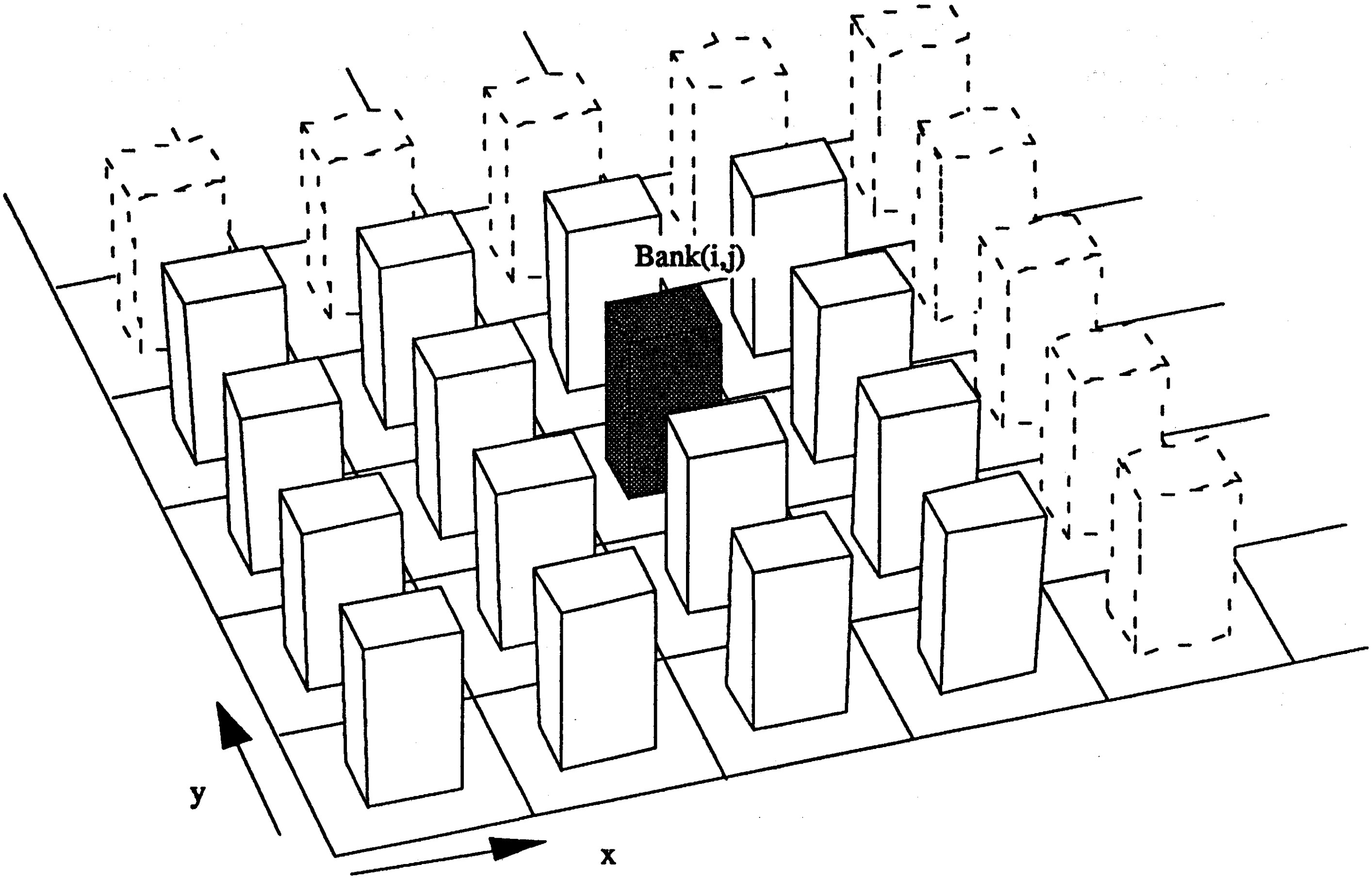


Figure 1. Two-Dimensional Model of a Banking System

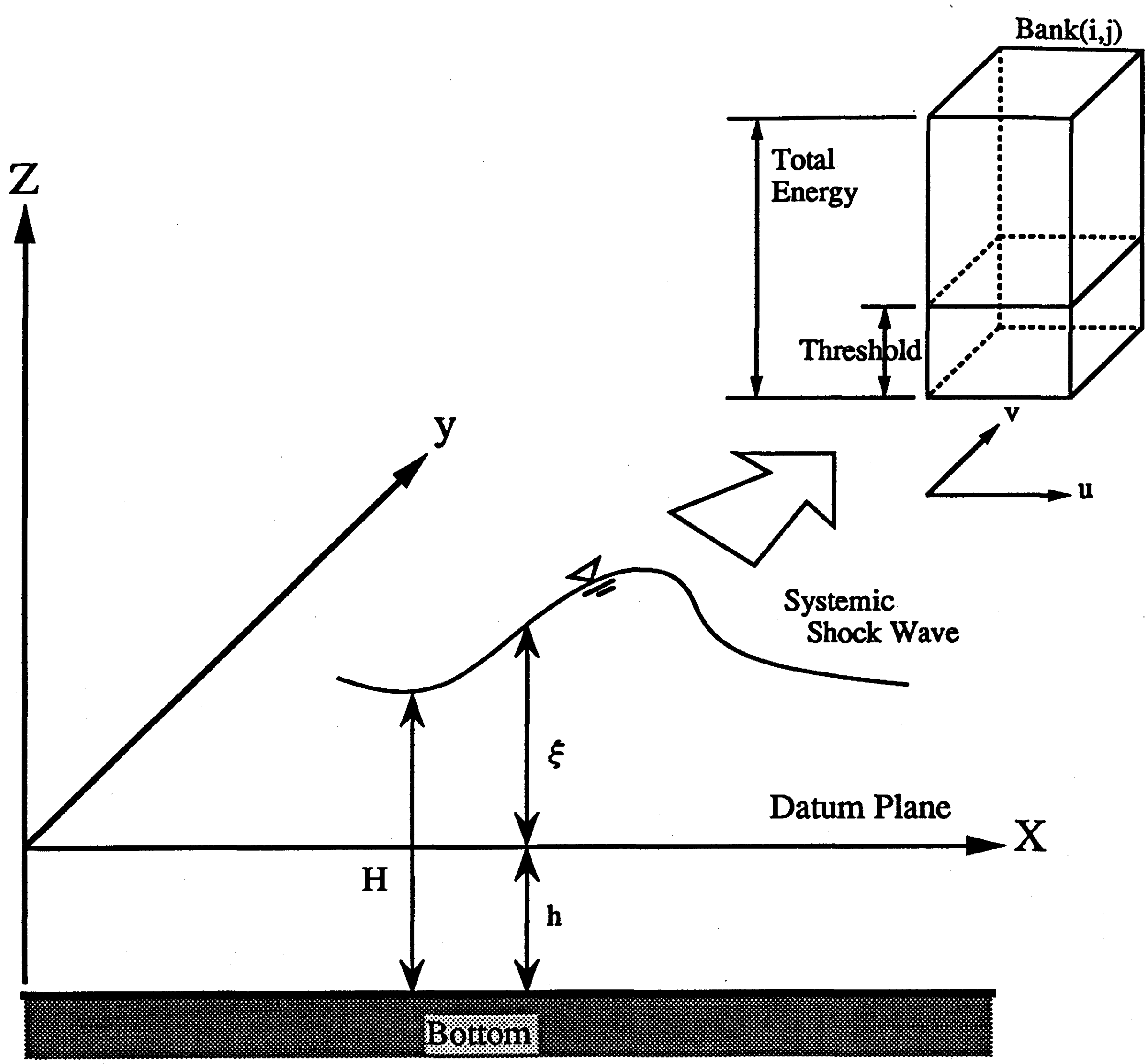
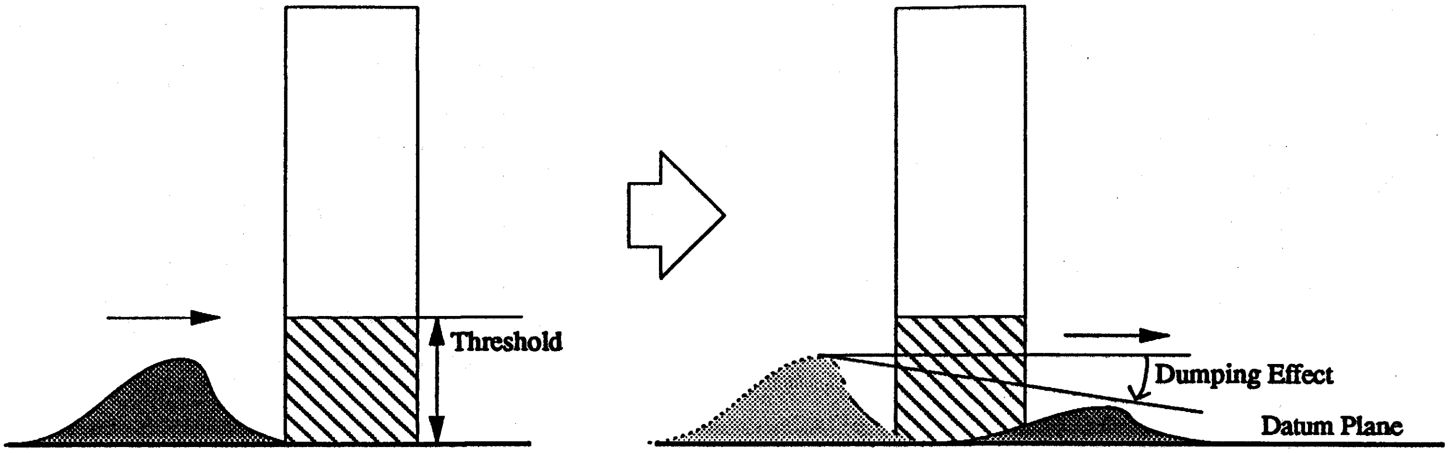
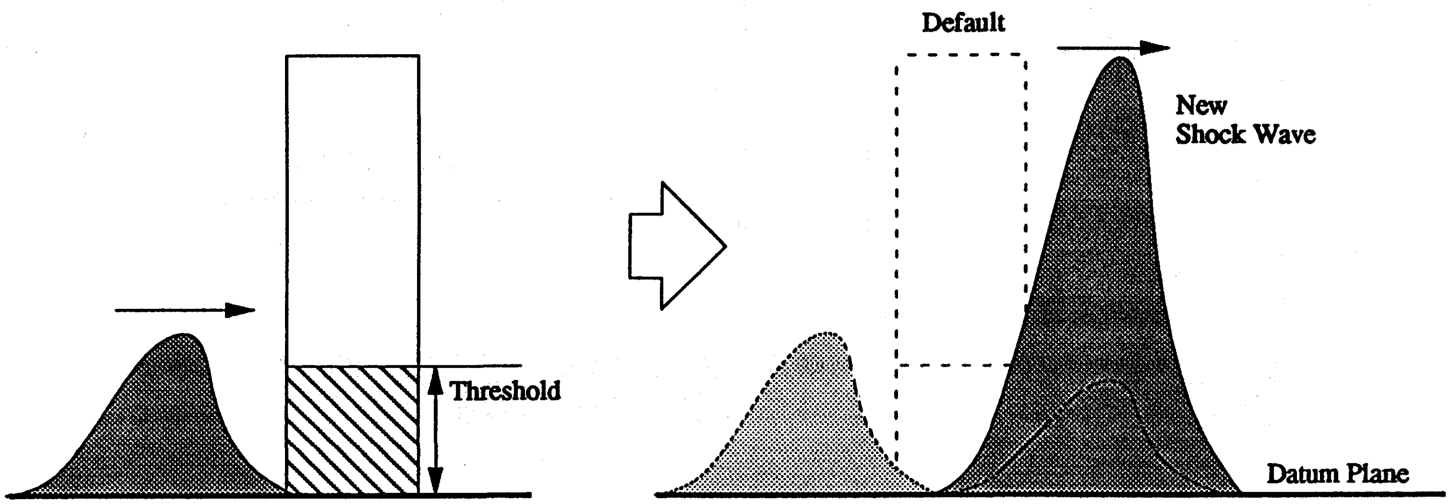


Figure 2. Schematic View of a Systemic Shock Wave and a Bank(i,j)



Case 1: Systemic Shock Wave ≦ Threshold



Case 2: Systemic Shock Wave > Threshold

Figure 3. Mechanism of Systemic Shock Wave Transfer

3. Finite Difference Equations

Finite difference approximation is a very useful method to solve complicated models. To simulate the energy transfer of the systemic shock wave through the banking system, the finite difference technique is almost the only technique because of the complicated boundary and initial conditions of the model.

The finite difference equations corresponding to the partial difference equations of (1), (2) and (3) are expressed in the Alternating Direction Implicit form, known as ADI, on a staggered grid in Figure 4. The formulas are known as central-difference approximations, and have the second order error terms on the assumption that each function is reasonably well approximated by an interpolating polynomial.

The method breaks the integration time step into two half time steps. For the first direction, velocity components in the x-direction u and energy surface elevation ξ are solved implicitly for every j-value from equations (1) and (3). The time steps of u and ξ are from n to n+1 and from n to n+1/2, respectively.

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n+1}\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2\Delta x}+\bar{v}_{i,j}^{n}\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2\Delta y}+g\frac{\xi_{i,j}^{n+1/2}-\xi_{i-1,j}^{n+1/2}}{\Delta x}$$
$$+g\frac{u_{i,j}^{n+1}\left\{(u_{i,j}^{n})^2+(\bar{v}_{i,j}^{n})^2\right\}^{1/2}}{C^2\bar{H}_{i,j}^{n}}=0 \qquad (4)$$

$$\frac{\xi_{i,j}^{n+1/2}-\xi_{i,j}^{n}}{\frac{1}{2}\Delta t}+\frac{\bar{H}_{i+1,j}^{n}u_{i+1,j}^{n+1}-\bar{H}_{i,j}^{n}u_{i,j}^{n+1}}{\Delta x}+\frac{\bar{H}_{i,j}^{\prime n}v_{i,j}^{n}-\bar{H}_{i,j-1}^{\prime n}v_{i,j-1}^{n}}{\Delta y}=0 \qquad (5)$$

where $\bar{v}_{i,j}^{n}=\frac{1}{4}(v_{i-1,j-1}^{n}+v_{i,j-1}^{n}+v_{i-1,j}^{n}+v_{i,j}^{n})$: average of v at $u_{i,j}$ node at time n

$\bar{H}_{i,j}^{n}=\frac{1}{2}(\xi_{i-1,j}^{n}+\xi_{i,j}^{n}+h_{i,j-1}+h_{i,j})$: average of H at $u_{i,j}$ node at time n

$\bar{H}_{i+1,j}^{n}=\frac{1}{2}(\xi_{i,j}^{n}+\xi_{i+1,j}^{n}+h_{i+1,j-1}+h_{i+1,j})$: average of H at $u_{i+1,j}$ node at time n

$\bar{H}_{i,j}^{\prime n}=\frac{1}{2}(\xi_{i,j}^{n}+\xi_{i,j+1}^{n}+h_{i,j}+h_{i+1,j})$: average of H at $v_{i,j}$ node at time n

$\bar{H}_{i,j-1}^{\prime n}=\frac{1}{2}(\xi_{i,j-1}^{n}+\xi_{i,j}^{n}+h_{i,j-1}+h_{i+1,j-1})$: average of H at $v_{i,j-1}$ node at time n

Unknown parameters are $u_{i,j}^{n+1}$, $u_{i+1,j}^{n+1}$, $\xi_{i,j-1}^{n+1/2}$ and $\xi_{i,j}^{n+1/2}$.

For the second direction, velocity component in the y-direction v and energy surface elevation ξ are solved implicitly for every i-value from equations (2) and (3). The time steps of v and ξ are from n to n+1 and from n+1/2 to n+1, respectively.

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}+\bar{u}_{i,j}^{n+1}\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2\Delta x}+v_{i,j}^{n+1}\frac{v_{i,j+1}^{n}-v_{i,j-1}^{n}}{2\Delta y}+g\frac{\xi_{i,j+1}^{n+1}-\xi_{i,j}^{n+1}}{\Delta y}+g\frac{v_{i,j}^{n+1}\left\{(\bar{u}_{i,j}^{n+1})^2+(v_{i,j}^{n})^2\right\}^{1/2}}{C^2\bar{H}_{i,j}^{\prime n+1/2}}=0 \tag{6}$$

$$\frac{\xi_{i,j}^{n+1}-\xi_{i,j}^{n+1/2}}{\frac{1}{2}\Delta t}+\frac{\bar{H}_{i+1,j}^{n+1/2}u_{i+1,j}^{n+1}-\bar{H}_{i,j}^{n+1/2}u_{i,j}^{n+1}}{\Delta x}+\frac{\bar{H}_{i,j}^{\prime n+1/2}v_{i,j}^{n+1}-\bar{H}_{i,j-1}^{\prime n+1/2}v_{i,j-1}^{n+1}}{\Delta y}=0 \tag{7}$$

where $\bar{u}_{i,j}^{n+1}=\frac{1}{4}(u_{i,j}^{n+1}+u_{i+1,j}^{n+1}+u_{i,j+1}^{n+1}+u_{i+1,j+1}^{n+1})$: average of u at $v_{i,j}$ node at time $n+1$

$\bar{H}_{i,j}^{\prime n+1/2}=\frac{1}{2}(\xi_{i,j}^{n+1/2}+\xi_{i,j+1}^{n+1/2}+h_{i,j}+h_{i+1,j})$: average of H at $v_{i,j}$ node at time $n+1/2$

$\bar{H}_{i,j-1}^{\prime n+1/2}=\frac{1}{2}(\xi_{i,j-1}^{n+1/2}+\xi_{i,j}^{n+1/2}+h_{i,j-1}+h_{i+1,j-1})$: average of H at $v_{i,j-1}$ node at time $n+1/2$

$\bar{H}_{i+1,j}^{n+1/2}=\frac{1}{2}(\xi_{i,j}^{n+1/2}+\xi_{i+1,j}^{n+1/2}+h_{i+1,j-1}+h_{i+1,j})$: average of H at $u_{i+1,j}$ node at time $n+1/2$

$\bar{H}_{i,j}^{n+1/2}=\frac{1}{2}(\xi_{i-1,j}^{n+1/2}+\xi_{i,j}^{n+1/2}+h_{i,j-1}+h_{i,j})$: average of H at $u_{i,j}$ node at time $n+1/2$

Unknown parameters are $v_{i,j}^{n+1}$, $v_{i,j-1}^{n+1}$, $\xi_{i,j+1}^{n+1}$ and $\xi_{i,j}^{n+1}$.

The scheme uses a "time splitting" technique to linearize the advection terms in the momentum equations. The finite difference equations form a system of linear algebraic equations of the form $\boldsymbol{Ax=b}$, where the $\boldsymbol{A}$ matrix is tridiagonal. This system is solved using a rapid Gauss Elimination Back Substitution routine specifically designed to take computational advantage of the structure of banded matrices.

Turbulent shear stress has been lumped into the bottom friction terms of the model's momentum equations using coefficient of roughness C. This is the only mechanism for dissipating the momentum of the energy.

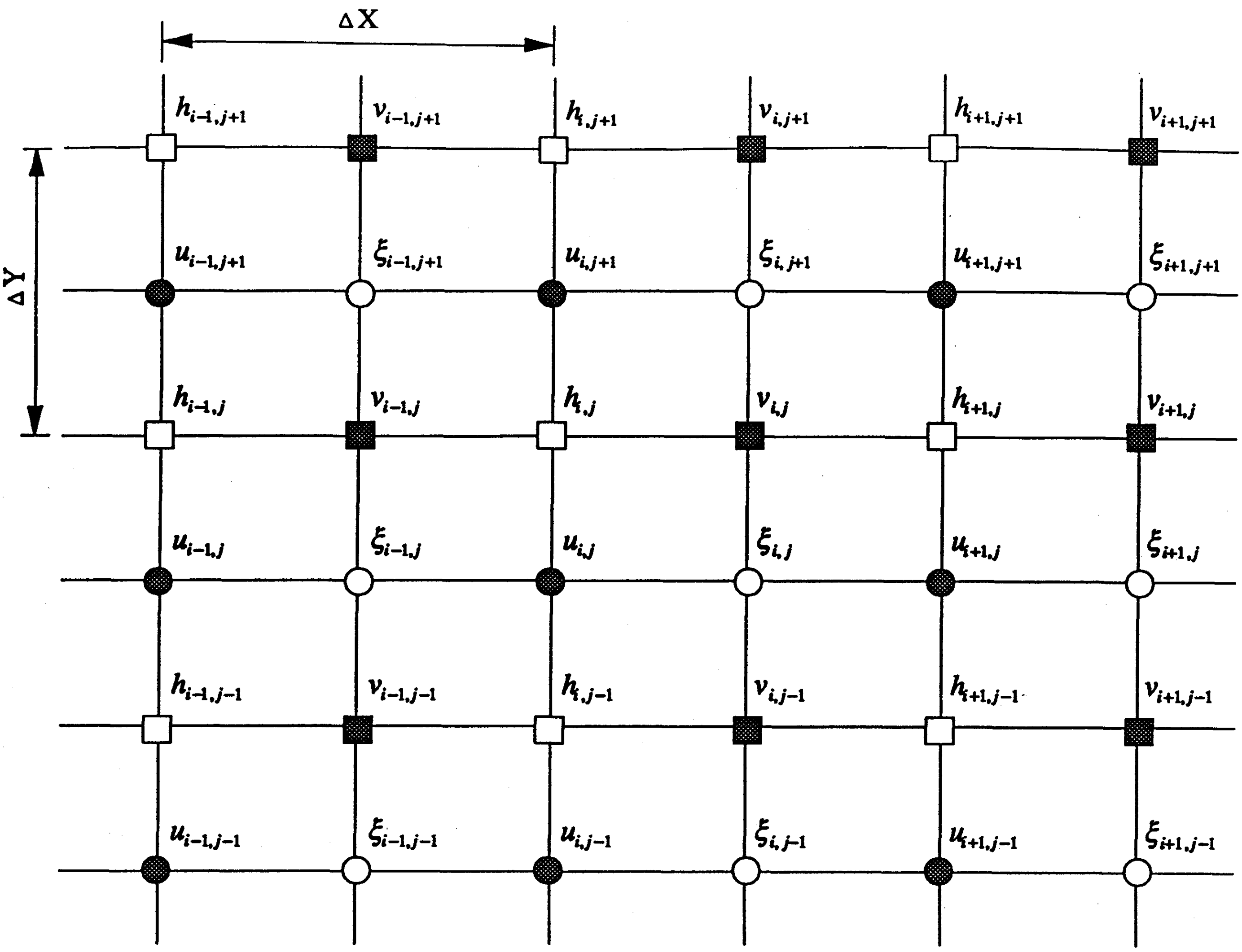


Figure 4. Staggered Finite Difference Grid

4. Simulation Results

In this paper, four types of hypothetical banks are considered, as shown in Table 1, and six cases of different conditions of the initial shock and mixture of the bank types are simulated, as shown in Table 2. In Cases 3 to 6, different types of banks are randomly distributed in the banking system. The 51 x 51 grid banking system was used, and the total number of banks is 2,601 in the x-y plane. In Cases 5 and 6, the zoned banking system was considered, as explained in Figure 5. For all the cases, the initial shock of default is given as occurring at the center of the banking system, and the parameters of equations (4), (5), (6) and (7) are given as follows; $\Delta t = 0.2\,\text{sec}$, $\Delta x = 500\,\text{ft}$, $\Delta y = 500\,\text{ft}$, $h = 10\,\text{ft}$, $g = 32.2\,\text{ft/sec}^2$ and $C = 60\,\text{ft}^{1/2}/\text{sec}$.

Cases 1 and 2 represent the situations that the banks are all well-capitalized and large-scale, all under-capitalized and large-scale, respectively. The stability of a banking system under different capitalization levels is examined by comparing these two cases.

In Cases 3 and 4, the mixture of well- and under- capitalized banks and large- and small-scale banks is the same but the magnitude of the initial shock is different. The stability of a banking system under different magnitudes of initial shock is examined by comparing these two cases.

In Cases 5 and 6, the zoned structure is used to stabilize the banking system. In these cases, the well-capitalized large-scale banks are mostly set up in the central zone, and the mixture of banks in the outer zone is the same as in Cases 3 and 4. The difference between Case 5 and 6 is the percentage of Type A and B Banks in the central zone.

The simulation results of the systemic shock wave and the defaulting banks after different period of time have elapsed are shown in Figures 6 to 8. In Cases 1 and 3, the height of the systemic shock wave dumps gradually after the initial default; therefore, the banking system is stable. In contrast, the systemic shock wave is transferred to the neighboring banks in Cases 2 and 4 by the mechanism explained in Figure 3; therefore, the banking system is unstable. This could be called as the "Domino Effect of Defaults". The conditions of the simulation differ between Cases 2 and 4 because of the different mixture of types of banks. Since the mixture of Case 4 is random, the shape of the systemic shock wave is not uniform. In Case 5, the banking system is stable because of the zoned structure. However, the banking system is not stable in Case 6 because its percentage of Type A Banks in the central zone is less than in Case 5.

Table 1. Height of Threshold and Total Energy

	Threshold	Total Energy	Note
Type A Banks	0.4 ft	1.0 ft	Well-Capitalized Large-Scale Banks
Type B Banks	0.2 ft	0.5 ft	Well-Capitalized Small-Scale Banks
Type C Banks	0.2 ft	1.0 ft	Under-Capitalized Large-Scale Banks
Type D Banks	0.1 ft	0.5 ft	Under-Capitalized Small-Scale Banks

Table 2. Conditions of Initial Shock and Mixture of Types of Banks

	Initial Shock	Zone	Type A Banks	Type B Banks	Type C Banks	Type D Banks
Case 1	1.0 ft	N.A.	100 %	0 %	0 %	0 %
Case 2	1.0 ft	N.A.	0 %	0 %	100 %	0 %
Case 3	0.5 ft	N.A.	50 %	5 %	40 %	5 %
Case 4	1.0 ft	N.A.	50 %	5 %	40 %	5 %
Case 5	1.0 ft	Central	80 %	20 %	0 %	0 %
		Outer	50 %	5 %	40 %	5 %
Case 6	1.0 ft	Central	70 %	30 %	0 %	0 %
		Outer	50 %	5 %	40 %	5 %

N.A. = Not Applicable

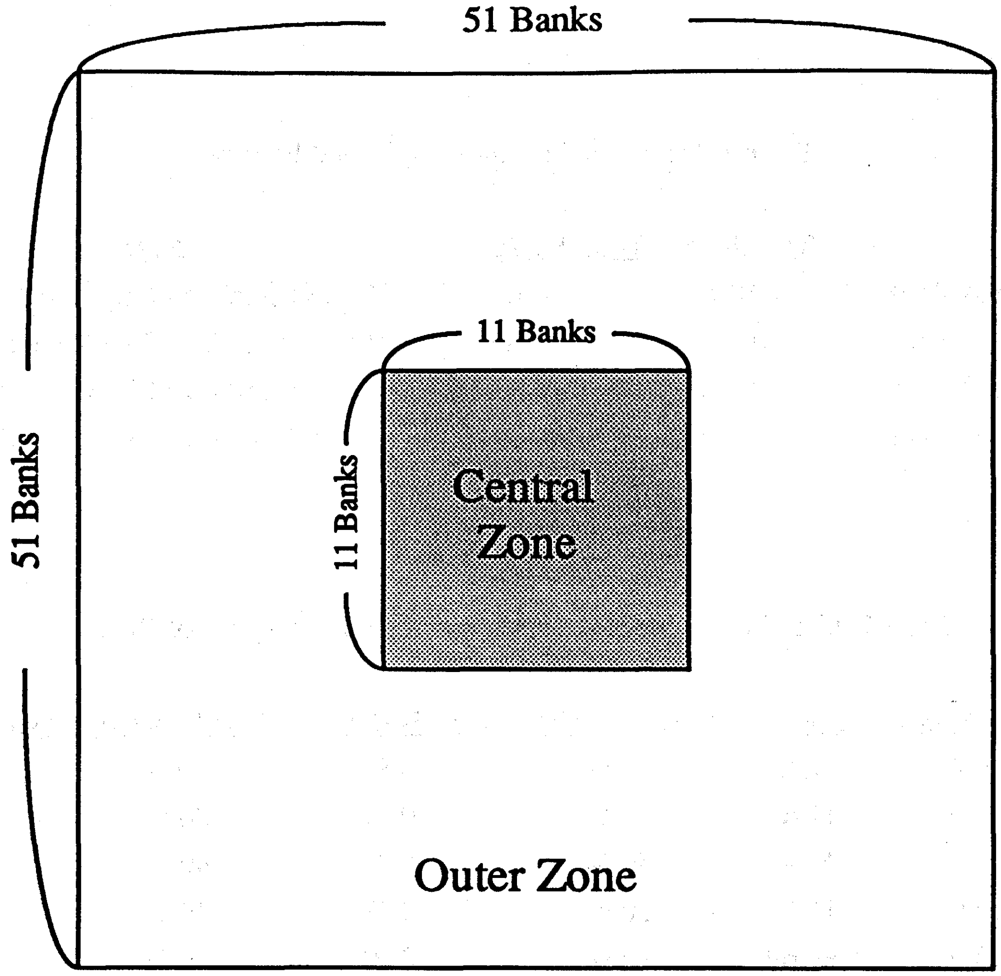


Figure 5. Zoned Banking System in Cases 5 and 6

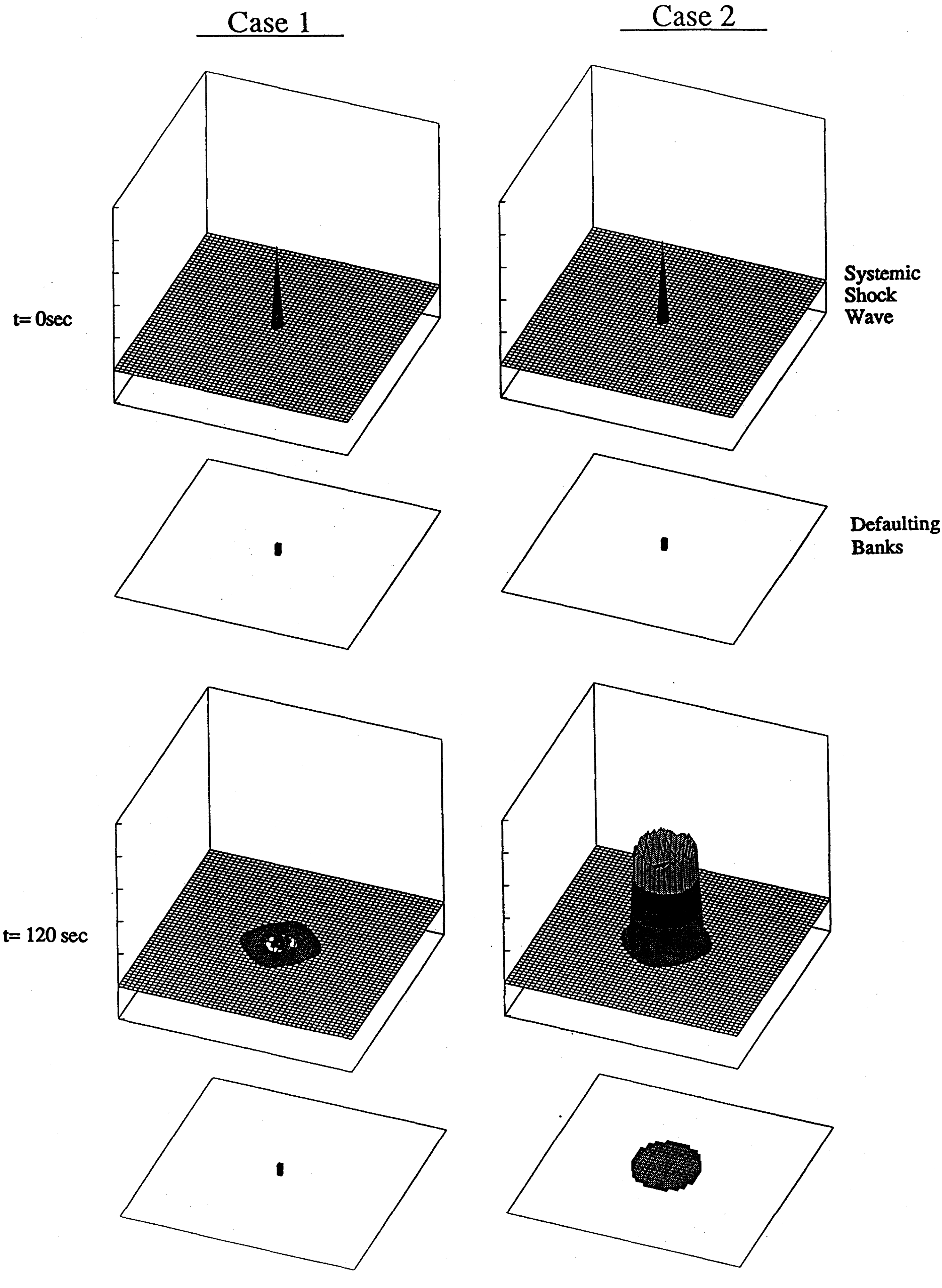


Figure 6 (1). Comparison of Simulation Results of Cases 1 and 2

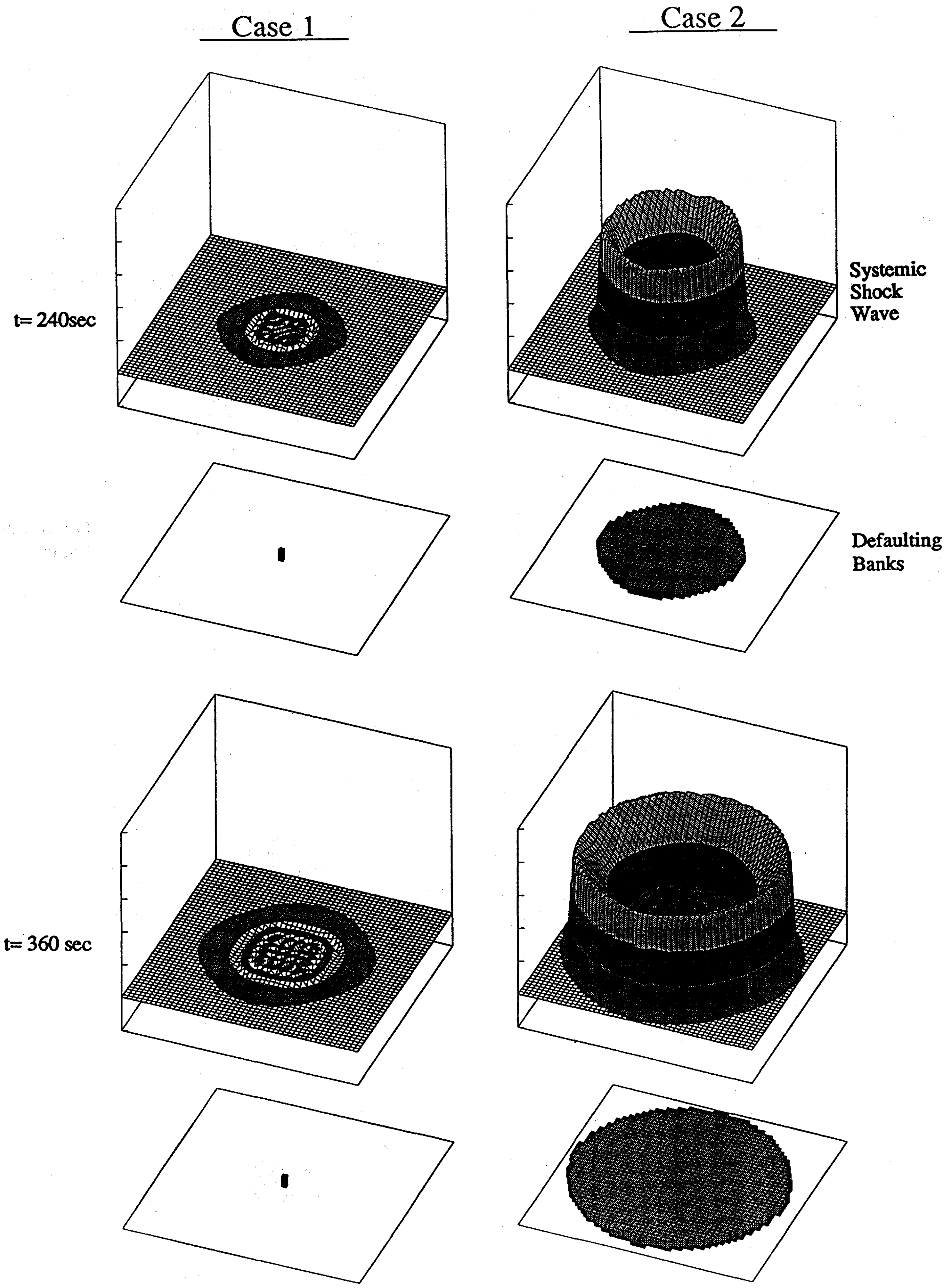


Figure 6 (2). Comparison of Simulation Results of Cases 1 and 2

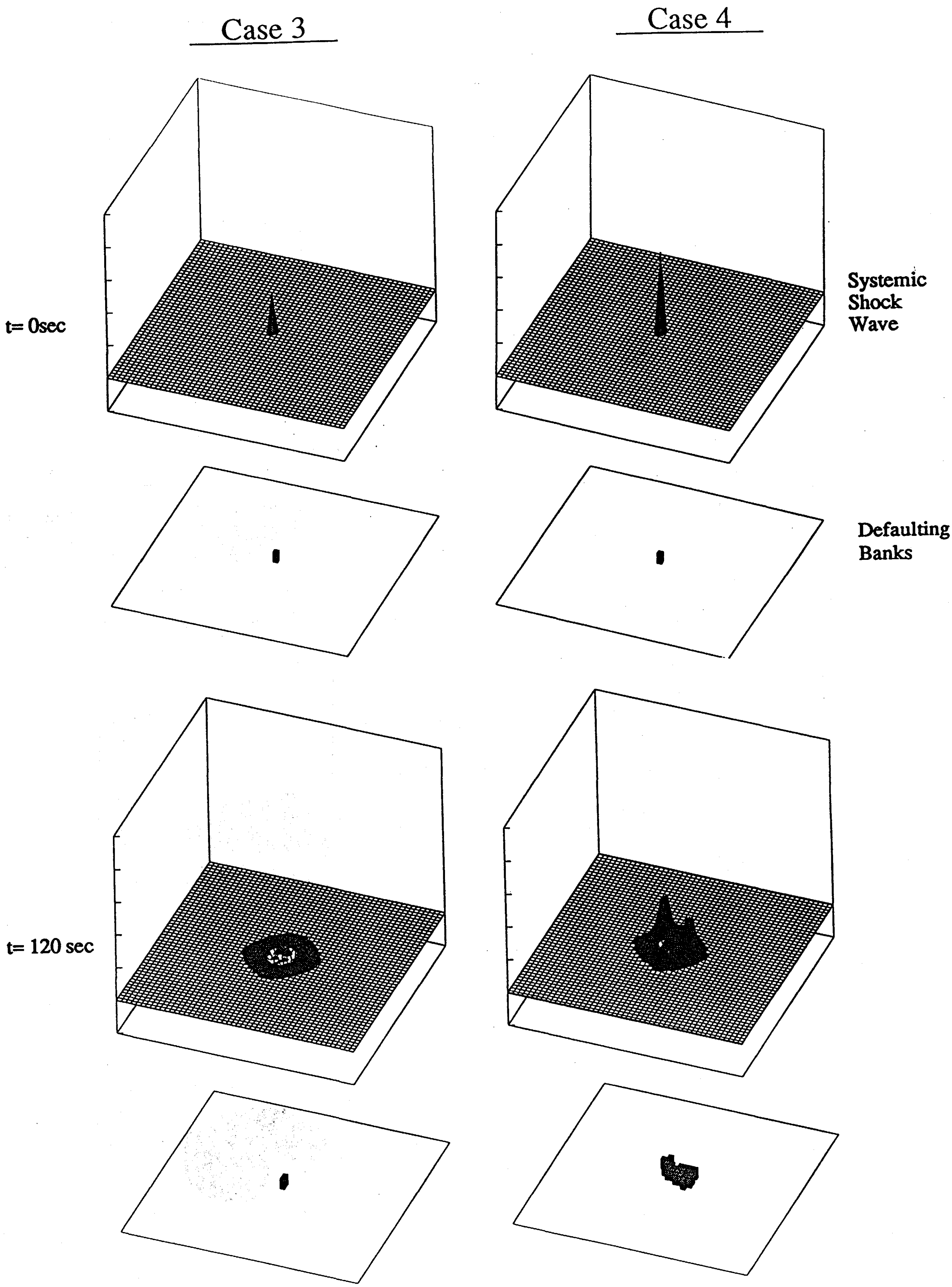


Figure 7 (1). Comparison of Simulation Results of Cases 3 and 4

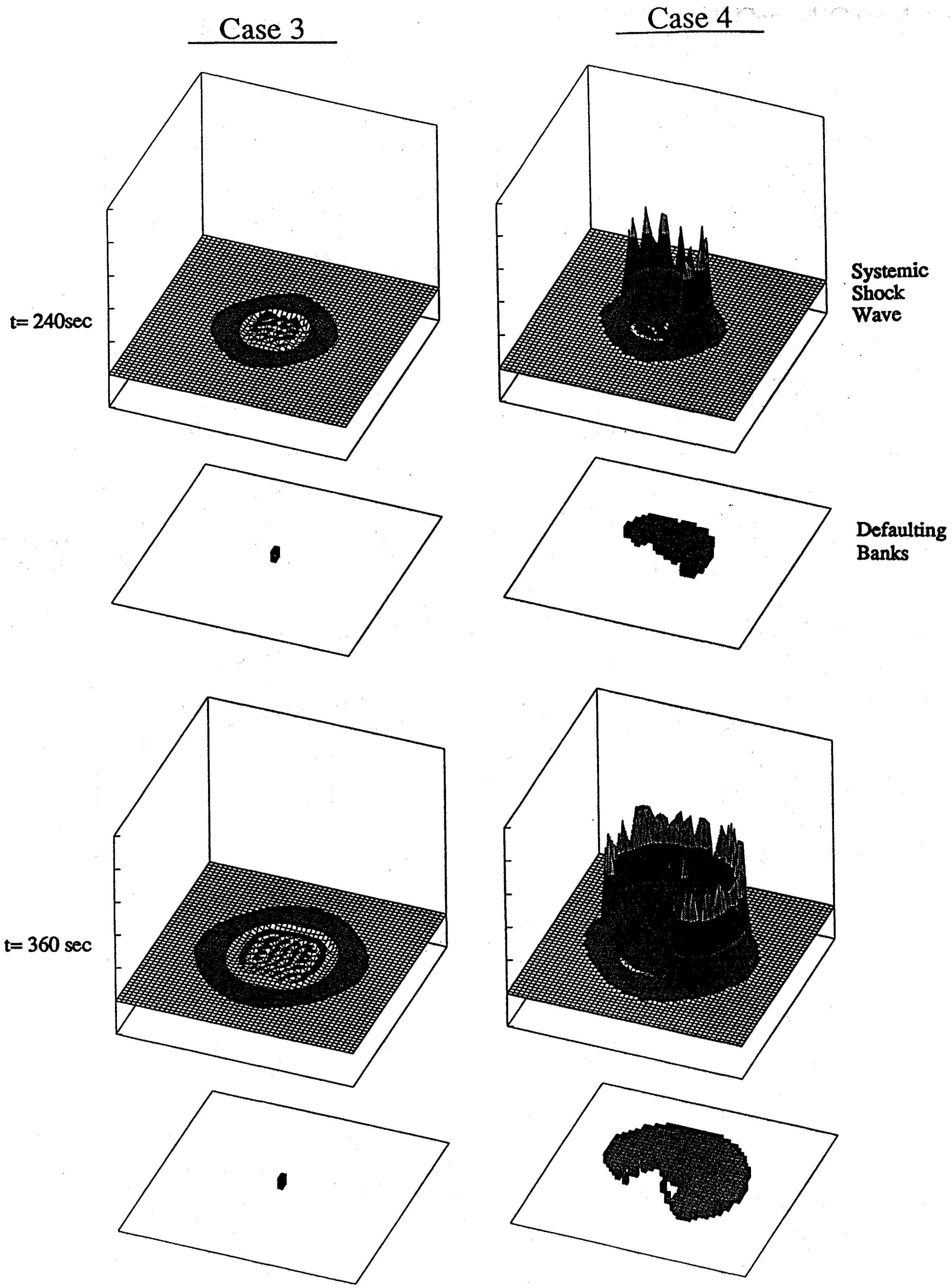


Figure 7 (2). Comparison of Simulation Results of Cases 3 and 4

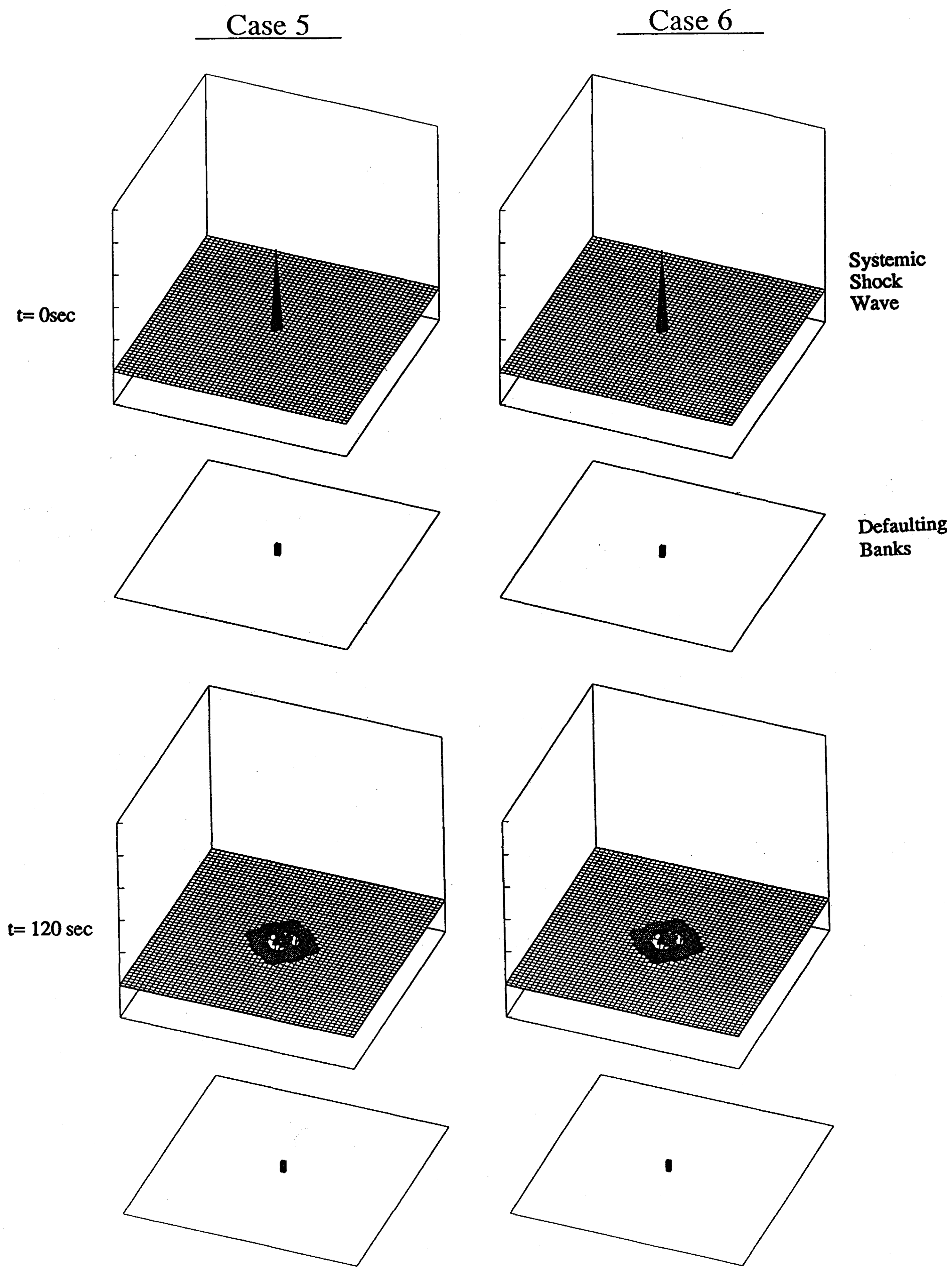


Figure 8 (1). Comparison of Simulation Results of Cases 5 and 6

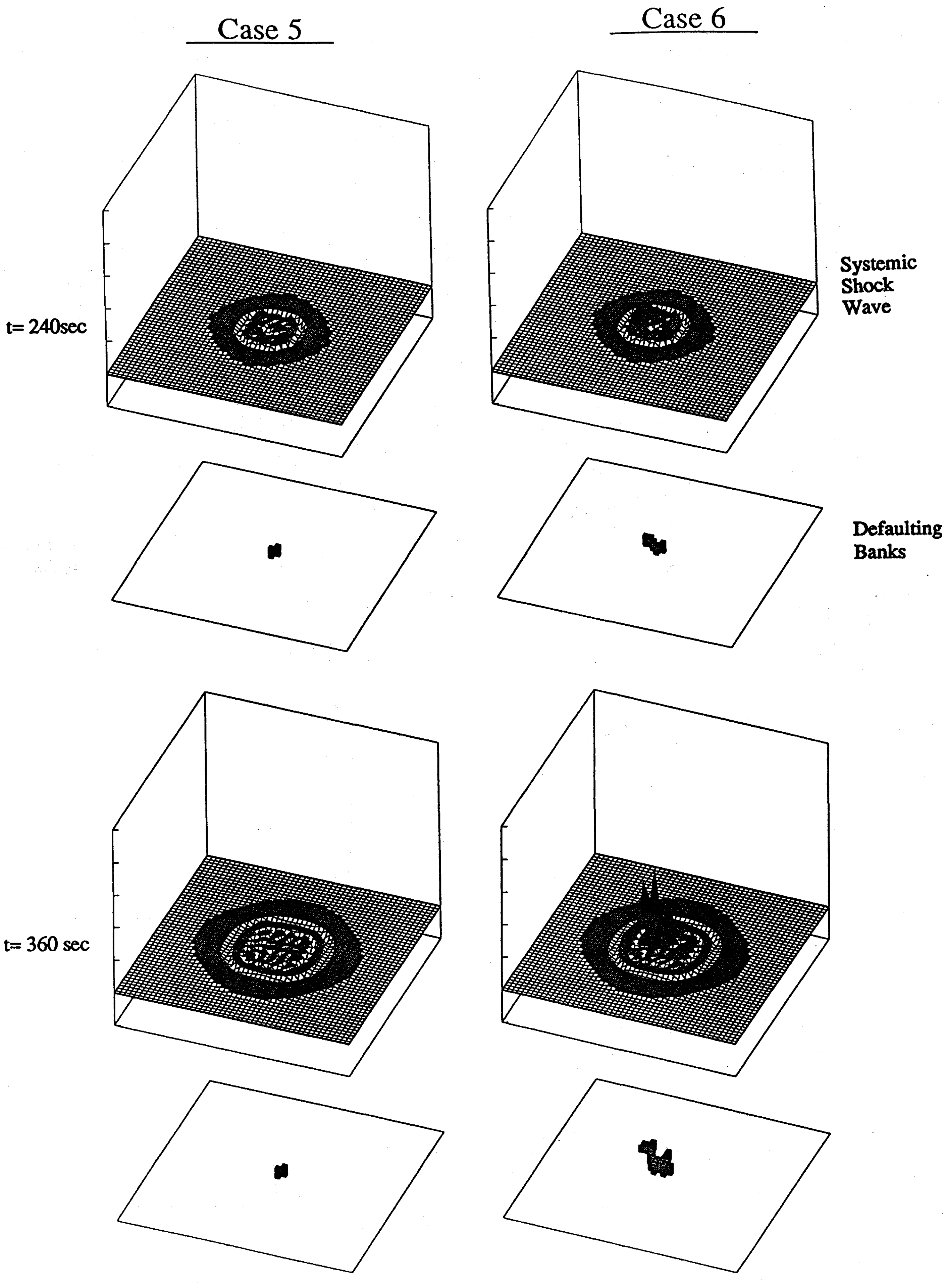


Figure 8 (2). Comparison of Simulation Results of Cases 5 and 6

5. Conclusions and Discussions

Before the legislation of FDICIA came into effect in 1991, "Too Big to Fail" was one of the most important policies to stabilize the banking system in the US. The simulation results of Cases 3 and 4 show how the banking system is affected by the magnitude of an initial shock. If the magnitude of the initial shock is large, the "Domino Effect of Defaults" will occur. Therefore, the regulators should take action using "Too Big to Fail" policy to rescue the banking system. This could be supported as a rational action based on the simulation results. However, the invocation of "Too Big to Fail" has been prohibited, in principle, since FDICIA legislation because the policy would decrease market discipline and cause a moral hazard. Since then, instead of "Too Big to Fail", "Prompt Regulatory Actions" has been used as the major regulatory action in the US.

The effect of "Prompt Regulatory Actions" could be interpreted as the zoned structure as shown in Figure 5. The idea of the policy is that if the bank does not have an adequate capital ratio, some of the management decisions and the business actions should be restricted. In terms of the model, if the bank does not have an adequate capital ratio, it should be set far away from the center of the two-dimensional banking system. The effectiveness of "Prompt Regulatory Actions" is clearly shown in the simulation results of Cases 5 and 6.

The author believes that "Too Big to Fail" is still a necessary policy even after the implementation of "Prompt Regulatory Actions". The reason is that without it, the banking system would be unstable if the magnitude of the initial shock is large. The simulation result of Case 6 shows that the "Domino Effect of Defaults" still occurs even with adoption of the zoned banking system. This is because the absolute value of the capital, which is treated as the threshold in the model, determines whether default occurs or not. Therefore, the resistance to default of Type B Banks, well-capitalized small-scale banks, is the same as Type C Banks, under-capitalized large-scale banks, and Type B Banks would be the trigger of the "Domino Effect of Defaults". So, if the percentage of well-capitalized large-scale banks is not high enough in the central zone, as in Case 6, the banking system is unstable.

To avoid the "Domino Effect of Defaults", the total exposure, which is treated as the total energy in the model, should be considered as the determining factor for "Prompt Regulatory Actions". This is because if the total exposure is large, the magnitude of the impact on the banking system is serious. Thus, the business of large-scale banks should be restricted more severely than that of small-scale banks. The idea of the new rule is that large-scale banks should have a higher capital ratio than small-scale banks. The effect of this type of rule is to ensure that most banks in the central zone are not only well-capitalized but also large-scale, as in Case 5, because only well-capitalized banks can be permitted to be large-scale. In other words, under-capitalized banks cannot be permitted to be large-scale, and cannot be permitted to be in the central zone. By doing this, the probability of occurrence of the "Domino Effect of Defaults" could be decreased. Until this type of rule is implemented, the regulators should consider invoking "Too Big to Fail" to stabilize the banking system.

The author is very clear that "Least Cost Resolution" without consideration of the "Domino Effect of Defaults" is a naive policy. The reason is obvious from the simulation results. If regulators neglect the "Domino Effect of Defaults", the calculated cost of resolution would not be guaranteed to be the minimum. Therefore, the invocation of "Least Cost Resolution" should be limited to small-scale banks.

Further research should be carried out into how this model could be applied to analyze the stability of the actual banking system. To achieve this goal, the theoretical parameter determination of the distance between banks, the height of the threshold and the total energy, and the posting of banks in the x-y plane are the crucial issues. Therefore, the further expansion and substantiality of disclosure of banks should be considered by the regulators to enhance the stability of the banking system. Especially, knowing the net exposure between banks and the total exposure of individual banks including off-balance contracts would be helpful for the progress of this type of research. For example, the VAR could be a valuable source for estimating these parameters.

Appendix: Boundary Conditions and Matrix Forms of ADI

The boundary conditions for the model can be provided either as the energy surface elevation or as the velocity of the systemic shock wave. For this simulation, the boundary condition should be provided as the energy surface elevation through the boundary being zero because the two-dimensional banking system should imaginary be open ended. Under the conditions, the systemic shock wave would pass through the boundary with velocity component.

The boundary conditions in the x-direction could be written as follows;

$$\begin{vmatrix} e_{u_{2,j}} & f_{\xi_{2,j}} & 0 & \cdots & \cdots & \cdots & 0 \\ d_{u_{2,j}} & e_{\xi_{2,j}} & f_{u_{3,j}} & 0 & \cdots & \cdots & 0 \\ 0 & d_{\xi_{2,j}} & e_{u_{3,j}} & f_{\xi_{3,j}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 0 & d_{\xi_{I-2,j}} & e_{u_{I-1,j}} & f_{\xi_{I-1,j}} & 0 \\ 0 & \cdots & \cdots & 0 & d_{u_{I-1,j}} & e_{\xi_{I-1,j}} & f_{u_{I,j}} \\ 0 & \cdots & \cdots & \cdots & 0 & d_{\xi_{I-1,j}} & e_{u_{I,j}} \end{vmatrix} \begin{vmatrix} u_{2,j}^{n+1} \\ \xi_{2,j}^{n+1/2} \\ u_{3,j}^{n+1} \\ \cdots \\ u_{I-1,j}^{n+1} \\ \xi_{I-1,j}^{n+1/2} \\ u_{I,j}^{n+1} \end{vmatrix} = \begin{vmatrix} b_{u_{2,j}} \\ b_{\xi_{2,j}} \\ b_{u_{3,j}} \\ \cdots \\ b_{u_{I-1,j}} \\ b_{\xi_{I-1,j}} \\ b_{u_{I,j}} \end{vmatrix} \qquad \text{(A1)}$$

where $d_{u_{i,j}} = -\frac{\overline{H}_{i,j}^n}{\Delta x}$: continuity at $\xi_{i,j}$ node

$e_{\xi_{i,j}} = \frac{2}{\Delta t}$: continuity at $\xi_{i,j}$ node

$f_{u_{i+1,j}} = \frac{\overline{H}_{i+1,j}^n}{\Delta x}$: continuity at $\xi_{i,j}$ node

$b_{\xi_{i,j}} = \frac{2\xi_{i,j}^n}{\Delta t} - \frac{\overline{H}_{i,j}'^n v_{i,j}^n - \overline{H}_{i,j-1}'^n v_{i,j-1}^n}{\Delta y}$: continuity at $\xi_{i,j}$ node

$d_{\xi_{i-1,j}} = -\frac{g}{\Delta x}$: momentum at $u_{i,j}$ node

$e_{u_{i,j}} = \frac{2}{\Delta t} + \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x} + g\frac{\left\{(u_{i,j}^n)^2 + (\overline{v}_{i,j}^n)^2\right\}^{1/2}}{C^2 \overline{H}_{i,j}^n}$: momentum at $u_{i,j}$ node

$f_{\xi_{i,j}} = \frac{g}{\Delta x}$: momentum at $u_{i,j}$ node

$b_{u_{i,j}} = \frac{2u_{i,j}^n}{\Delta t} - \overline{v}_{i,j}^n \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta y}$: momentum at $u_{i,j}$ node

The boundary conditions in the y-direction could be written as follows;

$$\begin{vmatrix} e_{v_{i,1}} & f_{\xi_{i,2}} & 0 & \cdots & \cdots & \cdots & 0 \\ d_{v_{i,1}} & e_{\xi_{i,2}} & f_{v_{i,2}} & 0 & \cdots & \cdots & 0 \\ 0 & d_{\xi_{i,2}} & e_{v_{i,2}} & f_{\xi_{i,3}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 0 & d_{\xi_{i,k-2}} & e_{v_{i,k-2}} & f_{\xi_{i,k-1}} & 0 \\ 0 & \cdots & \cdots & 0 & d_{v_{i,k-2}} & e_{\xi_{i,k-1}} & f_{v_{i,k-1}} \\ 0 & \cdots & \cdots & \cdots & 0 & d_{\xi_{i,k-1}} & e_{v_{i,k-1}} \end{vmatrix} \begin{vmatrix} v_{i,1}^{n+1} \\ \xi_{i,2}^{n+1} \\ v_{i,2}^{n+1} \\ \cdots \\ v_{i,k-2}^{n+1} \\ \xi_{i,k-1}^{n+1} \\ v_{i,k-1}^{n+1} \end{vmatrix} = \begin{vmatrix} b_{v_{i,1}} \\ b_{\xi_{i,2}} \\ b_{v_{i,2}} \\ \cdots \\ b_{v_{i,k-2}} \\ b_{\xi_{i,k-1}} \\ b_{v_{i,k-1}} \end{vmatrix} \qquad \text{(A2)}$$

where $d_{v_{i,j-1}} = -\dfrac{\overline{H}'^{n+1/2}_{i,j-1}}{\Delta y}$: continuity at $\xi_{i,j}$ node

$e_{\xi_{i,j}} = \dfrac{2}{\Delta t}$: continuity at $\xi_{i,j}$ node

$f_{v_{i,j}} = \dfrac{\overline{H}'^{n+1/2}_{i,j}}{\Delta y}$: continuity at $\xi_{i,j}$ node

$b_{\xi_{i,j}} = \dfrac{2\xi^{n+1/2}_{i,j}}{\Delta t} - \dfrac{\overline{H}^{n+1/2}_{i+1,j} u^{n+1}_{i+1,j} - \overline{H}^{n+1/2}_{i,j} u^{n+1}_{i,j}}{\Delta x}$: continuity at $\xi_{i,j}$ node

$d_{\xi_{i,j}} = -\dfrac{g}{\Delta y}$: momentum at $v_{i,j}$ node

$e_{v_{i,j}} = \dfrac{2}{\Delta t} + \dfrac{v^n_{i,j+1} - v^n_{i,j-1}}{2\Delta y} + g\dfrac{\left\{(\overline{u}^{n+1}_{i,j})^2 + (v^n_{i,j})^2\right\}^{1/2}}{C^2 \overline{H}'^{n+1/2}_{i,j}}$: momentum at $v_{i,j}$ node

$f_{\xi_{i,j+1}} = \dfrac{g}{\Delta y}$: momentum at $v_{i,j}$ node

$b_{v_{i,j}} = \dfrac{2v^n_{i,j}}{\Delta t} - \overline{u}^{n+1}_{i,j} \dfrac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x}$: momentum at $v_{i,j}$ node

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