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Abstract

The rise of digital money may bring about privately issued money that circulates across borders and coexists with public money. This paper uses an open-economy search model with multiple currencies to study the impact of such global money on monetary policy autonomy -- the capacity of central banks to set a policy instrument. I show that the circulation of global money can entail a loss of monetary policy autonomy, but it can be preserved if government policy that limits the amount or use of global money for transactions is introduced or if the global currency is subject to counterfeiting. The result suggests that global digital money and monetary policy autonomy can be compatible.

Keywords: Cryptocurrency; Monetary policy autonomy; Currency counterfeiting; Government transaction policy
JEL classification: D82, E4, E5, F31

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1 Introduction

Money and payment systems have evolved over time. In the ancient world, commodities such as grains were the primary form of money and payment. These were superceded by the use of precious metals such as gold, which in turn gave way to fiat money – currency and coin issued by the government but with no intrinsic value – along with deposits issued by private banks. Now a substantial change is underway. Although fiat money and bank deposits remain dominant in many countries, technological innovations have led to a rise in digital money all over the world. Global digital money, which can be used for transactions across borders, has yet to emerge, but it is technologically feasible and within the reach of the future economy.

The potential adoption of global digital money would bring substantial benefits, not least, inclusion and efficiency, but at the same time, it could pose challenges to the conduct of monetary policy. Specifically, should a global digital currency substantially advance and replace a national currency, the effectiveness of monetary policy in that economy would be weakened (G7 Working Group on Stablecoins, 2019; IMF Staff Report, 2020). Ikeda (2020) formalizes this concern by focusing on a unit of account as a role of money in a framework of new open economy macroeconomics (NOEM). Benigno et al. (2022) reinforce the concern by focusing on a medium of exchange as a role of money in a general setup that can be applied to many monetary models. They show that even in a benign scenario where the national currency and the global currency co-exist but do not necessarily dominate one another, monetary policy autonomy – the capacity of central banks to set a monetary policy instrument as they see fit – could be lost. This implication could pose a serious challenge for both policymakers and private digital money issuers.

In this paper, I revisit the issue of monetary policy autonomy in a possible future economy in which both the national currency and the global digital currency are valued and used for transaction. Specifically, I examine the following questions. Is a loss of monetary policy autonomy an inevitable outcome in such a future economy? If not, what mechanisms could overturn the result of Benigno et al. (2022) and restore monetary policy autonomy? What government policy, if any, could prevent a loss of monetary policy autonomy?

I tackle these questions by building a two-country open economy new monetarist model, modified to incorporate a global currency. The model has search and information frictions that make money essential for transactions. In the model, a global currency can be used in both home and foreign countries, but each national currency can be used only in the country of its origin. The interest of this paper is a global-money economy where both the national money and the global money are used for transaction in each country.

I use the model and study monetary policy autonomy in such a global-money economy. I show that if there is no additional friction, the economy features a loss of monetary policy autonomy, which confirms the result of Benigno et al. (2022). But, if additional frictions are introduced, it is not the only outcome. Such frictions include the combination of currency counterfeiting and the imperfect recognizability of currencies, and government policy that limits the type of currencies used for some transactions. Indeed, it only needs one of these frictions to exist for monetary policy autonomy to be restored: the central bank can set its monetary policy instrument within a vast area where both the national money and the global money are valued and used for transactions. Finally, in addition to government policy on transaction currencies, government policy that sets an upper limit on the amount of the global money used for a transaction can prevent a loss of monetary policy autonomy.

The result that monetary policy autonomy is lost in the global-money economy is an application of the nominal-exchange-rate indeterminacy result of Kareken and Wallace (1981). In the model with no additional friction (referred as the baseline model hereafter), an arbitrage-free condition between holding the national money and the global money is established in both the home and foreign countries. Thus, through holding the global money, the opportunity cost of holding the home money is tied to that of holding the foreign money. Hence, be it the risk-free interest rate or the money growth rate, the monetary policy instrument, which determines the opportunity cost of holding the national money, is tied between the two countries, resulting in monetary policy synchronization where a monetary instrument in one country is tied with that in another country. This situation is a loss of monetary policy autonomy. At the same time, the arbitrage-free condition implies the indeterminacy of the portfolio of real money balances and thus the indeterminacy of the nominal exchange rate, leading to the result of Kareken and Wallace (1981).

This observation suggests that restoring monetary policy autonomy comes down to resolving the indeterminacy result of Kareken and Wallace (1981). Following the literature, I consider two candidates:

- The combination of currency counterfeiting and imperfect recognizability of currencies (Rocheteau, 2011; Li and Rocheteau, 2011; Nosal and Rocheteau, 2017, Chapter 12).

- Government policy that limits the types of money used for transaction (Li and Wright, 1998; Zhang, 2014).

Currency and its counterfeiting are two sides of the same coin. Significant and continuous effort over the ages has been made to make fiat money and bank deposits less prone to counterfeiting and fraud. Although a global currency and its vulnerabilities have yet to be seen, its digital feature and wide-spread use in the world would almost certainly invite countless attacks of counterfeiting and fraud. For example, cryptocurrencies can be copied easily (i.e., it is subject to a double-spending problem), and it would be difficult for an individual to recognize 'true' cryptocurrencies.

Therefore, I introduce the counterfeiting of global digital currencies to the baseline model. Counterfeiting is a threat for sellers because of imperfect recognizability: they cannot distinguish between true digital currencies and counterfeits. Receiving counterfeits in exchange for goods is equivalent to giving goods for free. To prevent such an outcome, sellers restrict the amount of digital currencies for transaction, giving rise to an endogenous liquidity constraint. This is because while buyers can counterfeit digital currencies, doing so is costly. And when the amount of digital currencies that can be accepted for transaction is limited by the liquidity constraint, the cost of counterfeiting outweighs the benefits of producing and using counterfeits for buyers. Hence, in such a situation, buyers would not engage in counterfeiting. When the limit of the use of digital currencies is not so high, the liquidity constraint becomes binding and the portfolio of national money and digital money is pinned down uniquely, giving rise to determinate demand for the two kinds of money, and thus restoring monetary policy autonomy in the global-money economy. While these results are obtained by adding a simple model of counterfeiting (Nosal and Rocheteau, 2017, Chapter 12) to the baseline model, I show that the same results also hold by introducing a model of cryptocurrency with a proof-of-work protocol, such as Bitcoin, studied by Chiu and Koeppl (2019). The corollary of the results of counterfeiting is that government policy that restricts the amount of digital money used for a transaction can also preserve monetary policy autonomy.

Government engages in spending and collecting taxes. The currency with which it can make and receive payments is a choice of the government. In practice, in transactions with the government, the permitted currency is restricted to the domestic currency in most countries. I formalize this idea by introducing government agents who can accept only the domestic currency for transactions in the baseline model. These sellers can also be interpreted as behavioral sellers who do not want to use the global money for behavioral reasons. The presence of these sellers makes the national money and the global digital money imperfect substitutes and increases the liquidity services obtained by holding the national money. The amount of the national money demanded by private agents is adjusted to equate the liquidity service to the cost of holding that national money, which in turn is determined by monetary policy. In this way, the portfolio of real balances and money demand are uniquely determined, resolving the indeterminacy result of Kareken and Wallace (1981) and restoring monetary policy autonomy.

Adding frictions, including government policy, to break an indeterminacy or irrelevance result has tradition in macroeconomics and finance. Modigliani and Miller (1958) clarify conditions under which a firm's capital structure becomes irrelevant for its value, and the vast literature that followed has explored what types of frictions would overturn the irrelevance result. In the context of monetary policy, Wallace (1981) builds on the Ricardian equivalence result emphasized by Barro (1974) and shows the irrelevance of open market operations, and Eggertsson and Woodford (2003) the irrelevance of quantitative easing. To resolve irrelevance results, the literature has added frictions such as imperfect asset substitution (Andrés et al., 2004) and the preferred-habitat theory of term structure (Vayanos and Vila, 2021).[1] As discussed above, the lineage of this paper starts from the irrelevance result of Kareken and Wallace (1981) and aims at resolving the loss of monetary policy autonomy shown by Benigno et al. (2022). By adding frictions to money holdings and restoring monetary policy autonomy, this paper re-affirms the trilemma of the global-money economy – the impossibility of achieving monetary policy autonomy (independence), free capital flow, and free exchange rates at the same time, argued by Benigno et al. (2022).

Related literature This paper contributes to the emerging literature on cryptocurrency and its implications for monetary policy. From the perspective of open economies, in addition to Benigno et al. (2022) and Ikeda (2020), this paper is also related to Baughman and Flemming (2020) who study global demand for a basket-based stablecoin in a two-country open economy search model. Uhlig and Xie (2020) build a small-open economy model with global digital currencies in the NOEM framework and study the role of shocks that arise from the indeterminacy of exchange rates between any pair of global currencies. Ferrari et al. (2022) examine the issuance of a central bank digital currency (CBDC) in the NOEM framework and argue that it amplifies the international spillovers of shocks. In the context of a closed economy, papers that study currency competition with or among cryptocurrencies and its implications for monetary policy include Garratt and Wallace

[1] For the discussion of the (ir)relevance of unconventional monetary policy, see e.g. Gertler and Karadi (2013) and Christiano and Ikeda (2013).

(2018), Fernández-Villaverde and Sanches (2019), Schilling and Uhlig (2019), Zhu and Hendry (2019), and Saito (2020). For the microeconomics of cryptocurrencies, see the recent survey by Halaburda et al. (2021).

Broadly, this paper builds upon the search-theoretic literature on dual currencies. The earliest works on dual currencies include Kiyotaki and Wright (1989) for a closed economy and Matsuyama et al. (1993) for an open economy. Craig and Waller (2000) surveys the literature in the 1990s. Lagos et al. (2017) summarizes the development of a search-theoretic model. The basic framework of the models used in this paper is the two-country open economy model of Zhang (2014), which itself extends the closed-economy model of Lagos and Wright (2005). This paper introduces a global currency, which can be used in both countries, and derives implications for monetary policy autonomy.

Resting on the literature on nominal exchange rate indeterminacy that starts from Kareken and Wallace (1981), to overturn the result of a loss of monetary policy autonomy, in this paper I add currency counterfeiting and the imperfect recognizability of currencies in a similar spirit to Rocheteau (2011), Li and Rocheteau (2011), Nosal and Rocheteau (2017, Chapter 12), and Gomis-Porqueras et al. (2017). This leads to an endogenous liquidity constraint, which has the same form as a no-double-spending constraint, derived by Chiu and Koeppl (2019), for a cryptocurrency with a proof-of-work protocol. As another version of the model, I introduce government policy that limits the type of money used for transaction as studied by Li and Wright (1998) and Zhang (2014).

The rest of the paper is organized as follows. Section 2 presents the main idea about a global currency and monetary policy autonomy in a simple framework. Section 3 articulates the ideas by building two-country open economy search models with a global currency. Section 4 concludes with some policy implications for monetary policy autonomy.

2 Main Ideas in a Simple Framework

This section uses a simple framework to lay out the main idea about the global-money economy and the consequences for monetary policy. Section 2.1 describes the common environment used in the paper. Section 2.2 shows a loss of monetary policy autonomy in the global-money economy and clarifies the key assumptions that lead to such a result. Section 2.3 introduces different assumptions and shows that monetary policy autonomy is preserved in the global-money economy.

2.1 Environment

Time is discrete and continues forever. The economy consists of two countries, home and foreign, denoted as h and f, respectively. There are three currencies: a home currency, a foreign currency, and a global digital currency, which are denoted as currency h, f, g, respectively. In each country $i \in \{h, f\}$, the national money M_i denominated in currency i is supplied by the central bank of country i. The supply of the global money M_g is exogenously given. For simplicity, I assume that there is no interest payment on any currency. There is no aggregate uncertainty and thus the economy is deterministic.

There is a numeraire good and the price of currency $l \in \{h, f, g\}$ in units of the numeraire good is denoted as $\phi_l \equiv 1/P_l$, where P_l is the price of the numeraire good in units of currency l. Both countries face the same prices, so that the law of one price holds for the numeraire good. The nominal exchange rate of currency f in units of currency h is given by

$$\mathcal{E} = \frac{\phi_f}{\phi_h} = \frac{P_h}{P_f} \tag{1}$$

For example, an increase in $\mathcal{E}$ means depreciation of currency h.

Since there is no aggregate uncertainty, in this paper I focus on a stationary equilibrium where aggregate real balances are constant in each country. In such an economy, the money growth rate and the nominal interest rate on risk-free bonds are equivalent as a monetary policy instrument. Without loss of generality, the central bank sets the money growth rate, $\gamma_i = M_i'/M_i$, in each country, where a variable with superscript $'$ indicates the variable in the next period. The growth rate of the global money is denoted as $\gamma_g = M_g'/M_g$.

The situation of interest is such that both domestic and global currencies are valued and held in each country: $\phi_l > 0$ for all $l \in \{h, f, g\}$. In this paper I exclusively focus on such a global-money economy. In addition, to shed light on the role of global money, I follow Benigno et al. (2022) and consider a situation in which the national money can be used for transaction only in the country of its origin, while the global money can be used in both countries.[2]

In this environment, the question is whether the global-money economy restricts the conduct of monetary policy. Specifically, I am interested in whether a central bank continues to have autonomy in setting a monetary policy instrument in the global-money economy.

[2] If a national currency can be used in both countries, such a currency is regarded as an international currency. See Zhang (2014) and Gomis-Porqueras et al. (2017) for the analysis of an international currency in open economy new monetarist models.

To clarify the status of monetary policy, let me introduce the following two definitions.

Definition 1 : Monetary policy synchronization. *Monetary policy synchronization is a situation in which the setting of a monetary policy instrument in each country is tied to the setting of a monetary policy instrument in the other country.*

Definition 2 : Monetary policy autonomy. *Monetary policy autonomy is a situation in which the setting of a monetary policy instrument in each country is independent of the setting of a monetary policy instrument in the other country.*

Under *monetary policy synchronization*, the setting of a monetary policy instrument in one country is restricted to a certain value that depends on the setting of a monetary policy instrument in the other country. A special case is the equalization of the setting of a monetary policy instrument between the two countries, as studied by Benigno et al. (2022). Under *monetary policy autonomy*, the central bank can set a monetary policy instrument as it sees fit in each country. In the economy without the global currency, monetary policy autonomy holds as in the standard two-country monetary model. The primary focus of this paper is whether monetary policy synchronization is an inevitable outcome in the global-money economy, or if monetary policy autonomy can be preserved.

2.2 Monetary Policy Synchronization

In this subsection, I argue that monetary policy synchronization arises in the global-money economy with no restriction on the portfolio choice of money. In country h, a representative individual holds both currencies h and g and has real balances of $\phi_h m_h^h + \phi_g m_h^g$, where m_h^j denotes money denominated in currency j (money j for short hereafter) held by the individual in country h. Since both currencies are valued and held, and since there is no restriction on holding both kinds of money, the individual is indifferent about holding money h or g. Although the liquidity service provided by money is an endogenous object, for the sake of exposition in this subsection only, let me assume that the liquidity service is identical between money h and g.[3] Then, the return of holding money h, ϕ_h'/ϕ_h, has to be equated to that of holding money g, ϕ_g'/ϕ_g. Since this arbitrage prevails in country f as well, the returns of holding money h and f are equalized through the holding of the global

[3]In Section 3.2, I will show that an identical liquidity service can be obtained as an equilibrium outcome.

money in both countries:

$$\frac{\phi'_h}{\phi_h} = \frac{\phi'_g}{\phi_g} = \frac{\phi'_f}{\phi_f} \tag{2}$$

In a stationary equilibrium, aggregate real balances are constant so that the the return of holding money $l \in \{h, f, g\}$ is equal to the inverse of the money growth rate γ_l. Hence, equation (2) implies that the money growth rates are equalized between the two countries:

$$\gamma_h = \gamma_g = \gamma_f \tag{3}$$

Since equation (2) constitutes an equilibrium condition, in any stationary equilibrium in which the real balances of each currency is constant and positive over time, equation (2) has to hold and so does equation (3). Equation (3) shows a specific form of monetary policy synchronization, where the setting of the monetary policy instrument is equalized between the two countries.

Equation (2) is an application of Kareken and Wallace (1981) to the open economy with the global currency. Actually, equations (1) and (2) imply a constant nominal exchange rate:

$$\mathcal{E} = \frac{\phi_f}{\phi_h} = \frac{\phi'_f}{\phi'_h} = \mathcal{E}' \tag{4}$$

Hence, in this economy, monetary policy synchronization arises in the form of equation (3) and at the same time the level of the nominal exchange rate is indeterminate as in Kareken and Wallace (1981).[4]

2.3 Monetary Policy Autonomy

Monetary policy synchronization holds in the economy considered in Section 2.2, but the economy imposes specific assumptions on the environment as described in Section 2.1. Such assumptions comprise no restriction on holding money and an identical liquidity service between the national money and the global money.

In this subsection, I suggest that monetary policy autonomy can be restored in the same environment with different assumptions. The core equation that gives rise to monetary

[4]Monetary policy synchronization in terms of nominal interest rates can be shown as in Benigno et al. (2022) as follows. In the global capital market where risk-free nominal bonds in units of the national currency $i \in \{h, f\}$ are traded, the uncovered interest rate parity (UIP) holds: $R_h = R_f \mathcal{E}'/\mathcal{E}$, where R_i is the gross nominal interest rate on the nominal bonds in units of the national currency i. Since equation (4) shows the constant nominal exchange rate, the UIP condition implies the equalization of the nominal interest rates: $R_h = R_f$.

policy synchronization is equation (2), which implies that the returns of holding different kinds of money are identical and thereby that the portfolio of real money balances is indeterminate. If equation (2) fails, monetary policy synchronization will break down and monetary policy autonomy will prevail.

Let $z_i^l \equiv \phi_l m_i^l$ denote the real balances of money $l \in \{h, f, g\}$ held by a representative individual in country $i \in \{h, f\}$. Then, under the presumption that the individual in country i holds money i and g only, the total real balances held by that individual is given by $z_i = z_i^i + z_i^g$. Let me now drop the assumption of no restriction on holding money, and assume a specific form of friction that gives rise to the following constraint on the real balances of the global money: for $i \in \{h, f\}$

$$z_i^g \leq \bar{z}_i^g, \tag{5}$$

where $\bar{z}_i^g$ is an upper bound of the amount of the global money that can be used for transaction by an individual in country i. Here, the upper bound is exogenously given. It can be regarded as exogenous government policy that puts the upper limit on the use of the global money for a transaction. In Section 3.3, I will show how it can also be interpreted as an incentive constraint, a constraint that ensures no hidden action by buyers who pay using the global money. The intuition is that if a large volume of the global money is accepted by a seller, the buyer has an incentive to take a hidden action, such as counterfeiting, that will damage the seller. Constraint (5) prevents buyers from engaging in such an action.

Suppose that constraint (5) is binding. Then, the real balances of money g are given by $z_i^g = \bar{z}_i^g$ for $i \in \{h, f\}$. Then, the global money market clearing condition, $\phi_g M_g = z_h^g + z_f^g$, determines the price of currency g as

$$\phi_g = \frac{\bar{z}_h^g + \bar{z}_f^g}{M_g}. \tag{6}$$

In addition, since the total real balances z_i are determined in each country in a typical monetary model as will be shown in Section 3, the real balances of the national money $i \in \{h, f\}$ are given by $z_i^i = z_i - \bar{z}_i^g$. Then, the price of the national currency is determined by the national money market clearing condition as: for $i \in \{h, f\}$

$$\phi_i = \frac{z_i - \bar{z}_i^g}{M_i} \tag{7}$$

Combined with equation (1), equation (7) leads to the nominal exchange rate, given by

$$\mathcal{E} = \frac{M_h}{M_f}\frac{z_f - \bar{z}_f^g}{z_h - \bar{z}_h^g} \tag{8}$$

When the two countries are symmetric, equation (8) is reduced to $\mathcal{E} = M_h/M_f$, so that the nominal exchange rate is simply given by the ratio of monetary aggregates between the two countries.

In a stationary equilibrium, the portfolios of real balances, the nominal exchange rate, and the prices of currencies are all determinate. And nothing forces the money growth rates to be equated between the two countries. From equations (6) and (7), the return of holding money $l \in \{h, f, g\}$ is given by $\phi_l'/\phi_l = M_l/M_l' = \gamma_l^{-1}$. Since the binding constraint (5) prevents these returns from being equated, the central bank in each country can set the money growth rate without restrictions.

3 The Model

In this section, I formalize the main idea put forward in Section 2 using an open-economy search model with multiple currencies. Section 3.1 describes the common setup of the model and extend the two-country search model studied by Zhang (2014) to incorporate a global currency. Section 3.2 shows that monetary policy synchronization holds in a baseline model where there is no restriction on holding money. Each of the remaining two subsections shows that monetary policy synchronization is overturned and monetary policy autonomy restored if a specific restriction in holding or accepting money is added. Section 3.3 introduces the action of counterfeiting or double spending a specific currency as in Nosal and Rocheteau (2017, Chapter 12) and Chiu and Koeppl (2019). Section 3.4 considers government transaction policy as in Li and Wright (1998) and Zhang (2014).

3.1 Common Setup

Environment In addition to the model environment of Section 2.1, two countries, h and f, are populated with a continuum of $2n$ and 2 agents, respectively, where $n \in (0, 1]$ denotes a relative population size of country h. Agents in each country are evenly divided between *buyers* and *sellers*, so the size of sellers or buyers is n and 1 in country h and f, respectively. Each period consists of two sub-periods as in Lagos and Wright (2005).

In the first sub-period, agents meet bilaterally and at random in decentralized markets (DM) and trade the DM good. In the second sub-period, all agents meet in a friction-less centralized market (CM), trade the CM good, and exchange money. The CM good is taken as a numeraire. Both the DM and CM goods are perishable. In the DM, agents lack commitment and individual histories are private information, rendering unsecured credit unusable and making money essential as a medium of exchange.

Preferences and technologies In the DM, a seller in each country can produce the DM good, q, with the cost $c(q)$, but does not have utility over consumption. A buyer, on the other hand, has the utility, $u(q)$, by consuming q but cannot produce. In the CM, all agents get utility $U(y)$ by consuming the CM good, y, which is produced according to a linear one-to-one technology in hours worked x. Periodic utilities for a buyer and a seller, $\mathcal{U}^B$ and $\mathcal{U}^S$, are assumed to be additively separable between the DM and CM goods and quasi-linear in hours worked:

$$\mathcal{U}^B = u(q) + U(y) - x \tag{9}$$

$$\mathcal{U}^S = -c(q) + U(y) - x \tag{10}$$

I assume that $u(\cdot)$ and $U(\cdot)$ are strictly increasing and strictly concave, and satisfy the Inada conditions, $c(\cdot)$ is weakly convex, $u(0) = c(0) = c'(0) = U(0) = 0$, and $c(\bar{q}) = u(\bar{q})$ for some $\bar{q} > 0$. Agents discount the future between adjacent periods with a discount factor $\beta \in (0, 1)$.

Currencies As described in Section 2.1, three currencies are available: a home currency, a foreign currency, and a global currency. The corresponding monetary aggregate is denoted as M_l and the associated growth rate is denoted as $\gamma_l = M_l'/M_l$ for $l \in \{h, f, g\}$. I assume that holding money is costly: $\gamma_l > \beta$ for all l. Changes in the money supply are implemented through lump-sum monetary transfers in the CM to buyers. The way how lump-sum monetary transfers are distributed to buyers does not affect the results of this paper, but for simplicity, it is assumed that each buyer receives $\tau \equiv \sum_{l\in\{h,f,g\}}(\gamma_l - 1)\phi_l M_l/(1 + n)$ in both countries.

Matching In the DM, agents in both countries meet bilaterally and at random. Buyers are mobile while sellers are immobile. With probability $\alpha \in [0.5, 1]$, a buyer from one country stays in the country, and with probability $1-\alpha$, the buyer visits the other country. As in Zhang (2014), the number of matches in the DM of country $i \in \{h, f\}$ is given by the matching function

$$\mathcal{M}_i \equiv \mathcal{M}\left(\mathcal{B}_i, \mathcal{S}_i\right) = \frac{\mathcal{B}_i \mathcal{S}_i}{\mathcal{B}_i + \mathcal{S}_i}$$

where $\mathcal{B}_i$ and $\mathcal{S}_i$ are the measures of buyers and sellers, respectively, in the DM of country i. Then, the probability that a buyer from country i meets a seller in country j, denoted by μ_{ij}, is derived as[5]

$$\mu_{hh} = \frac{\alpha}{1+\alpha+(1-\alpha)/n}, \qquad \mu_{hf} = \frac{1-\alpha}{1+\alpha+(1-\alpha)n}$$
$$\mu_{fh} = \frac{1-\alpha}{1+\alpha+(1-\alpha)/n}, \qquad \mu_{ff} = \frac{\alpha}{1+\alpha+(1-\alpha)n}$$

If country h is smaller than country f and some buyers move to the different country, i.e., if $n < 1$ and $\alpha < 1$, the probability of matching becomes asymmetric in the two countries. In the special case of $\alpha = 1$ in which all buyers stay in their country, the probability of matching with a seller in the same country is reduced to $\mu_{hh} = \mu_{ff} = 0.5$, and the probability of matching with a seller in the different country is zero: $\mu_{hf} = \mu_{fh} = 0$.

Consumption and labor decisions in the CM Let $W_i^{\mathcal{B}}(\mathbf{z}_i)$ and $V_i^{\mathcal{B}}(\mathbf{z}_i)$ denote a value function of a buyer from country $i \in \{h, f\}$ with a portfolio of real balances $\mathbf{z}_i \equiv (z_i^h, z_i^f, z_i^g)$ in the CM and the DM, respectively. In the CM, the problem of a typical buyer from country i can be written as

$$W_i^{\mathcal{B}}(\mathbf{z}_i) = \max_{\{y,x,\mathbf{z}_i'\}} \left\{ U(y) - x + \beta V_i^{\mathcal{B}}(\mathbf{z}_i') \right\} \tag{11}$$

subject to the flow budget constraint

$$y + \phi_h m_i^{h\prime} + \phi_f m_i^{f\prime} + \phi_g m_i^{g\prime} = x + z_i^h + z_i^f + z_i^g + \tau \tag{12}$$

The first-order condition with respect to y yields

$$y = y^* \equiv U'^{-1}(1) \tag{13}$$

Since the real balances of money l purchased today, $\phi_l m_i^{l\prime}$, in the budget constraint (12) will be valued as $z_i^{l\prime} = \phi_l' m_i^{l\prime}$ tomorrow, today's real balances can be written as $\phi_l m_i^{l\prime} =$

[5] See Appendix A.1 for the derivation of these probabilities.

$(\phi_l/\phi_l')z_i^{l\prime} = \gamma_l z_i^{l\prime}$. By using this relationship and equation (13), substituting the budget constraint (12) into the problem (11) yields

$$W_i^{\mathcal{B}}(\mathbf{z}_i) = U(y^*) - y^* + z_i^h + z_i^f + z_i^g + \tau + \max_{\mathbf{z}_i'} \left\{ -\gamma_h z_i^{h\prime} - \gamma_f z_i^{f\prime} - \gamma_g z_i^{g\prime} + \beta V_i^{\mathcal{B}}(\mathbf{z}_i') \right\} \quad (14)$$

Equation (14) implies that the CM value function is linear in real balances, i.e., $W_i^{\mathcal{B}}(\mathbf{z}_i) = W_i^{\mathcal{B}}(\mathbf{0}) + z_i^h + z_i^f + z_i^g$, and that the portfolio choice in the CM is independent of the current holdings of real balances.

Since holding money is costly, $\gamma_l > \beta$ for all $l \in \{h, f, g\}$, sellers have no incentive to carry real balances either in the DM or the CM. Hence, the CM value function of a typical seller in country j is given by

$$W_j^{\mathcal{S}}(\mathbf{z}_j) = U(y^*) - y^* + z_j^h + z_j^f + z_j^g + \beta V_j^{\mathcal{S}}(\mathbf{0}) \quad (15)$$

Terms of trade in the DM Upon matching with a seller in country j, a buyer from country i makes a take-it-or-leave-it offer to the seller.[6] The offer consists of a pair, $(\boldsymbol{d}_{ij}, q_{ij})$, where $\boldsymbol{d}_{ij} \equiv (d_{ij}^j, d_{ij}^g)$ is a portfolio of real balances of monies j and g that the buyer pays to the seller, and q_{ij} is the quantity of the DM good that the seller produces in exchange for receiving the payment. The total payment is $d_{ij} \equiv d_{ij}^j + d_{ij}^g$. This offer takes into account the assumption that a seller in country j does not receive the currency of the other country.

Consider a buyer from country i who enters the DM with a portfolio of real balances, $\mathbf{z}_i \equiv (z_i^h, z_i^f, z_i^g)$. Since money is the only means of payment, the offer has to satisfy the payment constraint

$$d_{ij}^l \leq z_i^l \quad \text{for } l \in \{j, g\} \quad (16)$$

Since holding money is costly, the constraint (16) holds with equality in equilibrium. Also, the seller's value function (15) in the CM implies that the value function in the DM will also become linear in real balances, so that the seller accepts the offer if and only if the payment is no less than the cost of producing the DM good:

$$d_{ij}^j + d_{ij}^g - c(q_{ij}) \geq 0 \quad (17)$$

[6]Here, a take-it-or-leave-it offer is assumed for simple exposition. The main results of this paper will hold for a general terms-of-trade mechanism, studied by Gu and Wright (2016). The general mechanism includes e.g. Kalai's proportional bargaining and generalized Nash bargaining in addition to a take-it-or-leave-it offer.

Constraint (17) holds with equality in equilibrium, otherwise, the buyer would be strictly better off by decreasing d_{ij}^{j} or d_{ij}^{g}, or increasing q_{ij}.

3.2 The Baseline Model

Now I consider a case with no additional frictions, which is referred as the baseline model. In this model, only requirements for trade in the DM are constraints (16) and (17) that hold with equality. I show that monetary policy synchronization ensues in this global-money economy, articulating the idea presented in Section 2.2.

Value function in the DM Consider a buyer from country i who enters the DM with a portfolio of real balances $\mathbf{z}_i \equiv (z_i^h, z_i^f, z_i^g)$. The buyer is matched with a seller in country j with probability μ_{ij}. Upon the match, the buyer makes the take-it-or-leave-it offer of $(\boldsymbol{d}_{ij}, q_{ij})$ that satisfies constraints (16) and (17) with equality, and receives q_{ij} in exchange for payment $d_{ij} = d_{ij}^{j} + d_{ij}^{g} = z_i^j + z_i^g$. The buyer leaves the DM with utility $u(q_{ij})$ and the real balances of $z_i^{\hat{j}}$ unused for $\hat{j} \neq j$. Since the value function of a buyer in the CM is linear in real balances as implied by equation (14), given the match, the buyer's life-time utility in the DM can be written as $u(q_{ij}) + W_i^{\mathcal{B}}(\mathbf{0}) + z_i^{\hat{j}}$. With probability $1 - \mu_{ih} - \mu_{if}$, the buyer is not matched with a seller. In this case, the buyer leaves the DM with periodic utility $u(0) = 0$ and all real balances $\mathbf{z}_i$ unused. Then, the buyer's life-time utility in the DM can be written as $W_i^{\mathcal{B}}(\mathbf{0}) + z_i^h + z_i^f + z_i^g$. To summarize, the value function of the buyer in the DM is given by

$$V_i^{\mathcal{B}}(\mathbf{z}_i) = \mu_{ih}[u(q_{ih}) + z_i^f] + \mu_{if}[u(q_{if}) + z_i^h] + (1 - \mu_{ih} - \mu_{if})(z_i^h + z_i^f + z_i^g) + W_i^{\mathcal{B}}(\mathbf{0}) \quad (18)$$

Since constraints (16) and (17) hold with equality, the quantity of the DM good in equation (18) is given by

$$q_{ij} = c^{-1}(z_i^j + z_i^g) \quad (19)$$

Equations (18) and (19) imply that the value function of a buyer in the DM depends only on real balances brought from the CM in the previous period.

Portfolio choice in the CM With $V_i^{\mathcal{B}}(\mathbf{z}_i)$ on hand, consider the buyer's portfolio choice problem in the CM in equation (14). By pushing one period back and using equations (18) and (19), the problem can be written as

$$\max_{\{z_i^h, z_i^f, z_i^g\}} -\gamma_h z_i^h - \gamma_f z_i^f - \gamma_g z_i^g + \beta \Big\{ \mu_{ih} \left[u\left(c^{-1}(z_i^h + z_i^g)\right) + z_i^f \right] \\ + \mu_{if} \left[u\left(c^{-1}(z_i^f + z_i^g)\right) + z_i^h \right] + (1 - \mu_{ih} - \mu_{if})(z_i^h + z_i^f + z_i^g) \Big\} \tag{20}$$

Given that the matching probabilities, μ_{ih} and μ_{if}, are strictly positive, an interior solution has to satisfy the first-order conditions with respect to z_i^h, z_i^f, and z_i^g, which are given, respectively, as

$$\frac{\gamma_h}{\beta} - 1 = \mu_{ih} \left[\frac{u'\left(c^{-1}(z_i^h + z_i^g)\right)}{c'\left(c^{-1}(z_i^h + z_i^g)\right)} - 1 \right] \tag{21}$$

$$\frac{\gamma_f}{\beta} - 1 = \mu_{if} \left[\frac{u'\left(c^{-1}(z_i^f + z_i^g)\right)}{c'\left(c^{-1}(z_i^f + z_i^g)\right)} - 1 \right] \tag{22}$$

$$\frac{\gamma_g}{\beta} - 1 = \mu_{ih} \left[\frac{u'\left(c^{-1}(z_i^h + z_i^g)\right)}{c'\left(c^{-1}(z_i^h + z_i^g)\right)} - 1 \right] + \mu_{if} \left[\frac{u'\left(c^{-1}(z_i^f + z_i^g)\right)}{c'\left(c^{-1}(z_i^f + z_i^g)\right)} - 1 \right] \tag{23}$$

where $u'(\cdot)$ and $c'(\cdot)$ denote the derivatives of functions $u(\cdot)$ and $c(\cdot)$, respectively. In each equation, the left-hand-side is the opportunity cost of holding money and the right-hand-side captures the liquidity service of money – the expected marginal benefit of holding real balances, $u'(\cdot)/c'(\cdot)$, in excess of the benchmark of $1 = u'(q^*)/c'(q^*)$.

Monetary policy synchronization Combining equations (21)-(23) yields

$$\gamma_h + \gamma_f = \gamma_g + \beta \tag{24}$$

This condition requires that the sum of the growth rates of money in the two countries is constant for given γ_g. The setting of the monetary policy instrument in one country is dependent on the setting of the monetary policy instrument in the other country, and hence this global-money economy features monetary policy synchronization.

To obtain the intuition of condition (24), let $i_l \equiv \gamma_l/\beta - 1$ denote the opportunity cost of holding money l. Then, condition (24) can be written as

$$i_g = i_h + i_f \tag{25}$$

This equation implies that the cost of holding the global money is equated to the sum of the costs of holding money h and money f. Since the global money can be used in both countries while the national money $j \in \{h, f\}$ can be used only in country j, holding the global money brings the sum of the benefits of holding the two kinds of money, which corresponds to the right-hand-side of equation (23). The benefit of holding money $l \in \{h, f, g\}$ is equated to its opportunity cost in equilibrium, leading to equation (25).

Special case To map this model and the main idea described in Section 2.2, consider a special case of $\alpha = 1$ in which all buyers stay in their country: $\mu_{hf} = \mu_{fh} = 0$. In this case, buyers from country i hold money i and g only, as assumed in Section 2.2. Then, given an interior solution, the first-order conditions of the problem (20) are reduced to

$$\frac{\gamma_i}{\beta} - 1 = \mu_{ii}\left[\frac{u'\left(c^{-1}(z_i^i + z_i^g)\right)}{c'\left(c^{-1}(z_i^i + z_i^g)\right)} - 1\right] \qquad \text{for } i \in \{h, f\} \tag{26}$$

$$\frac{\gamma_g}{\beta} - 1 = \mu_{ii}\left[\frac{u'\left(c^{-1}(z_i^i + z_i^g)\right)}{c'\left(c^{-1}(z_i^i + z_i^g)\right)} - 1\right] \tag{27}$$

These equations imply the equalization of the money growth rates, $\gamma_h = \gamma_g = \gamma_f$, as derived in equation (3). In addition, the liquidity service of money, which is captured by the right-hand-side of equations (26) and (27), is identical between monies h, f, and g, as assumed in Section 2.2. Hence, this special case provides solid micro foundations for the idea of monetary policy synchronization presented in Section 2.2.

3.3 Currency Counterfeiting

In this subsection, I add a specific friction in the form of currency counterfeiting and imperfect recognizability to the common setup of Section 3.1 and show that, unlike the economy considered in Section 3.2, monetary policy autonomy is preserved in this global-money economy.

Recognizability problem The national currencies are perfectly recognizable by sellers so that there is no counterfeiting of the national currencies as in the previous subsection. But here I assume that the global currency can be counterfeited by buyers and is imperfectly recognizable for sellers.[7] Specifically, sellers are unable to distinguish genuine global

[7]This asymmetric treatment between the national currencies and the global currency is not critical for the result of monetary policy autonomy. Even if the national currencies are subject to counterfeiting as

money from counterfeit global money in the DM, and buyers can produce counterfeit digital currencies of the global currency at the end of each period – between the closure of the CM and the opening of the DM. For simplicity, following Nosal and Rocheteau (2017, Chapter 12), I assume that producing counterfeit digital currencies entails a fixed cost of $\psi > 0$ only and the variable cost is zero. Then, by spending $\psi > 0$, a buyer can produce any amount of counterfeit global digital currencies. In addition, counterfeits produced in the previous period are all detected and confiscated when agents enter the CM today. Hence, in each period a buyer will incur the fixed cost to enter the DM with counterfeits in hand.

Incentive constraint In general, a buyer chooses a portfolio of genuine global money and counterfeit global money, but this choice can be simplified as follows. Since increasing the holding of genuine money is costly but increasing the holding of counterfeit money is free if the fixed cost is already paid, a buyer holds either genuine global money or counterfeit global money, but not both. In addition, holding more counterfeits than those used in the DM generates no benefit since all the unused counterfeits will be confiscated in the following CM. Hence, a buyer from country i accumulates either the real balances z_i^g of the genuine global money or the real balances z_i^g of the counterfeit global money. The associated costs of doing so are $\gamma_g z_i^g$ and ψ, respectively.[8] Then, the buyer has no incentive to counterfeit if $\gamma_g z_i^g \leq \psi$, or

$$z_i^g \leq \frac{\psi}{\gamma_g} \tag{28}$$

Since the payment constraint (16) holds with equality, a take-it-or-leave-it offer made by the buyer satisfies $\boldsymbol{d}_{ij} \equiv (d_{ij}^j, d_{ij}^g) = (z_i^j, z_i^g)$. Then, the incentive constraint (28) can be written as

$$d_{ij}^g \leq \frac{\psi}{\gamma_g} \tag{29}$$

Upon receiving the offer, if the offer satisfies the incentive constraint (29), the seller finds that the global money used for the payment is genuine. If the constraint (29) does not hold, then I assume that the seller believes the global money used for the payment to be counterfeited. In this case, the seller rejects the offer even if it satisfies the participation constraint (17) with equality, since the constraint fails to hold ex-post once the seller has

well, applying the model of symmetric counterfeiting considered by Gomis-Porqueras et al. (2017) to this model would generate the same result – the preservation of monetary policy autonomy – although such a model would be more complicated than that presented here.

[8]From the portfolio choice problem in the CM value function (14), the costs of accumulating z_i^g are $\gamma_g z_i^g$.

found the global money to be counterfeited. Hence, the incentive constraint (29) constitutes a necessary condition for the seller to accept the offer.

No double spending While counterfeiting with the fixed cost of ψ is a parsimonous way to derive the liquidity constraint (29) endogenously, it can be derived as a no-double-spending constraint for a cryptocurrency that employs a consensus protocol based on proof-of-work. Here, double spending by a buyer is equivalent to faking payment $d_{ij}^g = z_i^g$ to a seller without using the actual real balances of digital money z_i^g, while receiving the good produced by the seller. As shown by Chiu and Koeppl (2019), for a cryptocurrency with a proof-of-work protocol, a buyer has no incentive to double spend if and only if the amount of the payment in the cryptocurrency is limited as

$$d_{ij}^g \leq R(N+1)N \tag{30}$$

where R is a reward for mining – updating a record of transactions (blockchain) – and N is a confirmation lag – the number of confirmations in updating blockchains, where the good is produced and transferred to the buyer after the Nth confirmation.

The no-double-spending constraint (30) is intuitive. The left-hand-side of (30) is the benefit obtained when suceeding in double spending, and the right-hand-side of (30) captures the cost of double spending. It is increasing in the reward R, since the incentives of miners to compete for updating blockchains increase as R increases. It is increasing exponentially in N, since winning the competition of mining for N consecutive times, which is required for double spending, becomes much more difficult as N increases.

The reward R and the confirmation lag N can be endogenized as in Chiu and Koeppl (2019), but doing so would not affect the main results of this paper as long as the constraint (30) is binding. For example, R can be financed by seignorage by assuming $\gamma_g > 1$. In this case the lump-sum monetary transfers do not include the seignorage and are given by $\tau \equiv \sum_{l \in \{h,f\}} (\gamma_l - 1)\phi_l M_l/(1+n)$. An important point here is that the parsimonious incentive constraint (29) can be obtained by considering a cryptocurrency with a proof-of-work protocol as a global digital currency in this model.

Portfolio choice in the CM Given that the offer is accepted by the seller, the DM value function of the buyer is given by equation (18) as in the baseline model in Section 3.2. With the DM value function on hand, the buyer's portfolio choice problem in the CM is given by (20) – the same maximization problem as in the baseline model – but subject to the incentive constraint (28).[9] Let ξ_i denote a Lagrange multiplier on the constraint. Given that the holding of real balances is positive for all kinds of money, the first-order conditions with respect to z_i^h, z_i^f, and z_i^g are given, respectively by equations (21) and (22), and

$$\frac{\gamma_g + \xi_i}{\beta} - 1 = \mu_{ih} \left[\frac{u'\left(c^{-1}(z_i^h + z_i^g)\right)}{c'\left(c^{-1}(z_i^h + z_i^g)\right)} - 1 \right] + \mu_{if} \left[\frac{u'\left(c^{-1}(z_i^f + z_i^g)\right)}{c'\left(c^{-1}(z_i^f + z_i^g)\right)} - 1 \right] \tag{31}$$

The only difference between condition (31) in this economy and condition (23) in the baseline model is the presence of the multiplier ξ_i in equation (31).

Monetary policy autonomy Combining equations (21), (22), and (31) yields

$$\xi_i = \gamma_h + \gamma_f - \gamma_g - \beta \tag{32}$$

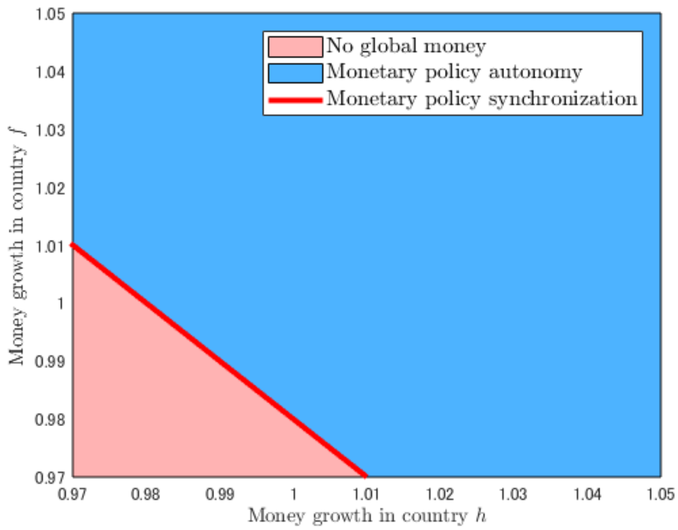
This condition shows that as long as $\xi_i > 0$, monetary policy autonomy holds. In other words, given $\gamma_g > \beta$, any pair of (γ_h, γ_f) that satisfies $\xi_i > 0$ and $\gamma_h, \gamma_f > \beta$ constitutes an equilibrium, given that the cost of counterfeiting is not too high.[10] If the right-hand-side of equation (32) is zero, condition (32) is reduced to condition (24) as in the baseline model studied in Section 3.2 and monetary policy synchronization arises. If the right-hand-side of equation (32) is negative, then the global money is not held in equilibrium: $z_i^g = 0$.

To illustrate, let me set $\beta = 0.97$ and $\gamma_g = 1.01$ and draw Figure 1, which plots the areas of monetary policy autonomy, monetary policy synchronization, and no global money on the (γ_h, γ_f) plane. In this economy with the possibility of counterfeiting of the global money, the region of monetary policy autonomy is vast, as shown in Figure 1. For example, given the setting of the money growth rate in the foreign country, the home central bank is free to set the money growth rate in the home country within the area of monetary policy autonomy. But, if the incentive constraint (28) is slack, an equilibrium with positive holdings of all kinds of money is restricted to the line indicated as "monetary policy synchronization" in Figure 1. This is the case when the right-hand-side of equation (32) is zero, i.e., when the opportunity cost of holding the global money is just equal to the sum of the costs of holding the home and foreign money.

[9] In the case of a cryptocurrency with a proof-of-work protocol as a global digital currency, the constraint (28) is replaced by $z_{ij}^g \leq R(N+1)N$, where the right-hand-side is the same as that of the no-double-spending constraint (30).

[10] For the condition that ensures the positive holdings of all kinds of money, see Appendix A.2.

Figure 1: Regions of monetary policy



Special case As in Section 3.2, consider a special case of $\alpha = 1$ in which all buyers stay in their country. Then, given that the holding of each money is positive, the first-order conditions of the buyer's portfolio choice problem in the DM are reduced to equation (26) and

$$\frac{\gamma_g + \xi_i}{\beta} - 1 = \mu_{ii} \left[\frac{u'\left(c^{-1}(z_i^i + z_i^g)\right)}{c'\left(c^{-1}(z_i^i + z_i^g)\right)} - 1 \right] \tag{33}$$

Combining equations (26) and (33) yields

$$\xi_i = \gamma_i - \gamma_g \tag{34}$$

This equation implies that, as long as $\gamma_i > \gamma_g$ – the return on holding the national money is smaller than that on holding the global money – and the cost of counterfeiting is relatively small, the incentive constraint (28) is binding and monetary policy autonomy holds. In this economy, the upper bound $\bar{z}_i^g$ in the ad-hoc constraint (5) in the simple framework discussed in Section 2.3 corresponds to ψ/γ_g in the incentive constraint (28), and the total real balances, $z_i = z_i^i + z_i^g$, are determined by equation (26). Hence, the price of the global currency, the price of each national currency, and the nominal exchange rate are determined by equations (6), (7), and (8), respectively, as in Section 2.3.

3.4 Government Transaction Policy

I now consider a different friction in the DM: government transaction policy. I add such a friction to the common setup of Section 3.1 and show that monetary policy autonomy can be preserved in the global-money economy.

Government sellers Thus far I have assumed that sellers are *private sellers* who behave in their own interests. In this subsection, I introduce *government sellers* who follow an exogenous policy over which currency they should accept. Specifically, I assume that government sellers in country j accept the national currency of country j only. In each country $j \in \{h, f\}$, a fraction of government sellers is assumed to be $0 \leq \tilde{\mu}_j \leq 1$. Then, for a buyer from country i, the probability of matching with a private seller in country j, who can accept both the national currency and the global currency, is $\mu_{ij}(1 - \tilde{\mu}_j)$, and the probability of matching with a government seller in country j, who can accept the national currency only, is $\mu_{ij}\tilde{\mu}_j$.

The terms of trade with a government seller in the DM differ from those with a private seller since a government seller can accept the national currency only. A take-it-or-leave-it offer made by a buyer from country i to a government seller in country j consists of a pair $(\tilde{d}^j_{ij}, \tilde{q}_{ij})$, where $\tilde{d}^j_{ij}$ is the payment using the national currency j in country j and $\tilde{q}_j$ is the quantity of the DM good produced by the government seller. This offer has to satisfy the payment constraint similar to (16),

$$\tilde{d}^j_{ij} \leq z^j_i \tag{35}$$

and the modified participation constraint,

$$\tilde{d}^j_{ij} - c(\tilde{q}_{ij}) \geq 0 \tag{36}$$

This constraint reflects the fact that the government seller can accept the national currency j only. The terms of trade with a private seller remain the same as in the common setup of Section 3.1.

Value function in the DM Consider a match in the DM for a buyer from country i with a portfolio of real balances $\mathbf{z}_i \equiv (z^h_i, z^f_i, z^g_i)$. For the purpose of exposition, I focus on a case where the payment constraint (16) is binding for all matches.[11] In this case, with

[11] For the case of non-binding payment constraints, see Appendix A.3.

probability $\mu_{ij}(1-\tilde{\mu}_j)$, the buyer is matched with a private seller in country j and makes an offer $(\boldsymbol{d}_{ij}, q_{ij})$ that satisfies conditions (16) and (17). The resulting life-time utility of the buyer is $u(q_{ij}) + W_i^{\mathcal{B}}(\mathbf{0}) + z_i^{\hat{j}}$ for $\hat{j} \neq j$ as in the baseline model in Section 3.2. With probability $\mu_{ij}\tilde{\mu}_j$, the buyer is matched with a government seller in country j and makes an offer $(\tilde{d}_{ij}^j, \tilde{q}_{ij})$ that satisfies conditions (35) and (36). The buyer leaves the DM with monies $\hat{j}$ and g unused, so that the buyer's life-time utility is given by $u(\tilde{q}_{ij}) + W_i^{\mathcal{B}}(\mathbf{0}) + z_i^{\hat{j}} + z_i^g$. With probability $1-\mu_{ih}-\mu_{if}$, the buyer is unmatched with a seller. Then, the DM value function of the buyer can be written as

$$\begin{aligned}V_i^{\mathcal{B}}(\mathbf{z}_i) &= \mu_{ih}(1-\tilde{\mu}_h)[u(q_{ih}) + z_i^f] + \mu_{ih}\tilde{\mu}_h[u(\tilde{q}_{ih}) + z_i^f + z_i^g] \\ &\quad + \mu_{if}(1-\tilde{\mu}_f)[u(q_{if}) + z_i^h] + \mu_{if}\tilde{\mu}_f[u(\tilde{q}_{if}) + z_i^h + z_i^g] \\ &\quad + (1-\mu_{ih}-\mu_{if})(z_i^h + z_i^f + z_i^g) + W_i^{\mathcal{B}}(\mathbf{0})\end{aligned}$$

Since the constraints (16), (17), and (36) hold with equality, the quantity of the DM good produced by a private seller and a government seller is given by equation (19) and $\tilde{q}_{ij} = c^{-1}(z_i^j)$, respectively.

Portfolio choice in the CM Similar to the formulation in Section 3.2, the buyer's portfolio choice problem in the CM can be written as

$$\begin{aligned}\max_{\{z_i^h, z_i^f, z_i^g\}} & -\gamma_h z_i^h - \gamma_f z_i^f - \gamma_g z_i^g + \beta\Big\{\mu_{ih}(1-\tilde{\mu}_h)\left[u\left(c^{-1}(z_i^h + z_i^g)\right) + z_i^f\right] \\ & + \mu_{ih}\tilde{\mu}_h\left[u\left(c^{-1}(z_i^h)\right) + z_i^f + z_i^g\right] + \mu_{if}(1-\tilde{\mu}_f)\left[u\left(c^{-1}(z_i^f + z_i^g)\right) + z_i^h\right] \\ & + \mu_{if}\tilde{\mu}_f\left[u\left(c^{-1}(z_i^f)\right) + z_i^h + z_i^g\right] + (1-\mu_{ih}-\mu_{if})(z_i^h + z_i^f + z_i^g)\Big\}\end{aligned}$$

The first-order conditions with respect to z_i^h, z_i^f, and z_i^g for their interior solution are given, respectively, as

$$\frac{\gamma_h}{\beta} - 1 = \mu_{ih}(1-\tilde{\mu}_h)\left[\frac{u'\left(c^{-1}(z_i^h + z_i^g)\right)}{c'\left(c^{-1}(z_i^h + z_i^g)\right)} - 1\right] + \mu_{ih}\tilde{\mu}_h\left[\frac{u'\left(c^{-1}(z_i^h)\right)}{c'\left(c^{-1}(z_i^h)\right)} - 1\right] \tag{37}$$

$$\frac{\gamma_f}{\beta} - 1 = \mu_{if}(1-\tilde{\mu}_f)\left[\frac{u'\left(c^{-1}(z_i^f + z_i^g)\right)}{c'\left(c^{-1}(z_i^f + z_i^g)\right)} - 1\right] + \mu_{if}\tilde{\mu}_f\left[\frac{u'\left(c^{-1}(z_i^f)\right)}{c'\left(c^{-1}(z_i^f)\right)} - 1\right] \tag{38}$$

$$\frac{\gamma_g}{\beta} - 1 = \mu_{ih}(1-\tilde{\mu}_h)\left[\frac{u'\left(c^{-1}(z_i^h + z_i^g)\right)}{c'\left(c^{-1}(z_i^h + z_i^g)\right)} - 1\right] + \mu_{if}(1-\tilde{\mu}_f)\left[\frac{u'\left(c^{-1}(z_i^f + z_i^g)\right)}{c'\left(c^{-1}(z_i^f + z_i^g)\right)} - 1\right] \quad (39)$$

Compared to equations (21)-(23) in the baseline model, holding the national money has additional benefits as appeared in the second term in the right-hand-side of equations (37) and (38) due to the presence of government sellers who accept the national money only. If there are no government sellers, i.e., $\tilde{\mu}_h = \tilde{\mu}_f = 0$, equations (37)-(39) are reduced to equations (21)-(23).

Monetary policy autonomy For given γ_h, γ_f, and γ_g, the three equations (37)-(39) provide a solution for z_i^h, z_i^f, and z_i^g. For given γ_g, a pair of (γ_h, γ_f) that ensures a solution constitutes an area of monetary policy autonomy. For simplicity, assume that a fraction of government sellers is identical between the two countries: $\tilde{\mu} = \tilde{\mu}_h = \tilde{\mu}_f$. By combining equations (37)-(39), a necessary condition for the existence of an equilibrium with positive holdings of the national money and the global money is derived as[12]

$$\gamma_h + \gamma_f - \gamma_g - \beta > \frac{\tilde{\mu}(\gamma_g - \beta)}{1 - \tilde{\mu}} \quad (40)$$

If there are no government sellers, $\tilde{\mu} = 0$, the left-hand-side of this inequality should be positive, but it does not hold since conditions (37)-(39) are reduced to condition (24) if $\tilde{\mu} = 0$. This observation highlights the role of government sellers for the existence of equilibrium with monetary policy autonomy. Condition (40) suggests that given γ_f and γ_g, the money growth rate of the home country within the range of $\gamma_h > \underline{\gamma}_h$ can support a global money equilibrium in which all the currences are valued and circulated, where the lower bound is implied by condition (40) as $\underline{\gamma}_h \equiv \gamma_g - \gamma_f + \beta - \tilde{\mu}(\gamma_g - \beta)/(1 - \tilde{\mu})$.

Special case Consider a special case of $\alpha = 1$ in which buyers are matched with sellers in the same country only. In this case, equations (37)-(39) are reduced to

$$\frac{\gamma_i}{\beta} - 1 = \mu_{ii}(1-\tilde{\mu}_i)\left[\frac{u'\left(c^{-1}(z_i^i + z_i^g)\right)}{c'\left(c^{-1}(z_i^i + z_i^g)\right)} - 1\right] + \mu_{ii}\tilde{\mu}_i\left[\frac{u'\left(c^{-1}(z_i^i)\right)}{c'\left(c^{-1}(z_i^i)\right)} - 1\right] \quad \text{for } i \in \{h, f\} \quad (41)$$

$$\frac{\gamma_g}{\beta} - 1 = \mu_{ii}(1-\tilde{\mu}_i)\left[\frac{u'\left(c^{-1}(z_i^i + z_i^g)\right)}{c'\left(c^{-1}(z_i^i + z_i^g)\right)} - 1\right] \quad (42)$$

[12]The proof of this claim can be found in Appendix A.4. The appendix also derives necessary and sufficient conditions for the existence and shows that condition (40) is a part of such conditions.

Combining equations (41) and (42) yields

$$\frac{\gamma_i}{\beta} - \frac{\gamma_g}{\beta} = \mu_{ii}\tilde{\mu}_i \left[\frac{u'\left(c^{-1}(z_i^i)\right)}{c'\left(c^{-1}(z_i^i)\right)} - 1 \right] \tag{43}$$

If there is no government seller, $\tilde{\mu}_i = 0$, this condition is reduced to the equalization of the money growth rates as in Section 3.2. Thus, the presence of government sellers plays a critical role in preventing the loss of monetary policy autonomy in the global-money economy.

Given that there are government sellers, $\tilde{\mu}_i > 0$, and given that the growth rate of the national money is greater than that of the global money, $\gamma_i - \gamma_g > 0$, the real balances of the national money is determined by equation (43). The total real balances of money, $z_i \equiv z_i^i + z_i^g$, can be solved from equation (42). The solution constitutes an equilibrium in the global-money economy if $z_i^g = z_i - z_i^i > 0$. From equations (42) and (43), this is the case if and only if

$$\gamma_i - \gamma_g > \frac{\tilde{\mu}_i(\gamma_g - \beta)}{1 - \tilde{\mu}_i} \tag{44}$$

In the region that satisfies condition (44), monetary policy autonomy prevails in the global money economy, which features the positive holdings of the national money and the global money. Since the national money provides a greater liquidity service – the right-hand-side of equation (41) – than the global money, in equilibrium the cost of holding the national money – the left-hand-side of equation (41) – also has to be greater than the cost of holding the global money. Thus, the growth rate of the national money has to be high enough to satisfy condition (44) in the global-money economy.

4 Conclusion

This paper has considered a possible future economy in which privately issued digital money is circulated across borders. It has explored the impact of such global money on monetary policy autonomy using a two-country open economy search model with a global currency. The theoretical analysis has shown that the emergence of global money will cause a loss of monetary policy autonomy if there is no limit on using global money for transaction, as in Benigno et al. (2022); but if using global money is subject to additional frictions such as currency counterfeiting and government transaction policy, then monetary policy autonomy will be preserved.

The result suggests that a loss of monetary policy autonomy is not an inevitable outcome of the circulation of global digital money and that monetary policy autonomy and global digital money can be compatible. Specifically, if an incentive mechanism or government policy limits the amount of global money used for transaction, central banks can set their monetary policy instrument as they see fit. This result points to the critical role of the design and regulation of global digital money in preserving the capacity of central banks to conduct monetary policy.

I would like to close the paper with a few caveats. First, this paper focuses exclusively on the impact of global digital money as a medium of exchange on monetary policy autonomy. Thus, it is silent about the effectiveness of monetary policy. Ikeda (2020) explores the impact of global digital money as a unit of account on the effectiveness of monetary policy. For analytical purposes, distinguishing the different roles of money, and between autonomy and effectiveness are useful in deriving sharp theoretical implications, but in practice the roles of money, autonomy, and effectiveness need to be considered simultaneously. Second, although this paper considers currency counterfeiting and government policy, there could be other important frictions such as imperfect financial markets (Gabaix and Maggiori, 2015) that could also help preserve monetary policy autonomy. Finally, this paper abstracts from the detailed design of global digital money, everything other than its amount used for transaction and its growth rate. In the case of stablecoins, for example, safe assets that support the value of stablecoins would need to be considered explicitly. Despite these caveats, I hope that this paper provides some insight into answering the questions posed by the rise in digital money.

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Appendix

A Proof and Derivation

A.1 Matching probabilities

As described in Section 3.1, a buyer stays in the domestic country with probability $\alpha \in [0.5, 1]$, and a buyer visits the foreign country with probability $1-\alpha$; the number of trade matches in the DM of country i is given by

$$\mathcal{M}_i \equiv \mathcal{M}(\mathcal{B}_i, \mathcal{S}_i) = \frac{\mathcal{B}_i \mathcal{S}_i}{\mathcal{B}_i + \mathcal{S}_i},$$

where $\mathcal{B}_i$ and $\mathcal{S}_i$ denote the measures of buyers and sellers in the DM of country i, respectively. In country h, the number of sellers is $\mathcal{S}_h = n$, while the number of buyers is $\mathcal{B}_h = \alpha n + 1 - \alpha$. In country f, the number of sellers is $\mathcal{S}_f = 1$, while the number of buyers is $\mathcal{B}_f = \alpha + n(1-\alpha)$. Then, the number of matches in countries h and f is given, respectively, by

$$\mathcal{M}_h = \frac{n(\alpha n + 1 - \alpha)}{n + \alpha n + 1 - \alpha}$$
$$\mathcal{M}_f = \frac{\alpha + n(1-\alpha)}{1 + \alpha + n(1-\alpha)}$$

The number of matches in country j where the matched buyers are from country i, denoted by $\mathcal{M}_{ij}$, is given by

$$\mathcal{M}_{hh} = \mathcal{M}_h \frac{\alpha n}{\alpha n + 1 - \alpha}, \qquad \mathcal{M}_{fh} = \mathcal{M}_h \frac{1-\alpha}{\alpha n + 1 - \alpha},$$
$$\mathcal{M}_{hf} = \mathcal{M}_f \frac{n(1-\alpha)}{\alpha + n(1-\alpha)}, \qquad \mathcal{M}_{ff} = \mathcal{M}_f \frac{\alpha}{\alpha + n(1-\alpha)}.$$

Conditional on being in country $j \in \{h, f\}$, a buyer meets a seller with probability

$$a_h \equiv \frac{\mathcal{M}_h}{\mathcal{B}_h} = \frac{1}{1 + \alpha + (1-\alpha)/n}$$
$$a_f \equiv \frac{\mathcal{M}_f}{\mathcal{B}_f} = \frac{1}{1 + \alpha + n(1-\alpha)}$$

Then, the probability that a buyer from country h meets a seller in country $j \in \{h, f\}$, denoted by μ_{hj}, is given by

$$\mu_{hh} = \alpha a_h, \qquad \mu_{hf} = (1-\alpha) a_f$$

Similarly, the probability that a buyer from country f meets a seller from country $j \in \{h, f\}$, denoted by μ_{fj}, is given by

$$\mu_{fh} = (1-\alpha)a_h, \qquad \mu_{ff} = \alpha a_f$$

A.2 Condition on the cost of counterfeiting

In the model of currency counterfeiting in Section 3.3, the equilibrium conditions regarding the portfolio choice of real balances consist of equations (21), (22), and (31), which are reproduced here for convenience:

$$i_h = \mu_{ih}\left[\frac{u'\left(c^{-1}\left(z_i^h + z_i^g\right)\right)}{c'\left(c^{-1}\left(z_i^h + z_i^g\right)\right)} - 1\right] \tag{45}$$

$$i_f = \mu_{if}\left[\frac{u'\left(c^{-1}\left(z_i^f + z_i^g\right)\right)}{c'\left(c^{-1}\left(z_i^f + z_i^g\right)\right)} - 1\right] \tag{46}$$

$$\frac{\xi_i}{\beta} = i_h + i_f - i_g \tag{47}$$

where $i_h, i_f, i_g > 0$ by assumption and ξ_i is a Lagrange multiplier on the incentive constraint (28): $z_i^g \leq \bar{z}^g \equiv \frac{\psi}{\gamma_g}$.

Assume $i_h + i_f - i_g > 0$ so that $\xi_i > 0$ and the constraint (28) is binding: $z_i^g = \bar{z}^g$. Let $z_i^{hg} \equiv z_i^h + z_i^g$ denote a solution for equation (45). Then, since $z_i^g = \bar{z}^g$, the real balancs of the home money is positive, $z_i^h > 0$, if $z_i^{hg} > \bar{z}^g$. Similarly, the real balances of the foreign money is positive, $z_i^f > 0$, if $z_i^{fg} > \bar{z}^g$, where $z_i^{fg} \equiv z_i^f + z_i^g$ is a solution to equation (46). Hence, both the real balances of the home money and foreign money are positive, $z_i^h, z_i^f > 0$, if $\min\{z_i^{hg}, z_i^{fg}\} > \bar{z}_i^g$ or the cost of counterfeiting is not too large, satisfying

$$\psi < \gamma_g \times \min\{z_i^{hg}, z_i^{fg}\}$$

This condition ensures the existence of an equilibrium in the global-money economy for any pair of (γ_h, γ_f) that satisifies $\xi_i > 0$ and $\gamma_h, \gamma_f > \beta$ for given $\gamma_g > \beta$.

A.3 Non-binding payment constraints in Section 3.4

There are four types of match regarding the types of seller: a home private seller, a home government seller, a foreign private seller, and a foreign government seller. Consider a buyer from the home country and matches with these sellers. It is straightforward to show that the payment constraint (16) is binding for a match with a government seller. Otherwise, the payment constraint is non-binding for a match with a private seller. The non-binding payment constraints for matches with both private and government sellers are inconsistent with $\gamma_h, \gamma_f > \beta$. Hence, the payment constraint may be non-binding for a match with a private seller. Suppose that the payment constraint is non-binding for matches with private sellers in both countries h and f. This implies that the marginal utility of holding the global money is zero, which is inconsistent with the positive cost of holding the global money, $\gamma_g > \beta$. Hence, the payment constraint may be non-binding for a match with a private seller either in the home country or in the foreign country, and not in the both countries.

Without loss of generality, consider a case in which the cash constraint is not binding for a match between a buyer from the home country and a private seller in the same country. In this case, the real money balance held by the buyer is great enough to satisfy $z_h^h + z_h^g > z^*$, where the satiation amount of the real balances, z^*, are given by

$$\frac{u'\left(c^{-1}\left(z^*\right)\right)}{c'\left(c^{-1}\left(z^*\right)\right)} = 1$$

Given $z_h^h + z_h^g > z^*$, the first-order conditions of the portfolio choice problem with respect to z_h^h, z_h^f, and z_h^g are given, respectively as

$$i_h = \mu_{hh}\tilde{\mu}_h \left[\frac{u'\left(c^{-1}\left(z_h^h\right)\right)}{c'\left(c^{-1}\left(z_h^h\right)\right)} - 1\right] \tag{48}$$

$$i_f = \mu_{hf}\left(1-\tilde{\mu}_f\right) \left[\frac{u'\left(c^{-1}\left(z_h^f + z_h^g\right)\right)}{c'\left(c^{-1}\left(z_h^f + z_h^g\right)\right)} - 1\right] + \mu_{hf}\tilde{\mu}_f \left[\frac{u'\left(c^{-1}\left(z_h^f\right)\right)}{c'\left(c^{-1}\left(z_h^f\right)\right)} - 1\right] \tag{49}$$

$$i_g = \mu_{hf}\left(1-\tilde{\mu}_f\right) \left[\frac{u'\left(c^{-1}\left(z_h^f + z_h^g\right)\right)}{c'\left(c^{-1}\left(z_h^f + z_h^g\right)\right)} - 1\right] \tag{50}$$

Equation (48) can be solved for z_h^h. Combining equations (49) and (50) yields

$$i_f - i_g = \mu_{hf}\tilde{\mu}_f \left[\frac{u'\left(c^{-1}\left(z_h^f\right)\right)}{c'\left(c^{-1}\left(z_h^f\right)\right)} - 1\right] \tag{51}$$

Given $i_f > i_g$, equation (51) can be solved for z_h^f. With z_h^f on hand, equation (50) can be solved for z_h^g. The solution has to satisfy $z_h^h + z_h^g > z^*$ and $z_h^f + z_h^g < z^*$. If these inequalities are satisfied, the solution of z_h^h, z_h^f, and z_h^g constitutes an equilibrium. Since the portfolio of real balances is determined, monetary policy autonomy is preserved in this economy.

A.4 Proof of monetary policy autonomy in Section 3.4

I show that condition (40) is a necessary condition for the existence of a stationary equilibrium with positive holdings of currencies h, f, and g in the economy with government sellers. I also derive necessary and sufficient conditions for the existence. Assume that a fraction of government sellers is identical between the two countries: $\tilde{\mu} = \tilde{\mu}_h = \tilde{\mu}_f$. The equilibrium conditions that characterize positive holdings of the three currencies are equations (37)-(39).

From equation (37), z_i^h can be written as a function of z_i^g, which is denoted as $z_i^h = Z_h(z_i^g)$. This function is decreasing but the degree of a decrease in less than unity, $-1 < Z_h'(\cdot) < 0$. Let $\bar{z}_i^h$ and $\underline{z}_i^h$ be defined as

$$\bar{z}_i^h = Z_h(0),$$
$$\underline{z}_i^h = Z_h(\bar{z}_i^g), \text{ where } \frac{u'\left(c^{-1}(\underline{z}_i^h + \bar{z}_i^g)\right)}{c'\left(c^{-1}(\underline{z}_i^h + \bar{z}_i^g)\right)} = 1$$

Similarly, from equation (38), function $z_i^f = Z_f(z_i^g)$ can be derived and it has the same properties as $z_i^h = Z_h(z_i^g)$. Define $\bar{z}^g = \min\{\bar{z}_h^g, \bar{z}_f^g\}$.

By substituting $z_i^h = Z_h(z_i^g)$ and $z_i^f = Z_f(z_i^g)$, equation (39) can be written as

$$\mu_{ih}\left[\frac{u'\left(c^{-1}(Z_h(z_i^g) + z_i^g)\right)}{c'\left(c^{-1}(Z_h(z_i^g) + z_i^g)\right)} - 1\right] + \mu_{if}\left[\frac{u'\left(c^{-1}(Z_f(z_i^g) + z_i^g)\right)}{c'\left(c^{-1}(Z_f(z_i^g) + z_i^g)\right)} - 1\right] - \frac{\gamma_g - \beta}{\beta(1-\tilde{\mu})} = 0 \qquad (52)$$

Let $Z_g(z_i^g)$ denote the left-hand-side of this equation as a function of z_i^g. Since $-1 < Z_j'(\cdot) < 0$ for $j \in \{h, f\}$, $Z_j(z_i^g) + z_i^g$ is increasing in z_i^g. Since $u''(\cdot) < 0$ and $c''(\cdot) \geq 0$, the function $Z_g(z_i^g)$ is strictly decreasing in z_i^g. Then, equation (52) can be formulated as a fixed-point problem for z_i^g as

$$Z_g(z_i^g) = 0 \qquad (53)$$

Since $Z_g(z_i^g)$ is strictly decreasing, a unique interior solution to (53) exists if and only if $Z_g(0) > 0$ and $Z_g(\bar{z}^g) < 0$. From equations (37)-(38) with $\tilde{\mu}_h = \tilde{\mu}_f = \tilde{\mu}$, condition $Z_g(0) > 0$ can be derived as condition (40), which is necessary but not sufficient. Necessary and sufficient conditions for the existence are condition (40) and $Z_g(\bar{z}^g) < 0$.