IMES DISCUSSION PAPER SERIES

Inflation Expectations and Central Bank Communication with Unknown Prior

Tatsushi Okuda and Tomohiro Tsuruga

Discussion Paper No. 2021-E-7

IMES

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

2-1-1 NIHONBASHI-HONGOKUCHO
CHUO-KU, TOKYO 103-8660
JAPAN

You can download this and other papers at the IMES Web site:

https://www.imes.boj.or.jp

IMES Discussion Paper Series 2021-E-7
July 2021

Inflation Expectations and Central Bank Communication with Unknown Prior

Tatsushi Okuda* and Tomohiro Tsuruga**

Abstract

We construct a noisy information model of central bank communication on future inflation rates and highlight an informational friction that plays a key role in explaining several empirical properties of firms' inflation expectations. Using a survey of Japanese firms' inflation expectations, we document new empirical facts related to the size of firms and their inflation expectations. We observe a persistent deviation of expectations from the central bank's inflation target and find that the deviation is monotonically increasing in firm size, while the degree of the forecasting imprecision, responsiveness to actual inflation, and the heterogeneity in firms' expectations are monotonically decreasing in firm size. To reconcile these empirical regularities, we construct a dynamic model of inflation expectation formation by Bayesian firms where the central bank's inflation forecast serves as a noisy public signal of future inflation rates and propose an informational friction in the communication about future inflation: the central bank's prior about the future inflation rate, which is unknown to firms. In this setup, the sluggishness of the adjustment of inflation expectations is amplified by the central bank's communication. Moreover, this friction drastically changes the role and the effect of central bank communication on firms' expectations formation. Firms utilize the inflation forecast as a signal not of the level but of changes in future inflation rates.

Keywords: Imperfect information; inflation expectations; communication
JEL classification: E50, D83, D84, D82

* Associate Director and Economist, Institute for Monetary and Economic Studies (currently, Director, Financial System and Bank Examination Department), Bank of Japan (E-mail: tatsushi.okuda@boj.or.jp)
** Director and Senior Economist, Institute for Monetary and Economic Studies, Bank of Japan (currently, International Monetary Fund, E-mail: TTsuruga@imf.org)

The authors are grateful to Robert Anderson, Thomas Chiang, Koichiro Kamada, James Morley, Taisuke Nakata, Athanasios Orphanides, Shigenori Shiratsuka, Georg Strasser, Yenan Wang, Francesco Zanetti, participants of the Midwest Macro Meetings 2018, the Computing in Economics and Finance 2018, the Japanese Economic Association 2018 Spring Meetings, the BOK-ERI/BOJ-IMES Joint Workshop 2018, the Western Economic Association 2019 International Conference, seminars at Hitotsubashi University and Waseda University, and the staff of the Bank of Japan for their invaluable comments. The authors particularly thank Wataru Tamura for his extremely useful suggestions. The views expressed in this paper are those of the authors and do not necessarily reflect the official views of the Bank of Japan and International Monetary Fund.

1 Introduction

Among the central issues of interest to central banks is how to manage inflation expectations through communication about *future inflation rates* that are consistent with their information regarding medium-to-long term economic fundamentals that reflects their policy intent, such as a target inflation rate.[1] Managing firms' inflation expectations is important since firms' inflation expectations could directly affect their price setting behavior, which in turn determines the inflation rate. The ability to manage expectations may rely on the credibility of the central bank's communication regarding the future conduct of monetary policy, which has been emphasized in the literature of the time-inconsistency problem in central bank's commitment policies (Kydland and Prescott 1977; Barro and Gordon 1983a, 1983b; Backus and Driffill 1985; Nakata 2015; Haberis, Harrison, and Waldron 2017). Moreover, the difficulty of managing expectations has recently been debated in a different context. As policy interest rates approach the effective lower bound in advanced economies, the liquidity trap could undermine the ability of the central bank to stabilize the inflation rate at the target level by manipulating interest rates, and this could harm the credibility of central bank communication (Eggertsson 2006; Adam and Billi 2007; Nakov 2008; Nakata and Schmidt 2016; Hills, Nakata, and Schmidt 2019).

Despite the importance of managing inflation expectations, as Coibion, Gorodnichenko, Kumar, and Pedemonte (2020) point out, little is known about how inflation expectations are formed by private agents, in particular by firms, and how central banks can effectively influence expectations through communication about its medium-to-long-term perspective of future inflation. We document several new stylized facts about firms' inflation expectations using a survey of Japanese firms that are potentially related to the effectiveness of central bank communication about future inflation rates and the use of information by firms in expectations formation. We find that the deviation of firms' inflation expectations from the central bank's inflation target is monotonically increasing in firm size. In addition, the imprecision, magnitude of responsiveness to the actual inflation rate, and degree of

[1] The *future inflation rate* in this context, represents a medium-to-long-term perspective of the inflation rate in the future. Communication about the future inflation rate is considered to be consistent with the central bank's intended inflation rate, while also reflecting the central bank's view on macroeconomic fluctuations. In Japan, these inflation rates are stated in "Understanding of Medium-to-Long-term Price Stability" (April 2009-), "The Price Stability Goal" (February 2012-), and "The Price Stability Target" (January 2013-).

heterogeneity in expectations are monotonically decreasing in firm size.[2]

To analyze the mechanism of firms' inflation expectations formation, we develop a dynamic model of inflation expectation formation by Bayesian firms.[3] The model is a pure communication model between firms and a central bank. Firms attempt to form precise expectations of the future inflation rate and the central bank sends its inflation forecast to firms as a public signal of the future inflation environment. Firms form their inflation expectations based on the public signal, their own private signals on future inflation rates, and another public signal, i.e. the actual inflation rate. The precision of the private signal received by firms affects their reliance on public signals, such as the actual inflation rate and the central bank's inflation forecast, and generates the differences in the formation process of inflation expectations across firms. On top of these signals, we introduce an informational friction in communication where the central bank's prior about the future inflation rate is unknown to firms. We then analyze the formation process and the properties of firms' inflation expectations.

The interpretation of the central bank's prior about the future inflation rate is that it reflects the central bank's view on the unobservable fundamentals of the inflation rate that shapes its inflation forecast. The view could deviate from the actual fundamentals because the central bank does not necessarily have perfect knowledge of the economic structure. For example, there is an ongoing debate about the formation of inflation expectations and the underlying drivers behind the changing trend of inflation dynamics (Coibion, Gorodnichenko, Kumar, and Pedemonte 2020; Candia, Coibion, and Gorodnichenko 2020). The discrepancy could also emerge as a result of the bank's strategic communication. For example, seminal papers on central bank communication by Morris and Shin (2002) and Angeletos and Pavan (2007) discuss the case where noisier public information from the central bank could improve social welfare.[4] If a central bank sends noisy signals and the noise is persistent, then the

[2]The responsiveness to past macroeconomic information has been investigated in the literature (see Gurkaynak *et al*, 2005, 2007; Gurkaynak, Levin, and Swanson 2010; Fuhrer 2012, 2017). Heterogeneity of inflation expectations is often called *disagreement* in the literature (Mankiw, Reis, and Wolfers 2004; Dovern, Fritshce, and Slacalek 2012; Armantier *et al.* 2013). For the implications of heterogeneity, see Morris and Shin (2006) and Angeletos and Lian (2019).

[3]Coibion and Gorodnichenko (2012, 2015), Coibion, Gorodnichenko, and Kamdar (2018), and Andrade, Crump, Eusepi, and Moench (2016) present evidence that imperfect information models exhibit a better fit of survey data than full-information rational expectations models.

[4]Fujiwara and Waki (2015, 2019) analyze the case where a central bank can improve social welfare by not disclosing its private information about future economic conditions under a standard New Keynesian setup. The literature on *information design* implies that it could be optimal for the central bank to commit to disclose imperfect information to private agents (Kamenica and Gentzkow 2011; Bergemann and Morris

persistent noise is regarded as the central bank's view by private agents. In both cases, private agents do not naively use the central bank's inflation forecast as a signal of the fundamentals of the inflation rate. To incorporate the central bank's view as an obstacle for communication while maintaining the simplicity of the model, we introduce the central bank's (uninformative) prior.[5]

Under the model's setting, we show that the friction in the central bank's unknown prior is useful for noisy information models in reconciling these empirical patterns and it poses a drastic change in the role of central bank communication in firms' expectations formation. With this friction, firms attempt to learn the persistent component of the noise in the central bank's prior from the track record of the central bank's inflation forecasts. Through this learning process, firms utilize the signals obtained from actual inflation rates and their own private signals. Their reliance on these signals depends on the firm's capacity to process information, which eventually generates the monotonic patterns. Importantly for us, even if the persistent component of the noise, i.e. the central bank's prior, is zero, these results hold as long as the bank's prior is unknown to firms.

More importantly, the role of central bank communication changes completely under this friction. With the friction of an unknown prior, firms utilize the central bank's inflation forecast as a signal not of the *level*, but of the *changes* in future inflation rates. Namely, if the values of the signals are the same between two periods, firms' interpretation of the signals is that the situation will not change. The dependency of expectations formation on past information is thus amplified and expectations can deviate from the target inflation rate even more persistently. This is in contrast to the case in which firms fully believe the central bank's inflation forecasts and the case in which they are merely unaware of its inflation forecasts.

Related literature. In the literature, the role of central bank communication in firms' inflation expectations has been actively studied, but many questions have remained open both empirically and theoretically. From an empirical point of view, in recent studies Kumar, Afrouzi, Coibion and Gorodnichenko (2015) and Coibion, Gorodnichenko, and Kumar

2013, 2019; Tamura 2016, 2018; Ui 2020).

[5] While this study does not explicitly analyze the central bank's strategic information transmission process, modeling it as cheap-talk (Crawford and Sobel 1982) or a persuasion game (Milgrom 1981 and Milgrom and Roberts 1986) could potentially provide us with intriguing results. Extending our model to a general equilibrium model would also generate more complex interactions between central bank communication and economic outcomes. We leave these extensions as future research topics.

(2018) conducted surveys of firms in New Zealand regarding their inflation expectations and confirmed a number of empirical regularities. They reported that firms' inflation expectations could persistently deviate from the target inflation rate, were responsive to the actual inflation rate, and slowly adjusted to the target inflation rate whereas the bank had not only adopted an inflation target, but also improved the transparency of its inflation forecasts in order to guide inflation expectations toward the inflation target.[6] These regularities were studied in light of informational frictions where firms are subject to noisy signals of economic variables. Their models were based on a rational inattention framework, where firms rationally choose not to fully acquire information about developments in the state of the aggregate economy.[7] However, these studies have been available to only a limited number of countries. As Coibion *et al.* (2020) points out, in most of the countries where the central bank adopts inflation targeting, country-wide empirical surveys on firms' inflation expectations are hardly available. One prominent exception is the Short-term Economic Survey of Enterprises in Japan (Tankan survey). As part of the Tankan survey, the Bank of Japan asks firms about both their future individual price inflation as well as future country-level aggregate price inflation, starting in April 2014. Based on the survey, we document that the imprecision of firms' inflation expectations, the magnitude of the responsiveness of their expectations to the actual inflation rate, and the degree of heterogeneity in expectations across firms are monotonically decreasing in firm size, while the deviation from the central bank's target inflation rate is monotonically increasing in firm size. This finding regarding monotonicity is novel and as we will see later, the monotonic pattern of heterogeneity is not straightforward for canonical imperfect information models where agents update their expectations following Bayes' law. We propose a plausible informational friction that is consistent with Bayesian updating and show that the friction is useful for noisy information models in reconciling these patterns.

Theoretically, many models have been proposed and explored in the literature.[8] Regarding expectations formation with an inflation target under imperfect information environments, Orphanides and Williams (2005), for example, proposed a model where agents hold

[6]Many of the empirical findings based on surveys of economists and households have pointed out similar empirical regularities (Fuhrer 2012, 2017; Ehrmann 2015; Lyziak and Palovitta 2017; Dovern and Kenny 2017; Corsello, Neri, and Tagliabracci 2019).

[7]With a slightly different theoretical framework, the responsiveness and slow adjustment were also reconciled by Eusepi, Moench, Preston, and Carvalho (2019), where firms have imperfect knowledge about the inflation target and adaptively learn about the target inflation rate.

[8]Eusepi and Preston (2018) provide an excellent review of this literature.

only imperfect knowledge of the central bank's target inflation rate and learn about it by evaluating observed economic shocks. They showed that inflation expectations become less stable if uncertainty about the target is higher. Kozicki and Tinsley (2005) analyzed the effects of permanent and transitory shocks in a model where private agents could perceive the target inflation rate differently from the true target. Demertzis and Viegi (2008, 2009) studied the stabilization effect on inflation expectations of having an explicit inflation target. Eusepi and Preston (2010) analyzed the effect of central bank communication about its targeting principles, including policy rules and a numerical target rate of inflation, on expectations stabilization. Hommes and Lustenhouwer (2019) investigated the effect of inflation targeting when credibility depends on the historical performance of achieving goals using a New Keynesian model with heterogeneous and bounded-rational expectations. Eusepi, Moench, Preston, and Carvalho (2019) proposed a structural model that generated persistent deviation of inflation expectations from the target inflation rate by temporarily shifting inflation expectations toward the actual inflation rate. Our paper is in the same vein but has a distinct feature, namely our model does not simply assume a lack of knowledge, but considers firms' fully rational information processing in a Bayesian manner when they attempt to learn about and eliminate the persistent component of the noise in the central bank's signals. To the authors' knowledge, this paper is the first attempt to investigate the consequences of using central bank communication to guide agents' inflation expectations toward the inflation target with a noisy information model when the central bank's prior is unknown to firms, which results in a different role for central bank communication in expectations formation. It turns out that this friction is useful in reconciling the aforementioned empirical facts.

This paper is organized as follows. Section 2 documents empirical facts using the Tankan survey. Section 3 lays out the model and shows expectations under a known prior. Section 4 explores the implications of the unknown prior for firms' expectations formation. Section 5 reconciles the empirical patterns. Section 6 concludes.

2 Stylized Empirical Facts

In this section, we present empirical facts regarding firms' expectations formation in relation to firm size that are observed in the Tankan survey of Japan. The Tankan survey is conducted by the Bank of Japan on a quarterly basis and contains around 10,000 sample firms that are chosen to represent the country-wide firm-size and industry distributions. Starting from April 2014, firms are asked about their forecasts of the inflation rates of their own business as well as the economy-wide inflation rate at an aggregate level.[9] The survey results are aggregated and presented both by firm size and by industry.

2.1 Deviation of Expectations from the Target Inflation rate

We first observe the deviations of firms' inflation expectations from the central bank's target inflation rate, i.e. a positive two percent change in the price level on a year-on-year basis. Panels (a), (b), and (c) of figure 1 show the average of Japanese firms' one-year, three-years and five-years ahead inflation expectations, respectively.

The panels indicate that firms' inflation expectations have persistently deviated from the inflation target, of two percent. In addition, they also reveal that the deviation of inflation expectations of larger firms have been larger and more persistent than those of smaller firms. This could be somewhat unpleasant for central bankers. Given that large firms have a higher ability to process information and larger exposure to macroeconomic variables, canonical rational inattention models (Sims 2003 and Mackowiak and Wiederholt 2009) imply that large firms' inflation expectations should be closer to the central bank's inflation forecasts than small firms' inflation expectations if the inflation forecasts are sufficiently credible.[10] On the contrary, our data suggest that larger firms' inflation expectations exhibit a more persistent gap with the central bank's inflation forecasts.[11]

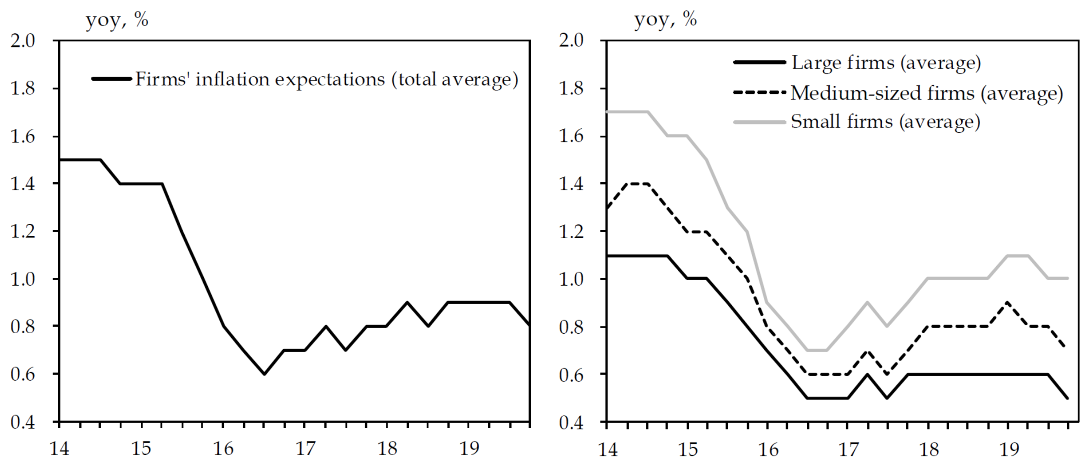
[9]The questions regarding inflation rate of firms' own businesses and economy-wide inflation are as follows: (1) Outlook for Output Prices: *Relative to the current level, what is your enterprise's expectations for the rate of change in the selling price of your main domestic products or services for one year ahead, three years ahead, and five years ahead, respectively? Please select the range nearest to your own expectation from the options below.* (2) Outlook for General Prices: *What is your enterprise's expectations of year-on-year rate of change in general prices (as measured by the Consumer Price Index) for one year ahead, three years ahead, and five years ahead, respectively? Please select the range nearest to your own expectation from the options below.* The full questionnaire and the data are publicly available here: https://www.boj.or.jp/en/statistics/tk/index.htm

[10]This is because larger firms with larger exposure to aggregate variables than to idiosyncratic variables should allocate more resources to process information regarding aggregate variables, such as economy-wide inflation rates, and fewer resources to process information regarding idiosyncratic variables. In Appendix A, we formally show that smaller firms would be less attentive to macroeconomic variables using a rational inattention framework.

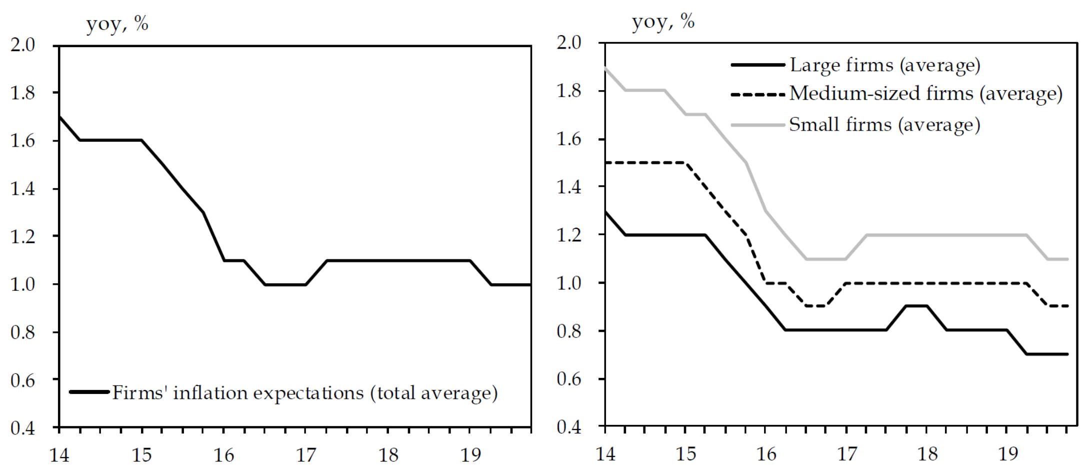
[11]The persistent deviation could be related to the so-called inflation bias in the context of time-inconsistency and incredibility of commitment policies (Kydland and Prescott 1977; Barro and Gordon 1983a, 1983b; Cukierman and Meltzer 1986; Backus and Driffill 1985; Blinder 2000; Fauset and Svensson 2001). In addition, more recent studies discuss the deflationary bias caused by the effective lower bound of policy interest rates (Eggertsson 2006; Adam and Billi 2007; Nakov 2008; Nakata and Schmidt 2016; Hills, Nakata, and Schmidt 2019). Our study highlights a distinct mechanism arising from central bank communication, which can coexist with these hypotheses.

Figure 1: Firms' inflation expectations in Japan

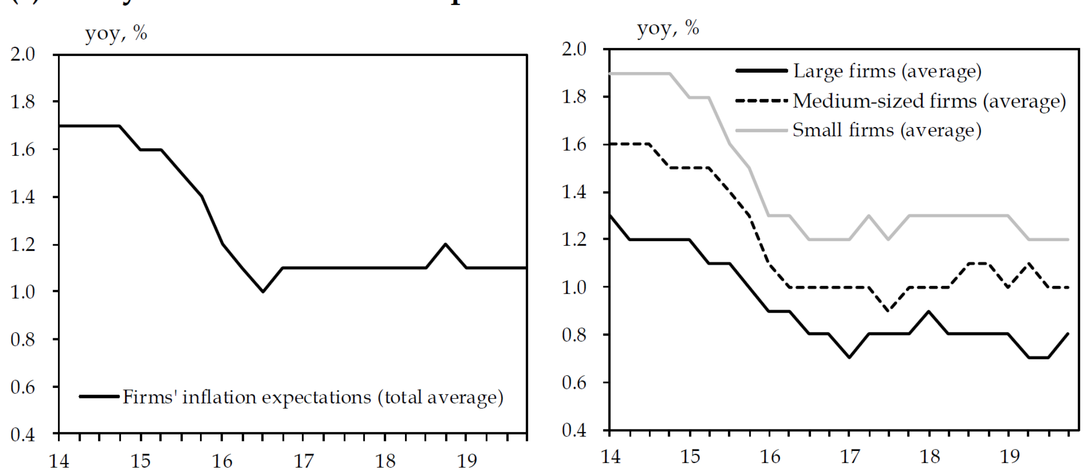
(a) One-year ahead inflation expectations



(b) Three-year ahead inflation expectations



(c) Five-year ahead inflation expectations



Source: Bank of Japan

2.2 Imprecision, Responsiveness, and Heterogeneity

Next, we observe patterns in the imprecision of inflation expectations measured by their forecasting errors, the responsiveness to the actual inflation rate, and the heterogeneity across firms.

Panels (a),(b), and (c) of figure 2 show the imprecision (averages of forecast errors) of firms' one-, three-, and five-year ahead inflation expectations across large, medium-size, and small firms. The forecast errors are defined as the Root Mean Squared Errors and the sample periods for the RMSEs of the one-, three-, and five-year ahead inflation expectations are 2014/1Q-2018/4Q, 2014/1Q-2016/4Q, and 2014/1Q-2014/4Q, respectively. They suggest that the forecast errors are monotonically decreasing in firm size.

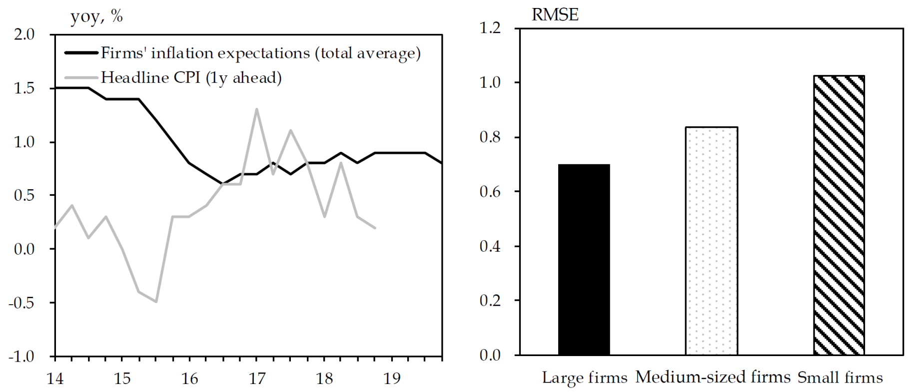
Panel (a) of figure 3 illustrates the developments in the economy-wide averages of firms' one-, three-, and five-year ahead inflation expectations during a period of declining inflation due to the plummetting of oil prices after 2014. During the period, average expectations adaptively declined along with actual inflation rates (i.e. Headline Consumer Price Indexes). Panels (b), (c) and (d) depict the cumulative changes in firms' one-, three-, and five-year ahead average inflation expectations across large, medium-sized, and small firms from June 2014 (the peak) to September 2016 (the trough). During this period, the decline in the inflation rate affected the inflation expectations of smaller firms more.

These monotonic patterns are consistent with the findings of existing empirical studies suggesting that agents who (are considered to) have less interest in macroeconomic variables exhibit less precise and adaptive expectations. Kumar, Afrouzi, Coibion, and Gorodnichenko (2015) argue that firms with more competitors tend to have more precise expectations. Using an experimental survey, Cavallo, Cruces, and Perez-Truglia (2017) show that people in high inflation countries seem to be more concerned and have harder priors about inflation than those in low inflation countries. Even within a low inflation country such as Japan, Tanaka *et al.* (2020) showed that larger firms make more accurate forecasts about aggregate variables based on the "Annual Survey of Corporate Behavior," conducted by the Cabinet Office of Japan.

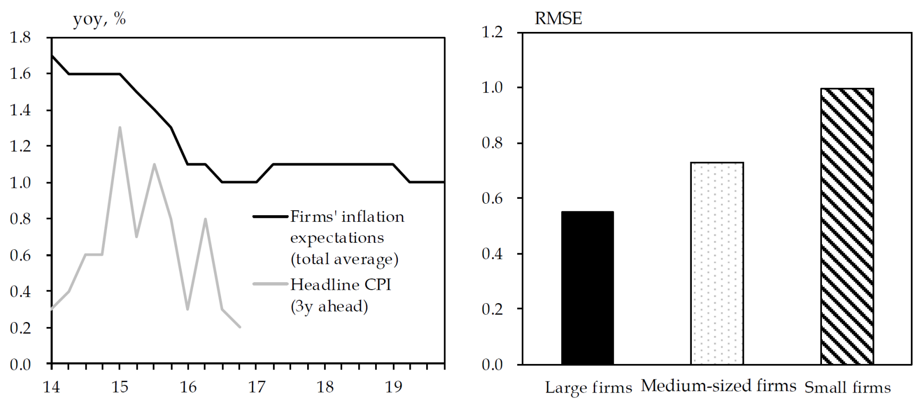
The relationship between imprecision and the responsiveness of firms' inflation expectations by firm size is consistent with canonical imperfect information models. However, canonical imperfect information models are inconsistent with our findings regarding heterogeneity. Panels (a), (b), and (c) of figure 4 show heterogeneity of firms' one-, three-,

Figure 2: Imprecision of firms' aggregate inflation expectations

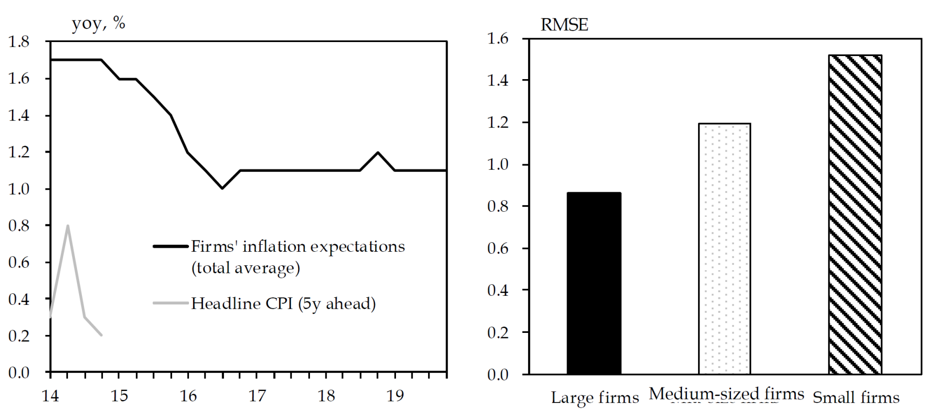
(a) One-year ahead inflation expectations



(b) Three-year ahead inflation expectations



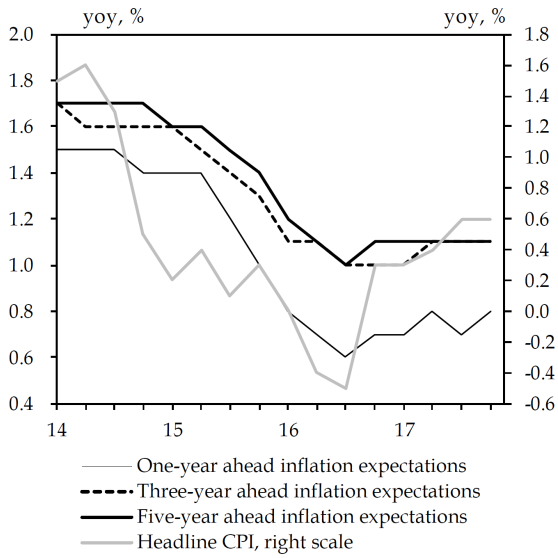
(c) Five-year ahead inflation expectations



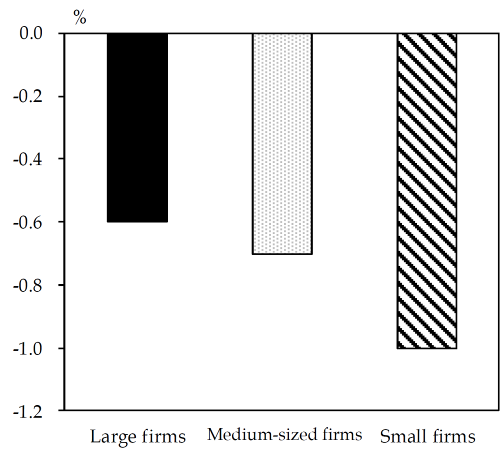
Note: The sample periods for RMSEs of one-, three-, and five-year ahead inflation expectations are 2014/1Q-2018/4Q, 2014/1Q-2016/4Q, and 2014/1Q-2014/4Q, respectively. CPI is adjusted for changes in the consumption tax rate and education fee system in public schools.
Source: Bank of Japan, Ministry of Internal Affairs and Communications

Figure 3: Responsiveness of firms' aggregate inflation expectations

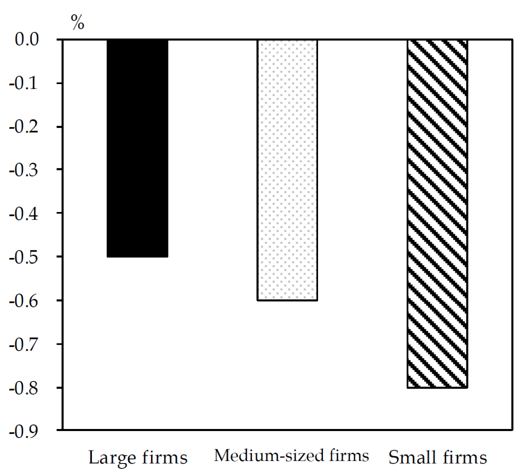
(a) Dynamics of firms' average expectations



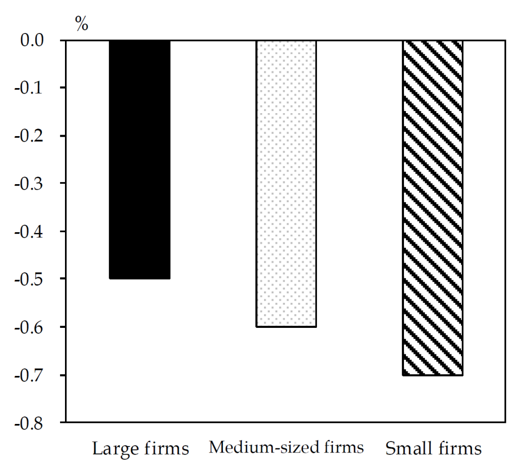
(b) Cumulative changes in firms' average expectations from June 2014 to Sep. 2016 (one-year ahead inflation expectations)



(c) Cumulative changes in firms' average expectations from June 2014 to Sep. 2016 (three-year ahead inflation expectations)



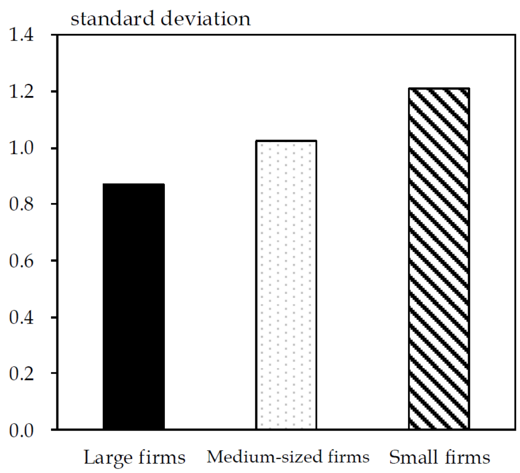
(d) Cumulative changes in firms' average expectations from June 2014 to Sep. 2016 (five-year ahead inflation expectations)



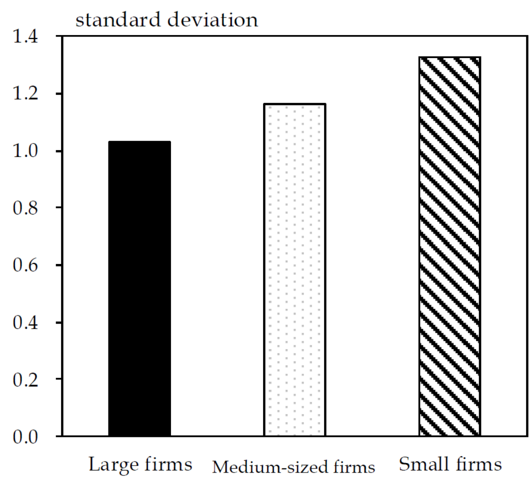
Note: CPI is adjusted for changes in the consumption tax rate and education fee system in public schools.
Source: Bank of Japan, Ministry of Internal Affairs and Communications

Figure 4: Heterogeneity of firms' aggregate inflation expectations

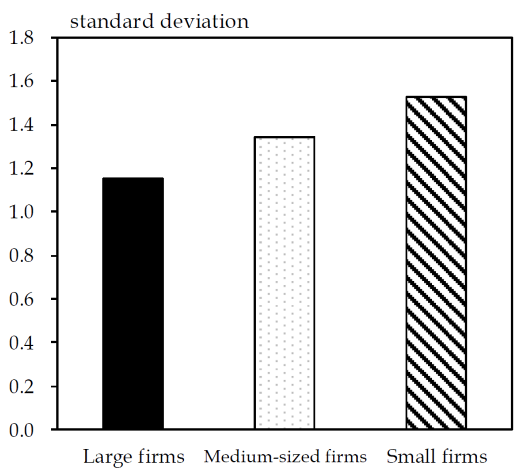
(a) One-year ahead inflation expectations



(b) Three-year ahead inflation expectations



(3) Five-year ahead inflation expectations



Note: The numbers are simple averages between 2014/1Q-2019/4Q.
Source: Bank of Japan

and five-year ahead inflation expectations across large, medium-sized, and small firms as the averages of all waves (2014/1Q-2019/4Q).The panels document that the heterogeneity is smaller across larger firms than smaller firms.[12]

This monotonicity in heterogeneity with respect to firm size is in line with Kumar, Afrouzi, Coibion, and Gorodnichenko (2015) and Coibion, Gorodnichenko, and Kamdar (2018), which find that firms with smaller information processing capacity exhibit more heterogeneous inflation expectations.[13] However, this requires further theoretical considerations. If a Bayesian firm has less precise private signals regarding aggregate variables, it should be more reliant on public signals such as the central bank's target inflation rate in its expectation formation. More reliance on public information should reduce the heterogeneity among smaller firms.[14]

3 Inflation Expectations and Central Bank Communication

To study the expectations formation mechanism behind the stylized facts, we develop a dynamic model of inflation expectations of firms where firms form expectations following Bayes' law and the central bank's inflation forecast serves as a noisy public signal about the future inflation rate.[15]

The outline of our two-period communication model is as follows. An economy is populated by a continuum of firms and a central bank. The inflation rate depends on economy-wide long-run inflation expectations, the output gap and temporary shocks (e.g., cost-push

[12] Angeletos and Lian (2018) shows that heterogeneity in agents' expectations significantly reduces the effectiveness of central bank communication.

[13] These monotonic patterns are basically maintained within each industry if industry-level disaggregated data are used (see Appendix B).

[14] The empirical patterns could be theoretically explained if the central bank's inflation target is a much noisier signal than other signals available to larger firms. In particular, large firms could virtually ignore the central bank's inflation target in making optimal decisions. However, this contradicts empirical studies, such as Romer and Romer (2000), that proved the usefulness of central bank information in predicting the inflation rate relative to private information obtained by the private sector. In the Japanese context, Fujiwara (2005) and Hattori, Kong, Packer, and Sekine (2016) suggest the influence of central bank forecasts on private agents' forecasts. In addition, a number of recent empirical studies discuss the information effect of forward guidance by central banks where private agents exploit the central bank's private information from the conduct of its monetary policy (Nakamura and Steinsson 2018 and Melosi 2017). These studies suggest the usefulness of signals sent by the central bank.

[15] This corresponds to a situation where the central bank operates a flexible inflation target under an environment where exogenous shocks and/or structural changes can occur. In such a situation, the long-term inflation rate is stochastic and time-varying.

shocks). Thus, the *actual inflation rate* (another type of *public signal*) endogenously generates information about the future inflation rate. In each period, firms observe the past inflation rate and receive *private signals* of their own and *public signals* sent by the central bank that are relevant for the fundamentals (in what follows, we refer to the signals as *CB signals*). Using the observation of past inflation rates, their own private signals, and the central bank's public signals, they form their expectations about the future inflation rate.

Next, we add one communication friction to the model. The central bank's prior about the future inflation rate (i.e. the persistent component of the noise in the *CB signal*) is unknown to firms.

With this friction, which is consistent with the information processing observed by Cavallo, Cruces, and Perez-Truglia (2016), firms attempt to identify the persistent component of the noise using their own information set, aiming to eliminate the persistent component of the noise from the central bank's signal. We model this communication friction by applying a technique employed by Sethi and Yildiz (2016, 2019) that analyzes information diffusion among people with mutually unknown priors.[16]

3.1 Set-up

Environment. An economy is populated by a central bank and various types of firms with different exposures to aggregate shocks (e.g., large firms, medium-sized firms, and small firms), indexed by $j \in [0, 1]$. Within each type, there exist a continuum of firms, indexed by the unit interval $i \in [0, 1]$. There are two periods, period 0 and period 1, and each period is indexed by $t \in \{0, 1\}$. The fundamentals related to future inflation rates in period 0 are randomly drawn and denoted as $\theta_{\infty|0} \in \mathbb{R}$, and the environment in period 1 is determined as

$$\theta_{\infty|1} = \theta_{\infty|0} + \varepsilon_1,$$

where $\varepsilon_1 \sim \mathcal{N}(0, \zeta^2)$ is the persistent stochastic shock in period t. ε_1 can be affected by various factors, including monetary policy and exogenous shifts in economic structures.[17] In what follows, for the sake of notational simplicity, we denote $\theta_{\infty|t}$ as θ_t.

Given θ_t, firms form inflation expectations about θ_t. We define

$$\overline{\mathbb{E}}[\theta_t|\mathbb{I}_t(j)] \equiv \int_{i\in[0,1]} \mathbb{E}[\theta_t|\mathbb{I}_t(i,j)]di,$$

[16] They name the prior "*perspectives*."

[17] Our focus here is on a communication problem, namely the transmission of information about $\theta_{\infty|t}$ after it has already been drawn.

as the average of inflation expectations based on the information set of firm i of type j in period t. Define the economy-wide inflation expectation as,

$$\pi_t^e \equiv \int_{j\in[0,1]} \overline{\mathbb{E}}[\theta_t|\mathbb{I}_t(j)]dj.$$

The inflation rate is then determined following an expectations-augmented Phillips curve:[18]

$$\pi_t = \pi_t^e + \kappa y_t + \xi_t,$$

where $y_t \in \mathbb{R}$ is the output gap, κ is the slope of the Phillips curve and $\xi_t \in \mathbb{R}$ is a cost-push shock following the distribution $\xi_t \sim \mathcal{N}\left(0, \widetilde{\delta}^2\right)$. Importantly, because θ_t is linearly mapped into π_t^e, firms can obtain information about θ_t based on their observation of π_t.[19] For example, the mapping can be expressed as,

$$\pi_t^e = \theta_t + \Upsilon_t,$$

where Υ_t is the noise in economy-wide average expectation. None of the firms knows the value of Υ_t but all of them know that Υ_t follows $\Upsilon_t \sim \mathcal{N}\left(0, \omega^2\right)$ because information structures are assumed common knowledge to all firms. For the sake of analytical simplicity, we suppose that firms have no knowledge of ξ_t. Firms are then able to extract information about θ_t as follows.

$$\pi_t = \theta_t + \Upsilon_t + \xi_t \Leftrightarrow \pi_t \sim \mathcal{N}\left(\theta_t, \delta^2\right),$$

where $\delta^2 \equiv \widetilde{\delta}^2 + \omega^2$. In what follows, we focus on firms of a particular type and define $\overline{\mathbb{E}}[\theta_t|\mathbb{I}_t] \equiv \overline{\mathbb{E}}[\theta_t|\mathbb{I}_t(j)]$.

Information structures and frictions in communication. In addition to π_t, two types of signals are assumed to exist in each period. One is the private signal held by firms for $t \in \{0, 1\}$, and is denoted by

$$x_t(i) = \theta_t + \eta_t(i),$$

where $\eta_t(i) \sim \mathcal{N}\left(0, \tau^2\right)$. τ is assumed to be different across firm type. It could be interpreted as the parameter describing firms' lack of incentive to collect information about aggregate variables. In particular, a larger τ indicates less attention to θ_t.

[18] By viewing inflation expectations as trend inflation, we can also interpret this setting as one type of unobserved component model which is more micro-founded (Stock and Watson 2007, Cogley and Sbordone 2008 and Mertens 2010). We can also interpret this as a New Keynesian Phillips curve.

[19] This mechanism follows Amador and Weill (2010, 2012) and Gorodnichenko (2010).

The other signal is provided by the central bank. The central bank has a prior about θ_t, denoted by $\theta_t \sim_{CB} \mathcal{N}(\mu_t, \upsilon^2)$ for $t \in \{0,1\}$, and the current prior depends, more or less, on the past value of the prior. For tractability, we assume that the prior follows the following random-walk process:

$$\mu_t \sim \mathcal{N}(\mu_{t-1}, \varphi^2), \tag{1}$$

where firms hold no knowledge of $\mu_0 \in \mathbb{R}$ in period 0, but they know the stochastic process of the central bank's prior.[20]

The bank also observes its private signal,

$$y_t = \theta_t + \epsilon_t,$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ and then reveals (the mean of) its posterior belief to firms as

$$\theta_t^* = \frac{\upsilon^2}{\upsilon^2 + \sigma^2} y_t + \frac{\sigma^2}{\upsilon^2 + \sigma^2} \mu_t, \tag{2}$$

for $t \in \{0,1\}$. Here, we assume that firms cannot directly observe $\{y_t, \mu_t\}$, but can only observe θ_t^*. Therefore, firms attempt to extract useful information y_t from θ_t^* by speculating about μ_t. It should be noted that we do not explicitly analyze the central bank's strategic communication policy.

Definitions. In what follows, for the sake of notational simplicity, we use the following definition of the information set: $\mathbb{I}_0(i)$, $\mathbb{I}_1'(i)$ and $\mathbb{I}_1(i)$ are three types of information sets of firm i about θ_t, that is,

$$\mathbb{I}_0(i) \equiv \{x_0(i), \theta_0^*\}, \mathbb{I}_1'(i) \equiv \{x_0(i), \pi_0, \theta_0^*, \theta_1^*\}, \mathbb{I}_1(i) \equiv \{x_0(i), x_1(i), \pi_0, \theta_0^*, \theta_1^*\}.$$

We also measure the degree of persistence of the deviation of inflation expectations from the target inflation rate as the dependency of inflation expectations in period one, $\mathbb{E}[\theta_1 | \mathbb{I}_1(i)]$, on the private signals in the previous period, $x_0(i)$, and the past inflation rate, π_0. This measure increases as inflation expectations become less responsive to the central bank's signals (θ_0^*, θ_1^*) and the newly arrived private signal in the current period $x_1(i)$. We denote the measure in the case of a known prior by $\hat{\phi}$ and that in the case of an unknown prior by ϕ.

[20] Our qualitative results remain intact as long as μ_t depends on μ_{t-1} to some extent.

3.2 Benchmark Case: Known Prior

As a benchmark, we derive firms' inflation expectations when they know the central bank's prior, i.e. firms directly observe y_t for $t \in \{0, 1\}$. In such a case, firms update their expectations based on the information set $\{x_1(i), y_1, \pi_0, x_0(i), y_0\}$ and they form inflation expectations as follows.

Proposition 1 *The inflation expectations of firm i in period 1 are given as follows:*

$$\mathbb{E}[\theta_1 | x_1(i), y_1, \pi_0, x_0(i), y_0] = \widetilde{\gamma}_1 y_1 + \widetilde{\gamma}_2 y_0 + \widetilde{\gamma}_3 x_1(i) + \widetilde{\gamma}_4 x_0(i) + \widetilde{\gamma}_5 \pi_0,$$

where,

$$\widetilde{\gamma}_1 \equiv \frac{\sigma^{-2}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}},$$

$$\begin{aligned}
\widetilde{\gamma}_2 &\equiv \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}} \frac{\sigma^{-2}}{\tau^{-2} + \delta^{-2} + \sigma^{-2}}, \\
\widetilde{\gamma}_3 &\equiv \frac{\tau^{-2}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}}, \\
\widetilde{\gamma}_4 &\equiv \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}} \frac{\tau^{-2}}{\tau^{-2} + \delta^{-2} + \sigma^{-2}}, \\
\widetilde{\gamma}_5 &\equiv \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}} \frac{\delta^{-2}}{\tau^{-2} + \delta^{-2} + \sigma^{-2}}.
\end{aligned}$$

Proof: See Appendix C.1. □

The expectations of firm i are composed of the weighted average of the five signals: the firm i's own signal $x_0(i)$, the inflation rate π_0, and the central bank's private information y_0 in period 0, as well as the firm i's own signal $x_1(i)$ and the central bank's private information y_1 in period 1. Note that the case in which firms completely ignore the *CB signals* is just an extreme case of this form as $\sigma \to \infty$.

3.3 Unknown Prior

Next, we explore the expectations formation under an unknown prior. Absent knowledge about the central bank's prior beliefs μ_1, firms learn about μ_1 in order to extract useful information y_1 from the *CB signal* θ_1^*.[21]

[21] Because $\mu_0 \in \mathbb{R}$ is completely unknown in period 0, firms cannot extract any information from *CB signals* in period 0.

Learning and information extraction process. The formation process of the central bank's expectations about the future inflation environment is given by equation (2), and thus the following equality holds:

$$\mu_0 = \theta_0^* + \frac{\upsilon^2}{\sigma^2}\left(\theta_0^* - y_0\right). \tag{3}$$

While θ_0^* is publicly observable, y_0 is not. Taking expectations with respect to information set $\mathbb{I}_1'(i)$, (3) is transformed into

$$\mathbb{E}[\mu_0|\mathbb{I}_1'(i)] = \theta_0^* + \frac{\upsilon^2}{\sigma^2}\left(\theta_0^* - \mathbb{E}[y_0|\mathbb{I}_1'(i)]\right). \tag{4}$$

Because y_0 is an unbiased signal of θ_0 and each firm is assumed to have no knowledge of μ_0, the following equality holds:

$$\mathbb{E}[y_0|\mathbb{I}_1'(i)] = \mathbb{E}[\theta_0|\ \mathbb{I}_1'(i)]. \tag{5}$$

By plugging (5) into (4), we obtain

$$\mathbb{E}[\mu_0|\mathbb{I}_1'(i)] = \theta_0^* + \frac{\upsilon^2}{\sigma^2}\left(\theta_0^* - \mathbb{E}[\theta_0|\mathbb{I}_1'(i)]\right).$$

In this way, firms estimate the persistent component of noise in the prior in each period by comparing *CB signal* θ_0^* with the firm's best guess of it ($\mathbb{E}[\theta_0|\mathbb{I}_1'(i)]$).

The imprecision of $\mathbb{E}[\mu_1|\mathbb{I}_1'(i)]$ is calculated as follows. Regarding the variance of y_0, we have:

$$\mathbb{V}[y_0|\mathbb{I}_1'(i)] = \mathbb{V}[\theta_0|\ \mathbb{I}_1'(i)] + \sigma^2. \tag{6}$$

The imprecision of firms' expectations of the central bank's information ($\mathbb{V}[y_0|\mathbb{I}_1'(i)]$) is the sum of the imprecision of firms' expectations of the inflation environment ($\mathbb{V}[\theta_0|\mathbb{I}_1'(i)]$) and the imprecision of the central bank's expectations of it (σ^2). Accordingly, using (6), we obtain,

$$\mathbb{V}[\mu_0|\mathbb{I}_1'(i)] = \left(\frac{\upsilon^2}{\sigma^2}\right)^2 \mathbb{V}[y_0|\mathbb{I}_1'(i)] = \left(\frac{\upsilon^2}{\sigma^2}\right)^2 \left[\mathbb{V}[\theta_0|\mathbb{I}_1'(i)] + \sigma^2\right].$$

Finally, from (1), $\mathbb{E}[\mu_1|\mathbb{I}_1'(i)]$ and $\mathbb{V}[\mu_1|\mathbb{I}_1'(i)]$ are obtained as follows:

$$\begin{aligned} \mathbb{E}[\mu_1|\mathbb{I}_1'(i)] &= \mathbb{E}[\mu_0|\mathbb{I}_1'(i)], \\ \mathbb{V}[\mu_1|\mathbb{I}_1'(i)] &= \mathbb{V}[\mu_0|\mathbb{I}_1'(i)] + \varphi^2. \end{aligned}$$

Given the above learning process and equation (2), firms' information extraction process based on the *CB signal* is given by

$$\mathbb{E}[y_1|\mathbb{I}_1'(i)] = \theta_1^* + \frac{\sigma^2}{\upsilon^2}\left(\theta_1^* - \mathbb{E}[\mu_1|\mathbb{I}_1'(i)]\right).$$

Note that equation (2) indicates that

$$y_1 = \theta_1^* + \frac{\sigma^2}{\upsilon^2}\left(\theta_1^* - \mu_1\right),$$

and the imprecision of $\mathbb{E}[y_1|\mathbb{I}_1'(i)]$ is thus given by

$$\mathbb{V}[y_1|\mathbb{I}_1'(i)] = \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \mathbb{V}[\mu_1|\mathbb{I}_1'(i)].$$

Combining all the equations above, we have

$$\begin{aligned}
\mathbb{E}[y_1|\mathbb{I}_1'(i)] &= \theta_1^* + \frac{\sigma^2}{\upsilon^2}\left(\theta_1^* - \mathbb{E}[\mu_1|\mathbb{I}_1'(i)]\right) \\
&= \theta_1^* + \frac{\sigma^2}{\upsilon^2}\left(\theta_1^* - \mathbb{E}[\mu_0|\mathbb{I}_1'(i)]\right) \\
&= \theta_1^* + \frac{\sigma^2}{\upsilon^2}\left(\theta_1^* - \theta_0^* - \frac{\upsilon^2}{\sigma^2}\left(\theta_0^* - \mathbb{E}[\theta_0|\mathbb{I}_1'(i)]\right)\right) \\
&= \frac{\upsilon^2 + \sigma^2}{\upsilon^2}\left(\theta_1^* - \theta_0^*\right) + \mathbb{E}[\theta_0|\mathbb{I}_1'(i)], \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}[y_1|\mathbb{I}_1'(i)] &= \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \mathbb{V}[\mu_1|\mathbb{I}_1'(i)] \\
&= \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \left(\mathbb{V}[\mu_0|\mathbb{I}_1'(i)] + \varphi^2\right) \\
&= \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \left[\left(\frac{\upsilon^2}{\sigma^2}\right)^2 \left[\mathbb{V}[\theta_0| \ \mathbb{I}_1'(i)] + \sigma^2\right] + \varphi^2\right] \\
&= \mathbb{V}[\theta_0|\mathbb{I}_1'(i)] + \sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2, \qquad (8)
\end{aligned}$$

where $\mathbb{E}[\theta_0|\mathbb{I}_1'(i)] = \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0$.

Expectations formation process. Besides the information based on the central bank's private signal y_1 described by equations (7) and (8), there exist two additional types of information. One is the information formed based on $\mathbb{E}[\theta_0|\mathbb{I}_1'(i)]$ and $\theta_1 \sim \mathcal{N}(\theta_0, \zeta^2)$, and the other is the *private signal* $x_1(i) \sim \mathcal{N}(\theta_1, \tau^2)$ obtained by the firm. By combining these three types of information optimally, firms' expectations $\mathbb{E}[\theta_1|\mathbb{I}_1(i)]$ are given as follows.

Proposition 2 *The inflation expectations of firm i in period 1 are given as follows:*

$$\mathbb{E}[\theta_1|\mathbb{I}_1(i)] = \gamma_1\theta_1^* + \gamma_2\theta_0^* + \gamma_3 x_1(i) + \gamma_4 x_0(i) + \gamma_5\pi_0,$$

where,

$$\gamma_1 \equiv \kappa\frac{\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\frac{\upsilon^2+\sigma^2}{\upsilon^2}, \gamma_2 \equiv -\kappa\frac{\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\frac{\upsilon^2+\sigma^2}{\upsilon^2},$$

$$\gamma_3 \equiv 1-\kappa, \gamma_4 \equiv \kappa\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}, \gamma_5 \equiv \kappa\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}},$$

$$\kappa \equiv \frac{\left[\frac{1}{\tau^{-2}+\delta^{-2}}+\frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1}}{\left[\frac{1}{\tau^{-2}+\delta^{-2}}+\frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1}+\tau^{-2}}.$$

Proof: See Appendix C.2. □

As is the case with the known prior, the expectations are the weighted averages of the five signals. However, the following key differences exist between the expectations in proposition 1 and those in proposition 2.

Corollary 1 *(i)* $\gamma_3+\gamma_4+\gamma_5 = 1$ *holds.*
(ii) Suppose $\zeta \to 0$ $(\theta_1 = \theta_0)$*. Then* $\gamma_1 \to 0$ *and* $\gamma_2 \to 0$ *hold.*
(iii) Suppose $\zeta \to \infty$ *(*θ_1 *is independent of* θ_0*). Then* $\gamma_5 = \frac{\delta^{-2}}{\delta^{-2}+2\tau^{-2}} > 0$.

First, the sum of the weights for the private signals $(x_1(i), x_0(i))$ and inflation rate π_0 is one $(\gamma_3+\gamma_4+\gamma_5 = 1)$ with the unknown prior, indicating that *CB signals* do not crowd out other signals. This is in contrast with the case with a known prior in which *CB signals* crowd out other signals, i.e., $\widetilde{\gamma}_3+\widetilde{\gamma}_4+\widetilde{\gamma}_5 = 1-\widetilde{\gamma}_1-\widetilde{\gamma}_2 < 1$.

Second, the *CB signals* (θ_1^*, θ_0^*) become completely uninformative if the fundamentals related to the future inflation rate is invariant $(\zeta \to 0)$.

Third, even when the fundamentals are perfectly random $(\zeta \to \infty)$ and do not depend on the fundamentals in the previous period, the *actual inflation rate* π_0, i.e. the signal about the fundamentals in the past, is incorporated into firms' inflation expectations $\mathbb{E}[\theta_1|\mathbb{I}_1(i)]$.

4 Anchoring Inflation Expectations under Unknown Prior

In this section, we show the two main implications of an unknown prior for central bank communication for the future inflation rate and firms' inflation expectations. One is the amplification effect on the sluggishness of inflation expectations. The other is the distinct role of central bank communication in firms' expectations formation.

4.1 Sluggish Adjustment of Inflation Expectations

One important implication of an unknown prior is the amplification effect regarding the dependency of expectations on the past information about fundamentals, including the actual inflation rate and firms' private signals observed in the past. With this friction, central bank communication about the future inflation environment increases the reliance of firms' expectations on past information, making the adjustment of firms' inflation expectations more sluggish than in the case in which the prior is perfectly known. In fact, this mechanism is distinct from the case in which firms receive only noisy public signals from the central bank, where *CB signals* to some extent anchor firms' inflation expectations, or the case in which firms simply ignore or are unaware of *CB signals*, where the signals do not affect firms' inflation expectations at all.

We define the reliance of firms' inflation expectations on past information under known and unknown priors as $\tilde{\phi} \equiv \tilde{\gamma}_4 + \tilde{\gamma}_5$ and $\phi \equiv \gamma_4 + \gamma_5$, respectively. We obtain the following analytical result.

Proposition 3 *(i) Suppose $\varphi \to 0$ and $\sigma \to 0$. Then $\phi > \hat{\phi}$ holds. (ii) Suppose $\sigma \to \infty$ and denote the dependencies $(\phi,\hat{\phi})$ in this case as $(\underline{\phi},\underline{\hat{\phi}})$. Then, $\phi > \underline{\phi} = \underline{\hat{\phi}} > \hat{\phi}$. (iii) ϕ is decreasing in σ. (iv) ϕ is decreasing in φ.*

Proof: See Appendix C.3. □

(i) of proposition 3 states that if the central bank's signal is perfectly informative ($\sigma \to 0$) and the bank's unknown prior is constant ($\varphi \to 0$), then inflation expectations with unknown prior adjust more sluggishly than those with a known prior. (ii) shows that, if the central bank's signal is informative, then the dependency is reduced under a known prior ($\underline{\hat{\phi}} > \hat{\phi}$). By contrast, under an unknown prior, the dependency is amplified ($\phi > \underline{\phi}$). Note that if the central bank's signal is completely uninformative, $\underline{\phi} = \underline{\hat{\phi}}$ holds, which corresponds to the case in which firms simply ignore the signal. The key result is (iii): ϕ decreases as the noise of the signal σ increases. Therefore, together with (ii), we conclude that *CB signals* could amplify the sluggish adjustment of inflation expectations under an unknown prior. Importantly, the amplification is more pronounced as the precision of the bank's signal, namely the bank's ability to acquire information, increases. (vi) claims that a more stable unknown prior amplifies the persistent deviation of firms' inflation expectations from the *CB signal*.

Figure 5 illustrates the relationship between ϕ and $\hat{\phi}$ for a wide range of σ and φ. The benchmark parametrization is $\zeta = 1, \tau = 1, \upsilon = 1, \sigma = 0.5$, and $\varphi = 1$. Panel (a) shows the relationship between (ϕ, $\hat{\phi}$) and $\sigma \in [0, 2]$. Importantly, ϕ is decreasing in σ while $\hat{\phi}$ is increasing in σ. Panel (b) shows the relationship between (ϕ , $\hat{\phi}$) and $\psi \in [0, 5]$. In this case, ϕ is decreasing in ψ while $\hat{\psi}$ is invariant to ψ. The figure illustrates that if the central bank's private signals are informative and the prior is stable, then firms' inflation expectations depend more on past fundamentals.

4.2 Persistent Deviation from the Target Inflation Rate

The second important implication regards the weaker anchoring effect of central bank communication in pining down the level of inflation expectations. Under an unknown prior, the role of central bank communication in firms' formation of future inflation expectations is completely different from that under a known prior. Under an unknown prior, the expectation formation process coincides with the case in which firms utilize *CB signals* as signals not of the *level*, but of *changes* in future inflation rates.

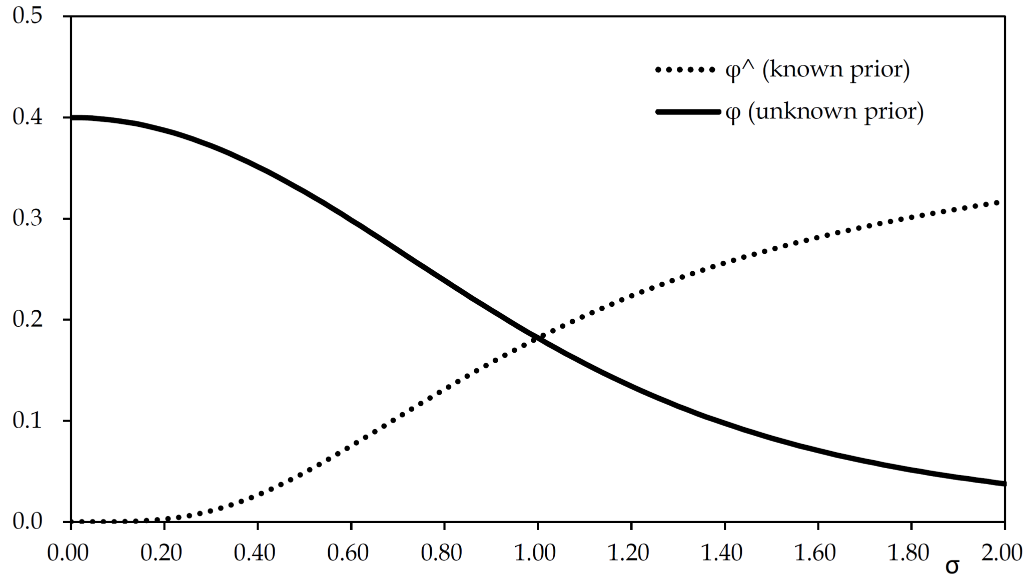
To see this, we reinterpret the information structures by focusing on the value of *CB signals*. Under an unknown prior, firms cannot extract information about θ_0 from the *CB signals* (θ_0^*, θ_1^*) themselves because

$$
\begin{aligned}
\theta_0^* &= \frac{\upsilon^2}{\upsilon^2+\sigma^2} y_0 + \frac{\sigma^2}{\upsilon^2+\sigma^2}\mu_0 = \frac{\upsilon^2}{\upsilon^2+\sigma^2}(\theta_0+\epsilon_0) + \frac{\sigma^2}{\upsilon^2+\sigma^2}\mu_0, \\
\theta_1^* &= \frac{\upsilon^2}{\upsilon^2+\sigma^2} y_1 + \frac{\sigma^2}{\upsilon^2+\sigma^2}\mu_1 = \frac{\upsilon^2}{\upsilon^2+\sigma^2}(\theta_1+\epsilon_1) + \frac{\sigma^2}{\upsilon^2+\sigma^2}\mu_1 \\
&= \frac{\upsilon^2}{\upsilon^2+\sigma^2}(\theta_0+(\theta_1-\theta_0)+\epsilon_1) + \frac{\sigma^2}{\upsilon^2+\sigma^2}(\mu_0+(\mu_1-\mu_0)),
\end{aligned}
$$

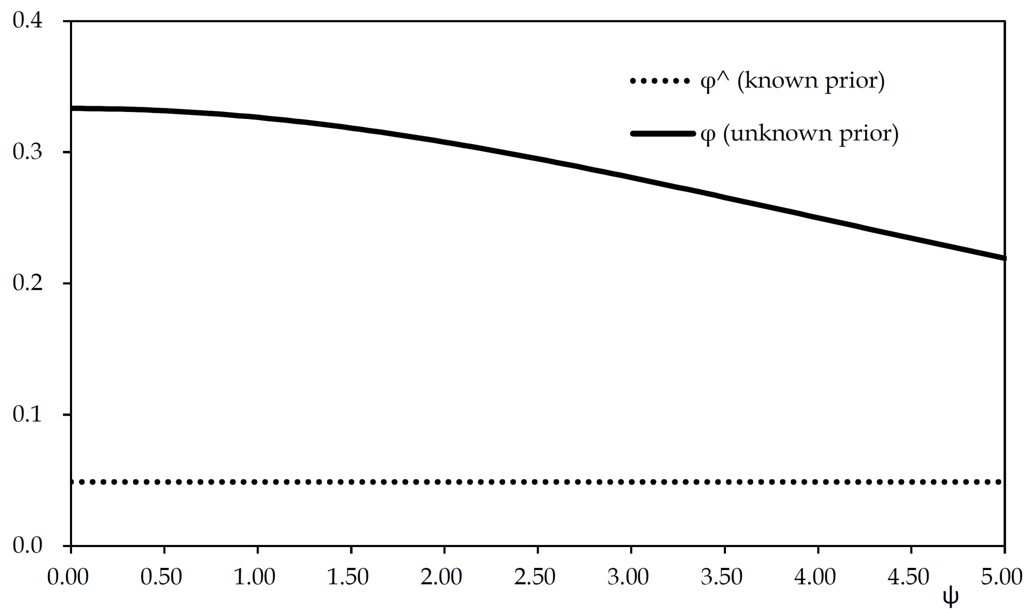
and μ_0 is a diffuse prior. However, by comparing θ_0^* and θ_1^*, firms obtain information useful

Figure 5: Firms' inflation expectations (Simulation)

(a) Imprecision of the central bank's private information (σ)



(b) Stability of the central bank's prior (ϕ)



for predicting θ_1 as follows,

$$\begin{aligned}\theta_1^* - \theta_0^* \quad &= \quad \frac{\upsilon^2}{\upsilon^2+\sigma^2}(\theta_1 - \theta_0 + \epsilon_1 - \epsilon_0) + \frac{\sigma^2}{\upsilon^2+\sigma^2}(\mu_1 - \mu_0) \\ &\Leftrightarrow \quad \theta_1 - \theta_0 = \frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^* - \theta_0^*) - (\epsilon_1 - \epsilon_0) + \frac{\sigma^2}{\upsilon^2}(\mu_1 - \mu_0) \\ &\Leftrightarrow \quad \frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^* - \theta_0^*) \sim \mathcal{N}\left(\theta_1 - \theta_0, 2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right), \end{aligned} \tag{9}$$

where $2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2 > \sigma^2$.

Under an unknown prior, the *CB signals* provide information not about the *level* of fundamentals but about *changes* in fundamentals.[22] Therefore, the information $\frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^* - \theta_0^*)$ regarding $\theta_1 - \theta_0$ becomes informative only if it is combined with firms' inflation expectations in the previous period θ_0. This interpretation provides us with the intuition behind corollary 1. Because $\theta_1 - \theta_0$ and θ_0 are independent of each other, information obtained from *CB signals* does not crowd out other information ((i) of corollary 1).

Moreover, if $\zeta \to 0$ ($\theta_1 = \theta_0$), then firms can accurately map their expectations of θ_0 into those of θ_1. In this way, information obtained from *CB signals* does not provide any additional information and thus $\gamma_1 \to 0$ and $\gamma_2 \to 0$ hold ((ii) of corollary 1).

Finally, even if $\zeta \to \infty$, namely θ_1 is independent of θ_0, firms' expectations of θ_0 become useful for inferring θ_1 because information extracted from the *CB signals* provide information about *changes* in the future inflation environment ($\theta_1 - \theta_0$), and thus firms can map the information on θ_0 into the information on θ_1 by using the extracted information ((iii) of corollary 1).

The following proposition confirms that this interpretation of the information structure provides the same results about firms' expectations as proposition 2. Interestingly, this mechanism is amplified as firms view the *CB signals* as more informative (i.e. private information of the bank is more precise and its prior is more stable).

Proposition 4 *Firm i's inflation expectations in period* 1 *formed with signals about* $\theta_0(x_0(i), \pi_0)$, $\theta_1 - \theta_0(\frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^* - \theta_0^*))$, *and* $\theta_1(x_1(i))$ *are equal to the expectations in proposition 2.*

Proof: See Appendix C.4. □

[22]Note that, under a known prior,

$$y_0 = \theta_0 + \epsilon_0, y_1 = \theta_1 + \epsilon_1,$$

and thus the signals provide useful information about the *levels* of fundamentals (θ_0, θ_1).

In what follows, we explain the mechanism behind (iii) of proposition 3 in more detail. As signal (9) shows, under an unknown prior, *CB signals* provide information about $\theta_1 - \theta_0$. Because we have $\theta_1 = \theta_0 + (\theta_1 - \theta_0)$, and θ_0 and $(\theta_1 - \theta_0)$ are independent variables, $\mathbb{E}[\theta_1|\mathbb{I}'_1(i)]$ and $\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]$ are formed as

$$\begin{aligned}
\mathbb{E}[\theta_1|\mathbb{I}'_1(i)] &= \mathbb{E}[\theta_0|\mathbb{I}'_1(i)] + \mathbb{E}[\theta_1 - \theta_0|\ \mathbb{I}'_1(i)], \\
\mathbb{V}[\theta_1|\mathbb{I}'_1(i)] &= \mathbb{V}[\theta_0|\mathbb{I}'_1(i)] + \mathbb{V}[\theta_1 - \theta_0|\ \mathbb{I}'_1(i)].
\end{aligned}$$

Moreover, the inflation expectations of firm i in period 1 ($\mathbb{E}[\theta_1|\mathbb{I}_1(i)]$) are formed as follows:

$$\begin{aligned}
\mathbb{E}[\theta_1|\mathbb{I}_1(i)] &= \frac{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1} + \tau^{-2}}\mathbb{E}[\theta_1|\mathbb{I}'_1(i)] + \frac{\tau^{-2}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1} + \tau^{-2}}x_1(i) \\
&= \frac{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1} + \tau^{-2}}\mathbb{E}[\theta_0|\mathbb{I}'_1(i)] \\
&\quad + \frac{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1} + \tau^{-2}}\mathbb{E}[\theta_1 - \theta_0|\mathbb{I}'_1(i)] \\
&\quad + \frac{\tau^{-2}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1} + \tau^{-2}}x_1(i).
\end{aligned}$$

The key observation here is that the weight on the expectations of fundamentals in the previous period ($\mathbb{E}[\theta_0|\mathbb{I}'_1(i)]$) are positively affected by the precision of signal (9). This occurs because firms can accurately map their inflation expectations in the previous period ($\mathbb{E}[\theta_0|\mathbb{I}'_1(i)]$) into fundamentals in the current period ($\mathbb{E}[\theta_1|\mathbb{I}'_1(i)]$) by exploiting the information about the changes in fundamentals ($\theta_1 - \theta_0$).

To summarize, under an unknown prior, *CB signals* facilitate firms' use of past signals in their formation process of inflation expectations. Because firms do not know the value of the persistent noise component in *CB signals*, the signal provides no information about the level of future inflation. On the other hand, by observing a sequence of *CB signals,* firms can extract useful information about changes in future inflation. This is the mechanism that allows for the persistent deviation of firms' inflation expectations from the target inflation rate. For example, if the values of *CB signals* in the first period are announced, firms cannot exploit any information since the level of the signal may be affected by the prior. However, if the signal in the second period is announced and it is close to the previous period's signal, then firms presume that the central bank's own private signals would indicate a similar value. This mechanism works even more strongly if firms view the central bank's private information as more informative.

5 Reconciling the Stylized Empirical Facts

In this section, we show that the model with an unknown prior can reconcile the empirical findings with a wider range of parameters than in the case without the friction. The learning process is the key to reconcile the stylized facts that as the firm's information processing capacity increases (i.e., the firm becomes larger) and their expectations become more precise, the responsiveness of inflation expectations to the actual inflation rate, and the heterogeneity of inflation expectations across firms become smaller. This is because larger firms tend to have larger resources to process information and rely less on the central bank's information, while firms need to extract the useful information from the central bank's signal by estimating and subtracting the persistent component of the noise in the bank's prior using their own dataset.

We define the following measures regarding the responsiveness and the heterogeneity of firms' inflation expectations.

Indicators. We set two measures: (1) the degree of *responsiveness* to the actual inflation rate

$$\frac{\partial\left(\int_{i\in[0,1]}\mathbb{E}[\theta_1|\mathbb{I}_1(i)]di\right)}{\partial\pi_0}\left(=\frac{\partial\mathbb{E}[\theta_1|\mathbb{I}_1(i)]}{\partial\pi_0}\right),$$

and (2) the *heterogeneity* of inflation expectations across firms

$$\int_{i\in[0,1]}\left(\mathbb{E}[\theta_1|\mathbb{I}_1(i)]-\int_{i\in[0,1]}\mathbb{E}[\theta_1|\mathbb{I}_1(i)]di\right)^2 di.$$

5.1 Benchmark Case: Known Prior

We first analyze the benchmark case where the central bank's prior is known to firms. We denote the responsiveness in the case in which the central bank's prior is known to firms and unknown to, firms by $\hat{\alpha}$ and α, respectively. We also denote the heterogeneity in the case in which the central bank's prior is known to firms and unknown to firms by $\hat{\beta}$ and β, respectively.

According to proposition 1, the weight on π_0 determines the responsiveness ($\hat{\alpha}=\widetilde{\gamma}_5$) and the weights on $x_1(i)$ and $x_0(i)$ affect the heterogeneity of expectations ($\hat{\beta}=\left[(\widetilde{\gamma}_3)^2+(\widetilde{\gamma}_4)^2\right]\tau^2$). We have the following results regarding the responsiveness of and heterogeneity in inflation expectations.

Proposition 5 *(i) Suppose $\zeta \to \infty$. Then the following relationships hold:*

$$\frac{\partial \hat{\alpha}}{\partial \tau} = 0 \ ,$$

and

$$\frac{\partial \hat{\beta}}{\partial \tau} \begin{cases} > 0 \text{ for } \tau < \sigma \\ = 0 \text{ for } \tau = \sigma \\ < 0 \text{ for } \tau > \sigma \end{cases} .$$

(ii) Suppose $\zeta \to 0$. Then the following relationships hold:

$$\frac{\partial \hat{\alpha}}{\partial \tau} > 0 \ ,$$

and for $\overline{\tau} \equiv \sigma \sqrt{\frac{2\delta^2}{\sigma^2 + 2\delta^2}} < \sigma$,

$$\frac{\partial \hat{\beta}}{\partial \tau} \begin{cases} > 0 \text{ for } \tau < \overline{\tau} \\ = 0 \text{ for } \tau = \overline{\tau} \\ < 0 \text{ for } \tau > \overline{\tau} \end{cases} .$$

(iii) Suppose $\zeta \in (0, \infty)$, then the following relationships hold:

$$\frac{\partial \hat{\alpha}}{\partial \tau} > 0 \ ,$$

and for $\tau \geq \sigma$,

$$\frac{\partial \hat{\beta}}{\partial \tau} < 0.$$

Proof: See Appendix C.5. □

As firms' private signals become less precise ($\tau \uparrow$), firms' expectations formation process depends less on their own *private signals* and more on *actual inflation rates* and *CB signals*. Therefore, the responsiveness of expectations ($\hat{\alpha}$), i.e. the dependency of inflation expectations on *actual inflation*, modestly increases along with τ unless the signals of future inflation rates received in the previous period are completely uninformative ($\zeta \to \infty$).

As for the degree of heterogeneity ($\hat{\beta}$), there exist two competing forces that affect the formation of expectations. One is the direct effect of the increase in heterogeneity of firms' private signals ($\tau \uparrow$). If the weight on the private signal remains unchanged, then heterogeneity increases together with the heterogeneity of *private signals*. The other is the effect of changes in τ on the weight of the *private signal*. As the *private signal* becomes more imprecise ($\tau \uparrow$), the informativeness of the signal about future inflation θ_1 and the weight of

θ_1 in the formation of expectations decreases. The key is that if $\tau \geqq \sigma$ holds, then the latter effect always dominates the former effect. Namely, as long as the central bank's private signal is more precise than firms' private signals, heterogeneity monotonically decreases with τ, which is inconsistent with our empirical observation.

5.2 Unknown Prior

Next, we explore the case where the persistent component of noise in the central bank's signal is unknown to firms. We obtain the following proposition.

Proposition 6 *Suppose (i) $\zeta \to \infty$ or (ii) $\zeta \to 0$. Then the following relationships hold:*

$$\frac{\partial \alpha}{\partial \tau} > 0,$$

and for $\delta \to \infty$ and $\varphi \to \infty$,

$$\frac{\partial \beta}{\partial \tau} > 0$$

holds in a range of $\tau \in (0, \infty)$.

Proof: See Appendix C.6. □

This proposition states that, under an unknown prior, the responsiveness (α) and the heterogeneity (β) are monotonically increasing in τ in a range of $\tau \in (0, \infty)$, consistent with observed empirical patterns. In this economy, empirically consistent patterns in the responsiveness and heterogeneity hold if the central bank does not disclose *CB signals* under a known prior. A similar information structure emerges under a known prior if the central bank announces *CB signals* and *CB signals* are sufficiently imprecise. However, if the central bank's private information is precise, then *CB signals* are precise as well and to the same extent. In such a case, the model predicts decreasing heterogeneity with respect to τ. By contrast, under an unknown prior, the information extracted from the *CB signals* can be sufficiently imprecise through the information extraction process even if a central bank's private information is precise. This endogenous amplification mechanism of the imprecision of *CB signals* plays a crucial role in generating the results in proposition 6.

5.3 Numerical Illustration

Finally, in this section we provide a numerical illustration to show our argument visually. In this exercise, we set the parameters for both panels to $\zeta = 1$ and $\delta = 2$, and the imprecision of the central bank's prior is set to $\nu = 1$.

Figure 6: Responsiveness and heterogeneity of firms' inflation expectations (Simulation)

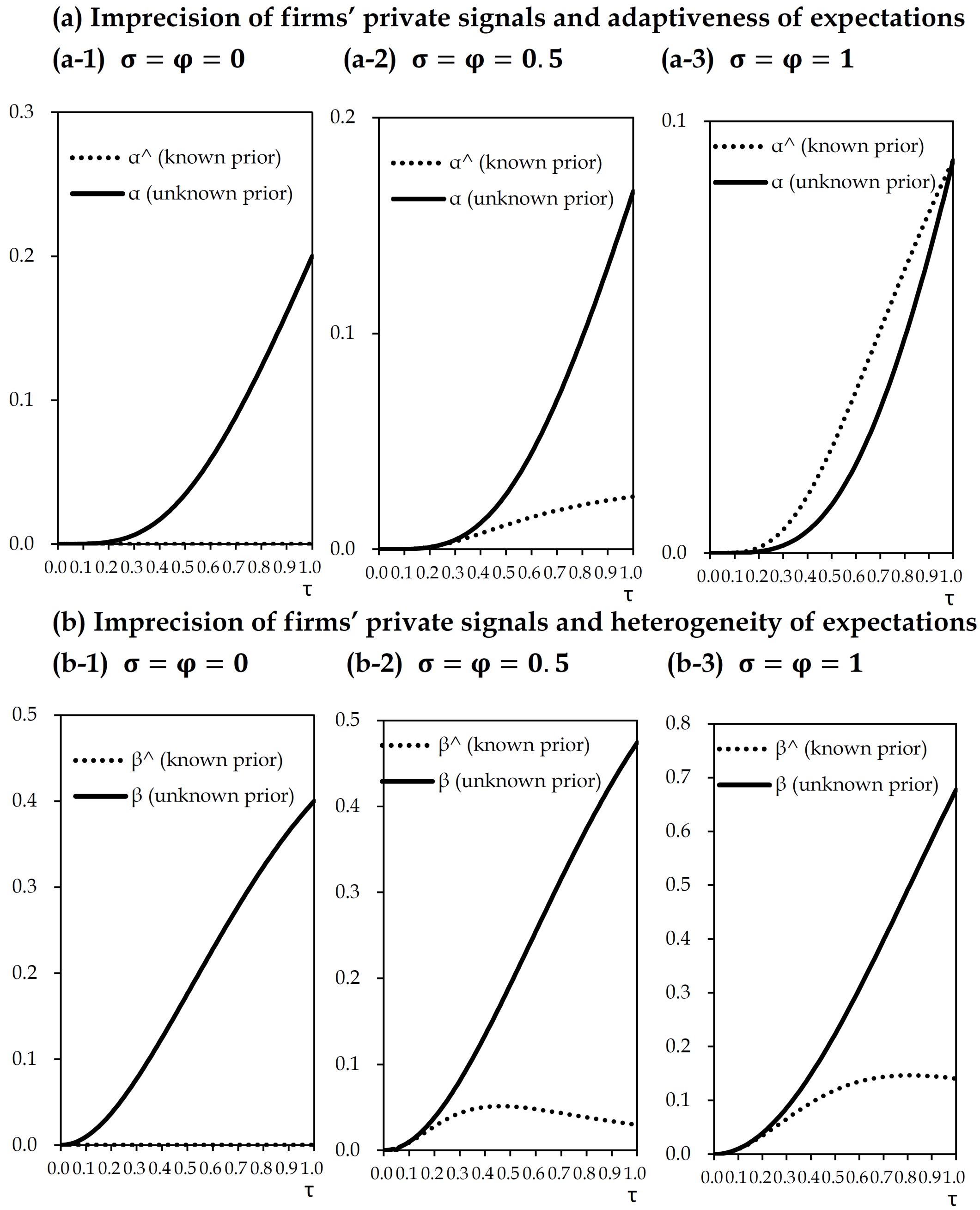


Figure 6 shows how the responsiveness (α, $\hat{\alpha}$) and the heterogeneity (β, $\hat{\beta}$) change along with the imprecision of firms' private signal τ when the central bank's prior is known and unknown, respectively. We examine the cases of $\sigma = \varphi = 0$, $\sigma = \varphi = 0.5$, and $\sigma = \varphi = 1$. In panel (a), cases (1), (2), and (3) indicate the relationship between the imprecision of firms' private signals (τ) and the responsiveness of firms' inflation expectations. If the central bank's private signal is noisy ($\sigma > 0$), then there is a monotonically upward-sloping relationship between τ and $(\alpha, \hat{\alpha})$, which is consistent with our empirical finding. However, if the central bank's signal is perfectly informative ($\sigma = 0$), then the model with a known prior fails to replicate the relationship in that $\hat{\alpha}$ is zero and invariant to τ. Therefore, the model with an unknown prior can generate empirically consistent theoretical predictions about the relationship between the imprecision and the responsiveness of inflation expectations with a wider range of parameters. In panel (b), cases (1), (2), and (3) indicate the relationship between the imprecision of firms' private signals (τ) and the responsiveness of firms' inflation expectations. The model with a known prior cannot generate a monotonically upward-sloping relationship between the imprecision of private signals (τ) and the heterogeneity (β, $\hat{\beta}$) in all cases while the model with an unknown prior can. The model with an unknown prior can explain our empirical observations about the relationship between the imprecision and the heterogeneity with a boarder range of parameters. These results support the existence of an unknown prior.

6 Concluding Remarks

In this paper, we construct a noisy information model of central bank communication on future inflation rates and highlight an informational friction that plays a key role in explaining empirical properties of firms' inflation expectations.

We document several empirical findings about firms' inflation expectations observed in Japan. The deviation from the central bank's target inflation rate is monotonically increasing in firm size, while the imprecision, the magnitude of the responsiveness of expectations to the actual inflation rate, and the heterogeneity of firms' expectations are monotonically decreasing in firm size.

To reconcile these empirical regularities, we construct a dynamic model of firms that form inflation expectations following Bayes' law where the firms gather information regarding future inflation both by themselves and from the central bank. We add an informational friction to central bank communication about the future inflation rate. Namely, the central bank's prior is unknown to firms. With this friction, central bank communication about the future inflation rate can amplify the dependency of inflation expectations on past information, make the heterogeneity of expectations monotonically increasing in the imprecision of the private signal, and hinder the anchoring effect of the communication, allowing for persistent deviations from the target level. We show that this mechanism is distinct from the cases in which firms are inattentive to the central bank's communication.

Our model can be extended in multiple directions. One extension is to develop a general equilibrium model with an unknown prior to explore the implications for macroeconomic fluctuations and social welfare. Another extension could be to endogenize the information structures following models of dynamic rational inattention (Mackowiak, Matejka, and Widerholt 2017) or information acquisition choice (Hellwig and Veldkamp 2009 and Myatt and Wallace 2012). Further, to examine the interplay between monetary policy and communication with an unknown prior is insightful as well. Such extensions may provide more helpful insights into the central bank's communication problem.

References

[1] Adam, Klaus, and Roberto Billi, 2007, "Discretionary monetary policy and the zero lower bound on nominal interest rates," *Journal of Monetary Economics* 54(3), pp. 728-752.

[2] Amador, Manuel, and Pierre-Olivier Weill, 2010, "Learning from Prices: Public Communication and Welfare," *Journal of Political Economy* 118(5), pp. 866-907.

[3] Amador, Manuel, and Pierre-Olivier Weill, 2012, "Learning from Private and Public Observations of Others' Actions," *Journal of Economic Theory* 147, pp. 910-940.

[4] Andrade, Philippe, Richard Crump, Stefano Eusepi, and Emanuel Moench, 2016, "Fundamental Disagreement," *Journal of Monetary Economics* 83, pp. 106-128.

[5] Angeletos, Gorge-Marios, and Chen Lian, 2018, "Forward Guidance without Common Knowledge," *American Economic Review* 108 (9), pp. 2477-2512.

[6] Angeletos, Gorge-Marios, and Alessandro Pavan, 2007, "Efficient Use of Information and Social Value of Information," *Econometrica* 75(4), pp. 1103-1142.

[7] Armantier, Oliver, Wandi Bruine de Bruin, Simon Potter, Giorgio Topa, Wilbert van der Klaauw, and Basit Zafar, 2013, "Measuring Inflation Expectations," *Annual Review of Economics* 5, pp. 273-301.

[8] Backus, David, and John Driffill, 1985, "Inflation and Reputation," *American Economic Review* 75(3), pp. 530-538.

[9] Barro, Robert, and David Gordon, 1983a, "A Positive Theory of Monetary Policy in a Natural-Rate Model," *Journal of Political Economy* 91(4), pp. 589-610.

[10] Barro, Robert, and David Gordon, 1983b, "Rules, Discretion and Reputation in a Model of Monetary Policy," *Journal of Monetary Economics* 12(1), pp. 101-121.

[11] Beechey, Meredith, Benjamin Johannsen, and Andrew Levin, 2011, "Are Long-Run Inflation Expectations Anchored More Firmly in the Euro Area than in the United States?" *American Economic Journal: Macroeconomics* 3, pp. 104-129.

[12] Bernanke, Ben, 2007, "Inflation Expectations and Inflation Forecasting," Speech at the Monetary Economics Workshop of the National Bureau of Economic Research Summer Institute, Cambridge, Massachusetts.

[13] Bergemann, Dirkm, and Stephen Morris, 2013, "Robust Predictions in Games With Incomplete Information," *Econometrica* 81(4), pp. 1251-1308.

[14] Bergemann, Dirk, and Stephen Morris, 2019, "Information Design: A Unified Perspective," *Journal of Economic Literature* 57 (1), pp. 44-95.

[15] Blinder, Alan, 2000, "Central-Bank Credibility: Why Do We Care? How Do We Build It?," *American Economic Review* 90(5), pp. 1421-1431.

[16] Bundick, Brent, and Lee Smith, 2018, "Does Communicating a Numerical Inflation Target Anchor Inflation Expectations? Evidence and Bond Market Implications," Federal Reserve Bank of Kansas City Research Working Papers, 18-01.

[17] Busetti, Fabio, Giuseppe Ferrero, Andrea Gerali, and Alberto Locarno. 2014, "Deflationary shocks and de-anchoring of inflation expectations," Occasional Papers 252, Bank of Italy.

[18] Candia, Bernardo, Olivier Coibion, and Yuriy Gorodnichenko, 2020, "Communication and the Beliefs of Economic Agents, " NBER Working Papers 27800.

[19] Cavallo, Alberto, Guillermo Cruces, and Ricardo Perez-Truglia, 2016, "Learning from Potentially Biased Statistics" *Brookings Papers on Economic Activity* (2016) Spring, pp. 59-108.

[20] Cavallo, Alberto, Guillermo Cruces, and Ricardo Perez-Truglia, 2017, "Inflation Expectations, Learning, and Supermarket Prices: Evidence from Survey Experiments," *American Economic Journal: Macroeconomics* 9(3), pp. 1-35.

[21] Carvalho, Carlos, Stefano Eusepi, Emanuel Moench, and Bruce Preston, 2019, "Anchored Inflation Expectations," mimeo.

[22] Cogley, Timothy, and Argia Sbordone, 2008, "Trend Inflation, Indexation, and Inflation Persistence in the New Keynesian Phillips Curve," *American Economic Review* 98(5), pp. 2101-2126.

[23] Coibion, Oliver, Yuriy Gorodnichenko, and Saten Kumar, 2018, "How Do Firms Form Their Expectations? New Survey Evidence," *American Economic Review* 108 (9), pp. 2671-2713.

[24] Coibion, Oliver, Yuriy Gorodnichenko, 2015, "Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts," *American Economic Review* 105 (8), pp. 2644-2678.

[25] Coibion, Oliver, Yuriy Gorodnichenko, and Rupal Kamdar, 2018, "The Formation of Expectations, Inflation, and the Phillips Curve," *Journal of Economic Literature*56 (4), pp. 1447-1491.

[26] Coibion, Olivier, Yuriy Gorodnichenko, Saten Kumar and Mathieu Pedemonte, 2020, "Inflation expectations as a policy tool?" *Journal of International Economics* 124C.

[27] Corsello, Francesco, Stefano Neri, and Alex Tagliabracci, 2019, "Anchored or de-anchored? That is question," Occasional Papers 516, Bank of Italy.

[28] Crawford, Vincent, and Joel Sobel, 1982, "Strategic Information Transmission," *Econometrica*50(6), pp. 1431-1451.

[29] Cukierman, Alex, and Allan Meltzer, 1986, "A Theory of Ambiguity, Credibility, and Inflation under Discretion and Asymmetric Information," *Econometrica* 54, pp. 1099-1128.

[30] Demertzis, Maria, and Nicola Viegi, 2008, "Inflation Targets as Focal Points," *International Journal of Central Banking* 4, pp. 55-87.

[31] Demertzis, Maria, and Nicola Viegi, 2009, "Inflation Targeting: A Framework for Communication," *The B.E. Journal of Macroeconomics* 9(1), pp. 1-32.

[32] Dovern, Jonas, and Geoff Kenny, 2017, "The Long-Term Distribution of Expected Inflation in the Euro Area: What has Changed Since the Great Recession?," ECB Working Paper No 1999.

[33] Dovern, Jonas, Ulrich Fritsche, and Jiri Slacalek, 2012, "Disagreement Among Forecasters in G7 Countries," *Review of Economics and Statistics* 94 (4), pp.1081-1096.

[34] Eggertsson, Gauti, 2006, "The Deflation Bias and Committing to Being Irresponsible," *Journal of Money, Credit and Banking* 38(2), pp. 283-321.

[35] Ehrmann, Michael, 2015, "Targeting Inflation from Below: How Do Inflation Expectations Behave?," *International Journal of Central Banking* 11(4), pp. 213-249.

[36] Eusepi, Stefano, and Bruce Preston, 2010, "Central Bank Communication and Macroeconomic Stabilization," *American Economic Journal: Macroeconomics* 2, pp. 235-271.

[37] Eusepi, Stefano, and Bruce Preston, 2018, "The Science of Monetary Policy: An Imperfect Knowledge Perspective," *Journal of Economic Literature* 56 (1), pp. 3-59.

[38] Eusepi, Stefano, Emanuel Moench, Bruce Preston and Carlos Viana de Carvalho, 2019, "Anchored Inflation Expectations," CEPR Discussion Papers 13900, C.E.P.R. Discussion Papers.

[39] Faust, Jon, and Lars Svensson, 2001, "Transparency and Credibility: Monetary Policy With Unobservable Goals," *International Economic Review* 42(2), pp. 369-397.

[40] Fuhrer, Jeffrey, 2012, "The Role of Expectations in Inflation Dynamics," *International Journal of Central Banking* 8(s1), pp. 137-166.

[41] Fuhrer, Jeffrey, 2017, "Expectations As a Source of Macroeconomic Persistence: Evidence from Survey Expectations in a Dynamic Macro Model," *Journal of Monetary Economics* 86, pp. 22-35.

[42] Fujiwara, Ippei, 2005, "Is the central bank's publication of economic forecasts influential?," *Economics Letters* 89(3), pp. 255-261.

[43] Fujiwara, Ippei, and Yuichiro Waki, 2015, "Private News and Monetary Policy Forward Guidance or (The Expected Virtue of Ignorance)," Federal Reserve Bank of Dallas, Globalization and Monetary Policy Institute Working Paper No. 238.

[44] Fujiwara, Ippei, and Yuichiro Waki, 2019, "Private News and Monetary Policy – Forward Guidance as Bayesian Persuasion," CAMA Working Papers 2019-91, Centre for Applied Macroeconomic Analysis, Crawford School of Public Policy, The Australian National University.

[45] Gorodnichenko, Yuriy, 2010 "Endogenous Information, Menu Costs and Inflation Persistence" mimeo.

[46] Gurkaynak, Refet, Andrew Levin, Brian Sack, and Eric Swanson, 2005, "The Sensitivity of Long-term Interest Rates to Economic News: Evidence and Implications for Macroeconomics Models," *American Economic Review* 95, pp. 425-436.

[47] Gurkaynak, Refet, Andrew Levin, and Eric Swanson, 2010, "Does Inflation Targeting Anchor Long-run Inflation Expectations? Evidence from the U.S., UK, and Sweden," *Journal of the European Economic Association* 8(6), pp. 1208-1242.

[48] Gurkaynak, Refet, Andrew Levin, Andrew Marder, and Eric Swanson, 2007, "Inflation Targeting and the Anchoring of Inflation Expectations in the Western Hemisphere," published in Frederic Mishkin and Klaus Schmidt-Hebbel (eds.), Series on Central Banking, Analysis and Economic Policies X: Monetary Policy under Inflation Targeting (Santiago, Chile: Banco Central de Chile), pp. 415-465.

[49] Haberis, Alex, Richard Harrison, and Matthew Waldron, 2017, "Uncertain forward guidance," Bank of England working papers 654, Bank of England.

[50] Hattori, Masazumi, Steven Kong, Frank Packer, and Toshitaka Sekine, 2016, "The effects of a central bank's inflation forecasts on private sector forecasts: Recent evidence from Japan," Bank of Japan Working Paper Series 16-E-11, Bank of Japan.

[51] Hellwig, Christian, and Laura Veldkamp, 2009, "Knowing What Others Know: Coordination Motives in Information Acquisition," *Review of Economic Studies* 76, pp. 223-251.

[52] Hills, Timothy, Taisuke Nakata and Sebastian Schmidt, 2019, "Effective Lower Bound Risk," *European Economic Review* 120(C).

[53] Hommes, Cars, and Joep Lustenhouwer, 2019, "Inflation Targeting and Liquidity Traps under Endogenous Credibility," *Journal of Monetary Economics* 107, 2019 (March), pp. 48-62.

[54] Kamenica, Emir, and Matthew Gentzkow, 2011, "Bayesian Persuasion," *American Economic Review* 101 (6), pp. 2590-2615.

[55] Kozicki, Sharon, and Peter Tinsley, 2001, "Shifting Endpoints in the Term Structure of Interest Rates," *Journal of Monetary Economics* 47(3), pp. 613-652.

[56] Kumar, Saten, Hassan Afrouzi, Olivier Coibion, and Yuriy Gorodnichenko, 2015, "Inflation Targeting Does Not Anchor Inflation Expectations: Evidence from Firms in New Zealand," *Brooking Papers on Economic Activity* 2015 (Fall), pp. 151-225.

[57] Kydland and Prescott, 1977, "Rules Rather than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy* 85(3), pp.473-492.

[58] Levin, Andrew, Fabio Natalucci, and Jeremy Piger, 2004, "The Macroeconomic Effects of Inflation Targeting," Federal Reserve Bank of St. Louis, Review 86, pp. 51-80.

[59] Lyziak, Tomasz, and Marttia Palovitta, 2017, "Anchoring Inflation Expectations in the Euro Area: Recent Evidence Based on Survey Data," *European Journal of Political Economy* 46, pp. 52-73.

[60] Levin, Andrew, Fabio Natalucci, and Jeremy Piger, 2004, "The Macroeconomic Effects of Inflation Targeting," Federal Reserve Bank of St. Louis, Review 86, pp. 51-80.

[61] Mackowiak, Bartosz, and Mirko Wiederholt, 2009, "Optimal Sticky Prices under Rational Inattention," *American Economic Review* 99, pp. 769-803.

[62] Mackowiak, Bartosz, Filip Matejka, and Mirko Wiederholt, 2018, "Dynamic rational inattention: Analytical results," *Journal of Economic Theory* 176, pp. 650-692.

[63] Mankiw, Gregory, Ricardo Reis, and Justin Wolfers, 2004, "Disagreement about Inflation Expectations," *NBER Macroeconomics Annual* 2003, pp. 209-248.

[64] Mehrotra, Aaron, and James Yetman, 2014, "Decaying Expectations: What Inflation Forecasts Tell Us About the Anchoring of Inflation Expectations," BIS Working Papers, 464.

[65] Melosi, Leonardo, 2017, “Signaling Effects of Monetary Policy,” *Review of Economic Studies* 84(2), pp. 853-884.

[66] Mertens, Elmar, 2010, “Measuring the Level and Uncertainty of Trend Inflation,” *Review of Economics and Statistics* 98(5), pp. 950-967.

[67] Milgrom, Paul, 1981, “Good News and Bad News: Representation Theorems and Applications,” *Bell Journal of Economics* 12(2), pp. 380–391.

[68] Milgrom, Paul, and John Roberts, 1986, “Relying on the Information of Interested Parties,” *RAND Journal of Economics* 17(1), pp. 18–32.

[69] Morris, Stephen, and Hyun Song Shin, 2002. “Social Value of Public Information,” *American Economic Review* 92(5), pp. 1521-1534.

[70] Morris, Stephen, and Hyun Song Shin, 2006, “Inertia of Forward-Looking Expectations,” *American Economic Review* 96(2), pp. 152-157.

[71] Myatt, David, and Chris Wallace, 2012, “Endogenous Information Acquisition in Coordination Games,” *The Review of Economic Studies* 79, pp. 340-374.

[72] Nakamura, Emi, and Jon Steinsson, 2018, “High-Frequency Identification of Monetary Non-Neutrality: The Information Effect,” *The Quarterly Journal of Economics* 133(3), pp. 1283-1330.

[73] Nakata, Taisuke, 2015, “Credibility of optimal forward guidance at the interest rate lower bound,” FEDS Notes, Board of Governors of the Federal Reserve System. 104(C), pp. 37-47.

[74] Nakata, Taisuke, and Sebastian Schmidt, 2019, “Conservatism and Liquidity Traps,” *Journal of Monetary Economics* 104(C), pp. 37-47.

[75] Nakov, Anton, 2008, “Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate,” *International Journal of Central Banking* 4(2), pp. 73-127.

[76] Nautz, Dieter, and Till Strohsal, 2015, “Are US Inflation Expectations Re-anchored?” *Economics Letters* 127, pp. 6-9.

[77] Nakata, Taisuke, and Sebastian Schmidt, 2019, “Conservatism and Liquidity Traps,” *Journal of Monetary Economics* 104(C), pp. 37-47.

[78] Neri, Stefano, and Tiziano Ropele, 2019, “Deflationary Shocks and Inflation Target Uncertainty,” Bank of Italy Working Paper No 1230.

[79] Nautz, Dieter, Laura Pagenhardt, and Till Strohsal, 2017, “The (De-)Anchoring of Inflation Expectations: New Evidence from the Euro Area,” *The North American Journal of Economics and Finance* 40, pp. 103-115.

[80] Orphanides, Athanasios, and John Williams, 2005, "Inflation Scares and Forecast-based Monetary Policy," *Review of Economic Dynamics* 8(2), pp. 498-527.

[81] Romer, Christina, and David Romer, 2000, "Federal Reserve Information and the Behavior of Interest Rates," *American Economic Review* 90 (3), pp. 429-457.

[82] Sethi, Rajiv, and Muhamet Yildiz, 2016, "Communication with Unknown Perspectives," *Econometrica* 84(6), pp. 2029-2069.

[83] Sethi, Rajiv, and Muhamet Yildiz, 2019, "Culture and Communication," mimeo.

[84] Sims, Christopher, 2003, "Implications of Rational Inattention," *Journal of Monetary Economics* 50(3), pp. 665-690.

[85] Stock, James, and Mark Watson, 2007, "Why has U.S. inflation become harder to forecast?" *Journal of Money, Credit and Banking* 39(S1), pp. 3-33.

[86] Strohsal, Till, and Lars Winkelmann, 2015, "Assessing the Anchoring of Inflation Expectations," *Journal of International Money and Finance* 50, pp. 33-48.

[87] Tamura, Wataru, 2016, "Optimal Monetary Policy and Transparency under Informational Frictions," *Journal of Money, Credit and Banking* 48(6), pp. 1293-1314.

[88] Tamura, Wataru, 2018, "Information Design, Signaling, and Central Bank Transparency," *Journal of Money, Credit and Banking* December 2018, pp. 223-258.

[89] Tanaka, Mari, Nicholas Bloom, Joel David, and Maiko Koga, 2020, "Firm Performance and Macro Forecast Accuracy," *Journal of Monetary Economics* 114, pp. 26-41.

[90] Ui, Takashi, 2020 "LQG Information Design," mimeo.

A Firm Size and Rational Inattention

This appendix develops the simplified and extended versions of Mackowiak and Wiederholt (2009). Suppose that there is a firm that produces and sells N number of differentiated goods $i \in \{1, 2, ..., N\}$. We suppose that the firm's best response in the market for good i is symmetric and is given as follows:

$$p_i^* = \theta_m + \phi\theta_i,$$

where θ_m is the macroeconomic variable and θ_i is the (sector-specific or) idiosyncratic variable. ϕ is a parameter that determines the relative importance of the idiosyncratic variable to the macroeconomic variable.

Next, we assume the following information structure.

$$\begin{aligned} s_{1i} &= \theta_m + \epsilon_{1i}, \quad \theta_m \sim \mathcal{N}\left(0, \sigma_m^2\right), \quad \epsilon_{1i} \sim \mathcal{N}\left(0, \sigma_{\epsilon_1}^2\right), \\ s_{2i} &= \theta_i + \epsilon_{2i}, \quad \theta_i \sim \mathcal{N}\left(0, \sigma_i^2\right), \quad \epsilon_{2i} \sim \mathcal{N}\left(0, \sigma_{\epsilon_2}^2\right). \end{aligned}$$

Given the best response and the information structure, the rational inattention problem of the firm is given as follows.

$$\begin{aligned} &\min_{\widehat{\sigma}^2_{\theta_m|\epsilon_1}, \widehat{\sigma}^2_{\theta_i|\epsilon_2}} \mathbb{E}\left[\frac{\omega}{2}\sum\nolimits_{i=1}^{N}\left(p_i - p_i^*\right)^2\right] \\ \text{subject to } p_i^* &= \theta_m + \phi\theta_i \text{ and } \frac{1}{2}\log_2\left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right) + \sum\nolimits_{i=1}^{N}\frac{1}{2}\log_2\left(\frac{\sigma_i^2}{\widehat{\sigma}^2_{\theta_i|\epsilon_2}}\right) di \leq N\kappa. \end{aligned}$$

Here $\widehat{\sigma}^{-2}_{\theta_m|\epsilon_1} \equiv \sigma_m^{-2} + \sigma_{\epsilon_1}^{-2}$ and $\widehat{\sigma}^{-2}_{\theta_i|\epsilon_2} \equiv \sigma_i^{-2} + \sigma_{\epsilon 2}^{-2}$. Because

$$\begin{aligned} &\frac{1}{2}\log_2\left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right) + \sum\nolimits_{i=1}^{N}\frac{1}{2}\log_2\left(\frac{\sigma_i^2}{\widehat{\sigma}^2_{\theta_i|\epsilon_2}}\right) \leq N\kappa, \\ \Leftrightarrow &\left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right)\left(\frac{\sigma_1^2}{\widehat{\sigma}^2_{\theta_1|\epsilon_2}}\right)\cdots\left(\frac{\sigma_N^2}{\widehat{\sigma}^2_{\theta_1|\epsilon_2}}\right) \leq 2^{2N\kappa} \Leftrightarrow \left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right)\left(\frac{\sigma_i^2}{\widehat{\sigma}^2_{\theta_i|\epsilon_2}}\right)^N \leq 2^{2N\kappa}, \end{aligned}$$

and the inequality binds,

$$\widehat{\sigma}^2_{\theta_i|\epsilon_2} = \sigma_i^2\left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right)^{(1/N)}\left(\frac{1}{2^{2\kappa}}\right),$$

holds.

The loss function is given by,

$$\mathbb{E}\left[\frac{\omega}{2}\sum\nolimits_{i=1}^{N}\left(\widehat{\sigma}^2_{\theta_m|\epsilon_1}+\phi^2\widehat{\sigma}^2_{\theta_i|\epsilon_2}\right)^2\right]=\mathbb{E}\left[\frac{\omega}{2}\sum\nolimits_{i=1}^{N}\left(\widehat{\sigma}^2_{\theta_m|\epsilon_1}+\phi^2\sigma_i^2\left(\frac{\sigma_m^2}{\widehat{\sigma}^2_{\theta_m|\epsilon_1}}\right)^{(1/N)}\left(\frac{1}{2^{2\kappa}}\right)\right)^2\right],$$

and thus the first-order condition with respect to $\widehat{\sigma}^2_{\theta_m|\epsilon_1}$ is derived as,

$$\frac{\omega}{2}\sum\nolimits_{i=1}^{N}\left(1-\frac{1}{N}\phi^2\left(\frac{\sigma_m^2}{\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1}}\right)^{(1/N)}\left(\frac{\sigma_i^2}{\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1}}\right)\left(\frac{1}{2^{2\kappa}}\right)\right)^2=0$$

$$\Leftrightarrow\quad 1=\frac{1}{N}\phi^2\left(\frac{1}{2^{2\kappa}}\right)\left(\frac{\sigma_m^2}{\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1}}\right)^{(1/N)}\left(\frac{\sigma_i^2}{\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1}}\right),$$

where $\sigma_m^2/\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1}>1$.

Therefore, the attention allocation to the macroeconomic variable $(1/\widehat{\sigma}^{2*}_{\theta_m|\epsilon_1})$ is increasing in the size of the firm, denoted by N. This occurs because information about the macroeconomic factor can be used for pricing in all of the markets while information about each idiosyncratic variable can be used for pricing in only one specific market.

B Industry-level Evidence

In this appendix we confirm the empirical patterns observed in Section 2 by investigating the industry-level evidence of 28 industries. The definitions of the imprecision, responsiveness, and heterogeneity of firms' inflation expectations are equivalent to those in figures 2, 3, and 4, respectively.

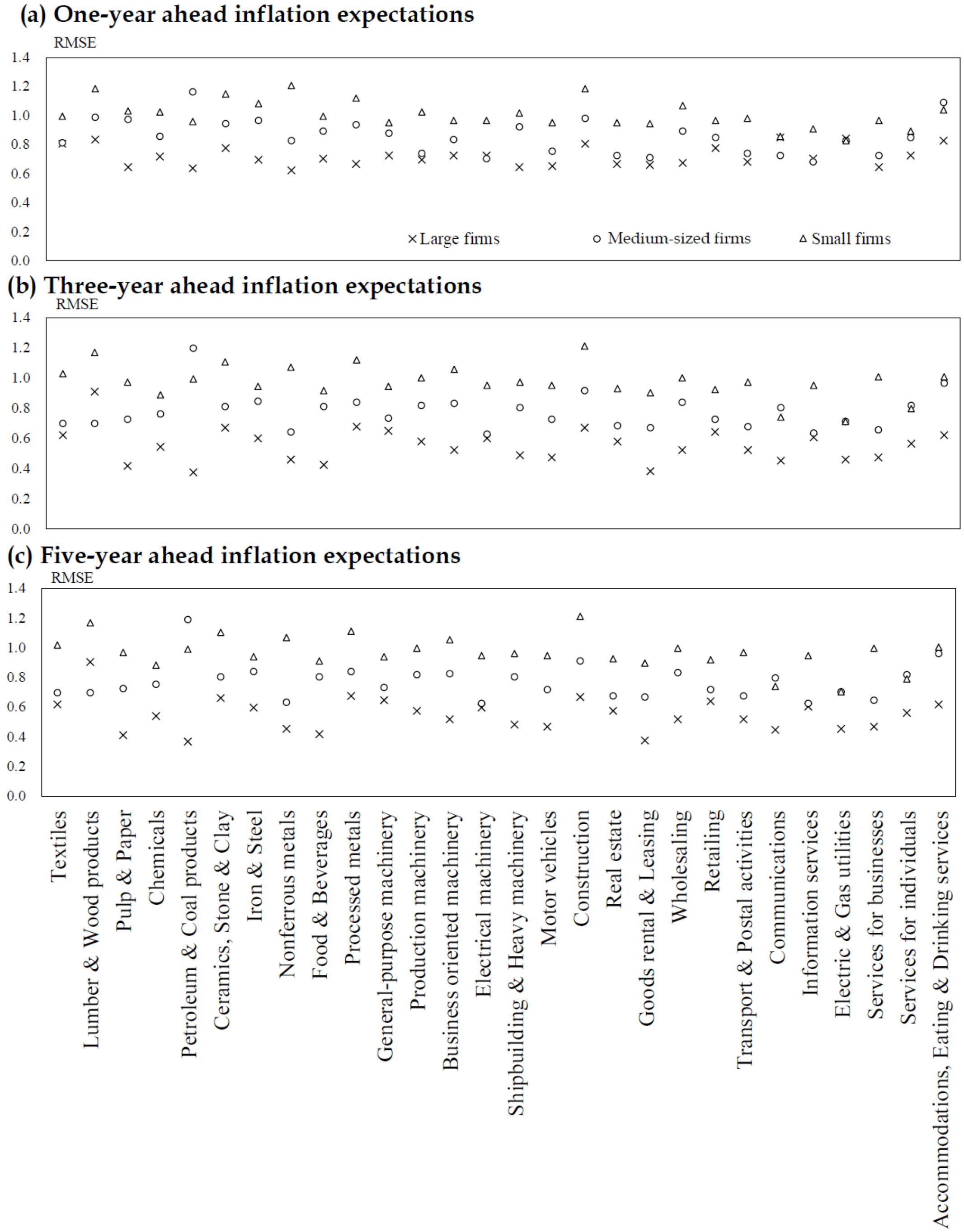
To begin with, panels (a), (b), and (c) of figure 7 show the imprecision of firms' one-, three-, and five-year ahead inflation expectations in each industry and show that smaller firms' inflation expectations tend to be more imprecise than larger firms' inflation expectations, as is observed in the aggregate evidence of figure 2.

Next, panels (a), (b), and (c) of figure 8 show the degree of responsiveness of firms' one-, three-, and five-year ahead inflation expectations in each industry and indicate that the dynamics of smaller firms' inflation expectations, indicated by the triangles, are more responsive than larger firms' inflation expectations, shown by the crosses and circles. This is consistent with the aggregate evidence of figure 3.

Finally, panels (a), (b), and (c) of figure 9 plot the degree of heterogeneity in firms' one-, three-, and five-year ahead inflation expectations in each industry. The panels indicate that smaller firms' inflation expectations, indicated by the triangles, are more heterogeneous than larger firms' inflation expectations, shown by the crosses and circles. This also matches the observations from the aggregate data of figure 4.

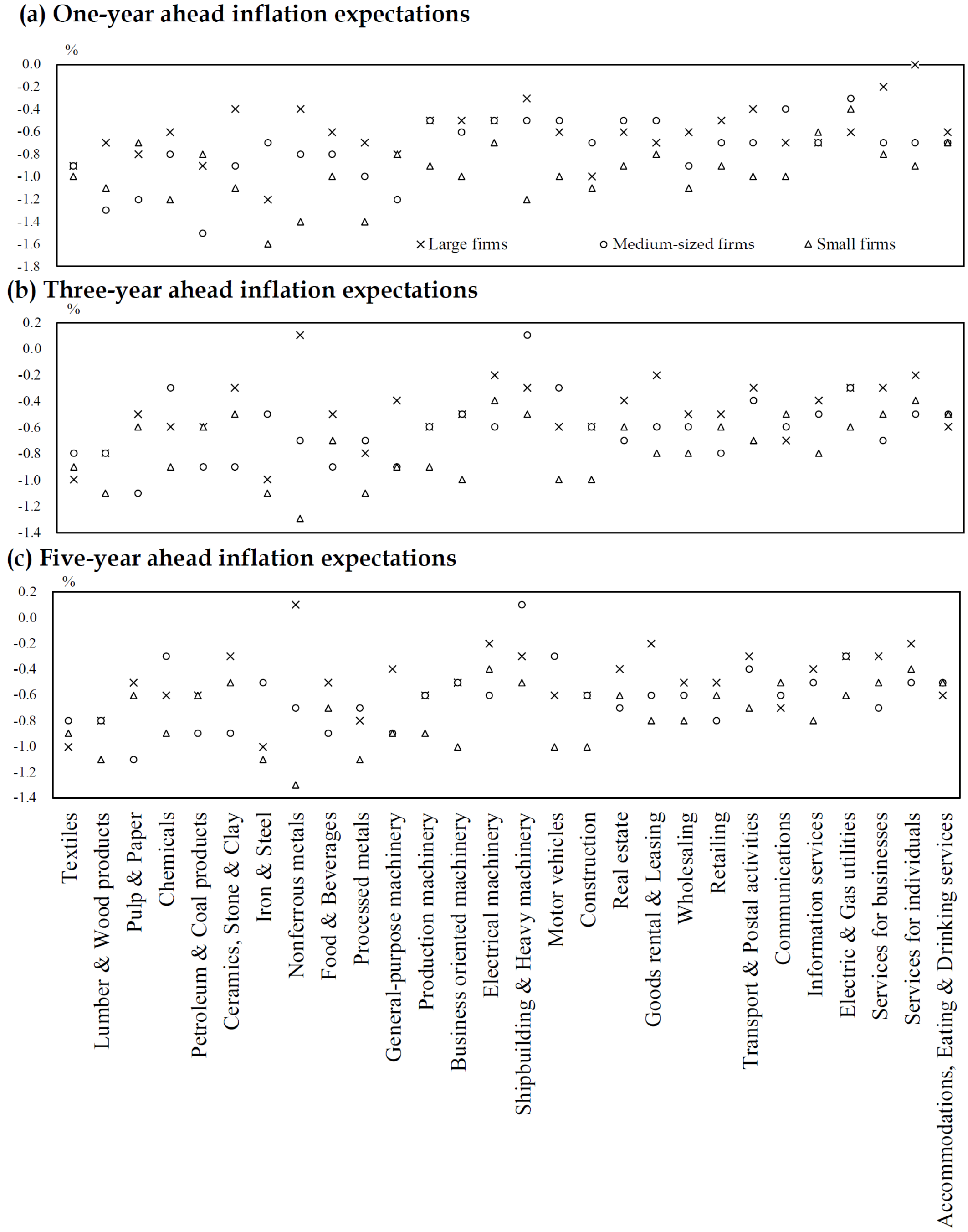
In conclusion, industry-level disaggregated data provide observations that are consistent with those in the aggregate data: smaller firms' inflation expectations are more imprecise, responsive, and heterogeneous.

Figure 7: Imprecision of firms' industry-level inflation expectations



Note: The sample periods for the RMSEs of one-, three-, and five-year ahead inflation expectations are 2014/1Q-2018/4Q, 2014/1Q-2016/4Q, and 2014/1Q-2014/4Q, respectively.
Source: Bank of Japan

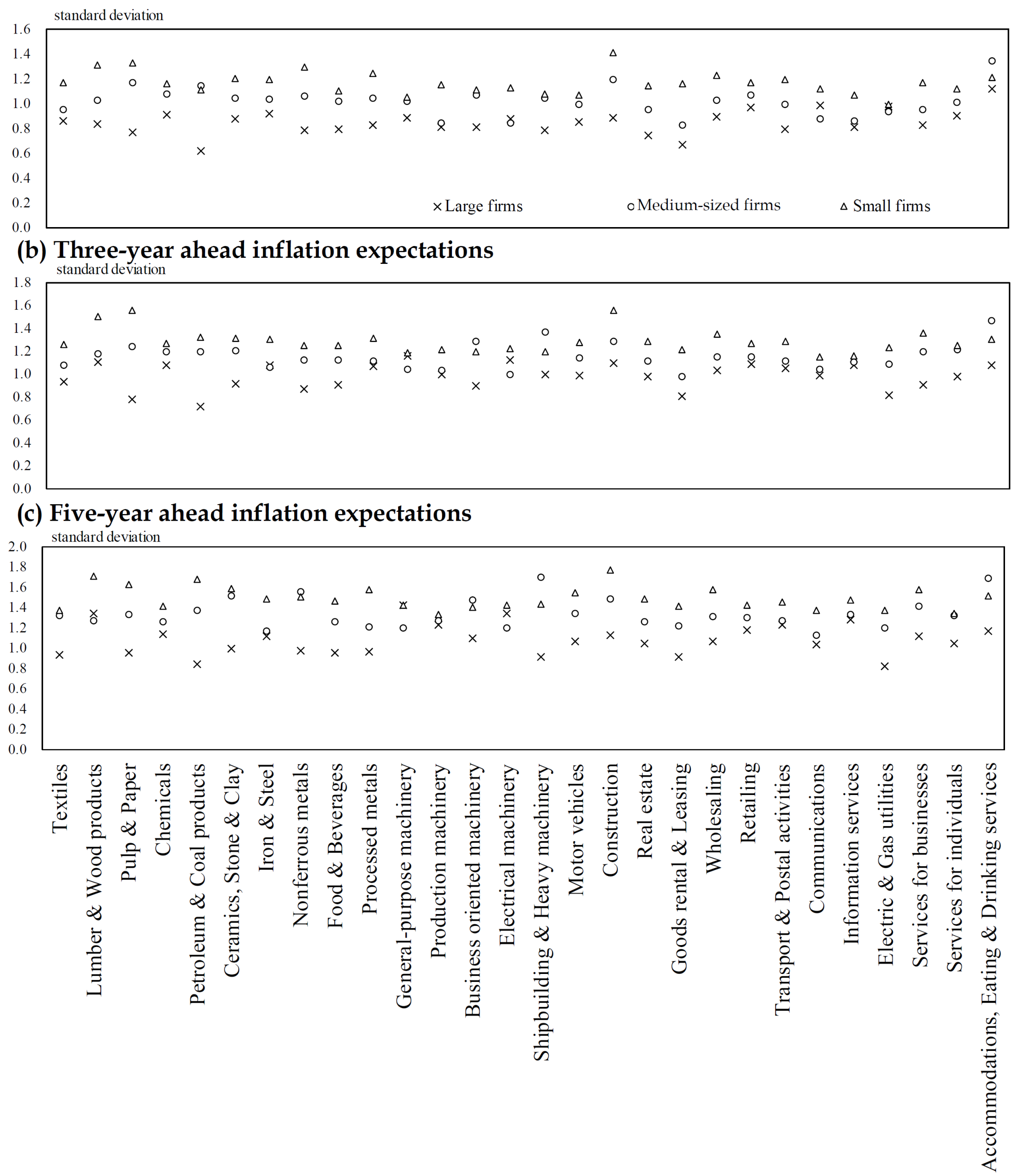
Figure 8: Responsiveness of firms' industry-level inflation expectations, cumulative changes from June 2014 to Sep. 2016



Note: CPI is adjusted for changes in the consumption tax rate and education fee system in public schools.
Source: Bank of Japan, Ministry of Internal Affairs and Communications

Figure 9: Heterogeneity of firms' industry-level inflation expectations

(a) One-year ahead inflation expectations



Note: The numbers are simple averages between 2014/1Q-2019/4Q.
Source: Bank of Japan

C Proofs

C.1 Proof of Proposition 1

First, expectations of firm i about θ_0 are calculated as,

$$\begin{aligned}
\mathbb{E}[\theta_0|x_0(i),\pi_0,y_0] &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}x_0(i)+\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}\pi_0+\frac{\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}y_0,\\
\mathbb{V}[\theta_0|x_0(i),\pi_0,y_0] &= \frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}.
\end{aligned}$$

Moreover, the following equations hold:

$$\begin{aligned}
\mathbb{E}[\theta_1|x_0(i),\pi_0,y_0] &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}x_0(i)+\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}\pi_0+\frac{\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}y_0,\\
\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0] &= \frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2.
\end{aligned}$$

Therefore, using $\mathbb{E}[\theta_1|x_0(i),\pi_0,y_0]$, expectations of firm i about θ_1 are calculated as follows.

$$\begin{aligned}
&\mathbb{E}[\theta_1|y_1,x_1(i),y_0,x_0(i),\pi_0]\\
=\ & \frac{\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0]^{-1}}{\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0]^{-1}+\tau^{-2}+\sigma^{-2}}\mathbb{E}[\theta_1|x_0(i),\pi_0,y_0]\\
&+\frac{\tau^{-2}}{\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0]^{-1}+\tau^{-2}+\sigma^{-2}}x_1(i)+\frac{\sigma^{-2}}{\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0]^{-1}+\tau^{-2}+\sigma^{-2}}y_1\\
=\ & \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}x_0(i)\\
&+\frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}\pi_0\\
&+\frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}\frac{\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}y_0\\
&+\frac{\tau^{-2}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}x_1(i)+\frac{\sigma^{-2}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}y_1,
\end{aligned}$$

$$\mathbb{V}[\theta_1|y_1,x_1(i),y_0,x_0(i),\pi_0]=\frac{1}{\mathbb{V}[\theta_1|x_0(i),\pi_0,y_0]^{-1}+\tau^{-2}+\sigma^{-2}}.\square$$

C.2 Proof of Proposition 2

First, we consider $\mathbb{E}[\theta_1|\mathbb{I}'_1(i)]$. By optimally combining the information about y_1, given by (7) and (8), and the prior formed from $\mathbb{E}[\theta_0|\mathbb{I}'_1(i)]$ and $\theta_1 \sim \mathcal{N}(\theta_0, \zeta^2)$ as,

$$\theta_1 = \theta_0 + \varepsilon_1 \Leftrightarrow \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + 0 \sim \mathcal{N}\left(\theta_1, \frac{1}{\tau^{-2}+\delta^{-2}} + \zeta^2\right),$$

$\mathbb{E}[\theta_1|\mathbb{I}'_1(i)]$ is calculated as

$$\begin{aligned}
\mathbb{E}[\theta_1|\mathbb{I}'_1(i)] &= \omega\left[\frac{\upsilon^2+\sigma^2}{\upsilon^2}\left(\theta_1^* - \theta_0^*\right) + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0\right] \\
&\quad + (1-\omega)\left[\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + 0\right] \\
&= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + \omega\frac{\upsilon^2+\sigma^2}{\upsilon^2}\left(\theta_1^* - \theta_0^*\right) + (1-\omega)\,0,
\end{aligned}$$

where

$$\mathbb{V}[\theta_1|\mathbb{I}'_1(i)] = \left(\frac{1}{\tau^{-2}+\delta^{-2}}\right)^2 + \omega^2\left(\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right) + (1-\omega)^2\zeta^2.$$

The optimal ω is determined as

$$\begin{aligned}
\frac{\partial \mathbb{V}[\theta_1|\mathbb{I}'_1(i)]}{\partial\omega} &= 2\omega\left(\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right) - 2(1-\omega)\zeta^2 = 0 \\
\Leftrightarrow \quad \omega &= \frac{\zeta^2}{\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2 + \zeta^2}.
\end{aligned}$$

Finally, $\mathbb{E}[\theta_1|\mathbb{I}_1(i)]$ is obtained as

$$\mathbb{E}[\theta_1|\mathbb{I}_1(i)] = \frac{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}+\tau^{-2}}\mathbb{E}[\theta_1|\mathbb{I}'_1(i)] + \frac{\tau^{-2}}{\mathbb{V}[\theta_1|\mathbb{I}'_1(i)]^{-1}+\tau^{-2}}x_1(i).\square$$

C.3 Proof of Proposition 3

First, $\hat{\phi} = \widetilde{\gamma}_4 + \widetilde{\gamma}_5$ is given by

$$\hat{\phi} = \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} + \zeta^2\right)^{-1} + \tau^{-2} + \sigma^{-2}}$$

and $\phi = \gamma_4 + \gamma_5 = \kappa$ is

$$\phi = \frac{\left[\frac{1}{\tau^{-2}+\delta^{-2}} + \frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1}}{\left[\frac{1}{\tau^{-2}+\delta^{-2}} + \frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1} + \tau^{-2}}.$$

Proof of (i). Suppose $\varphi \to 0$ and $\sigma \to 0$. Then

$$\hat{\phi} \to 0, \phi \to \frac{\tau^{-2}+\delta^{-2}}{2\tau^{-2}+\delta^{-2}} > 0.$$

Proof of (ii). Suppose $\sigma \to \infty$. Then

$$\underline{\hat{\phi}} = \underline{\phi} = \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}}+\zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}}.$$

We can transform $\underline{\hat{\phi}}$ and $\hat{\phi}$ as follows:

$$\begin{aligned}\underline{\hat{\phi}}\,(=\underline{\phi}) &= \frac{1}{1+\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}+\tau^{-2}\zeta^2},\\ \hat{\phi} &= \frac{1}{1+\frac{\tau^{-2}+\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\left(\tau^{-2}+\sigma^{-2}\right)\zeta^2}.\end{aligned}$$

Then, $\underline{\hat{\phi}} > \hat{\phi}$ because

$$\begin{aligned}\underline{\hat{\phi}} &> \hat{\phi} \Leftrightarrow \frac{\tau^{-2}+\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\left(\tau^{-2}+\sigma^{-2}\right)\zeta^2 > \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}+\tau^{-2}\zeta^2,\\ &\Leftrightarrow \frac{\tau^{-2}+\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\sigma^{-2}\zeta^2 > \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}},\end{aligned}$$

where $\sigma^{-2}\zeta^2 > 0$ holds by definition and $\frac{\tau^{-2}+\sigma^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} > \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}$ holds for $\delta^{-2} > 0$.

Next, we can transform ϕ as follows:

$$\phi = \frac{1}{1+\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}+\frac{\tau^{-2}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}}.$$

We can then prove $\phi > \underline{\phi}$ as follows:

$$\begin{aligned}\phi &> \underline{\phi} \Leftrightarrow \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}+\tau^{-2}\zeta^2 > \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}+\frac{\tau^{-2}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}},\\ &\Leftrightarrow \zeta^2 > \frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}} \Leftrightarrow 1+\zeta^2\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1} > 1,\end{aligned}$$

where $\zeta^2\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1} > 0$ holds.

Proof of (iii) and (vi). $\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}$ is decreasing in σ and φ and $\left[\frac{1}{\tau^{-2}+\delta^{-2}}+\frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1}$ is increasing in $\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}$.

Finally, ϕ is increasing in $\left[\frac{1}{\tau^{-2}+\delta^{-2}}+\frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right]^{-1}$.□

C.4 Proof of Proposition 4

From

$$\begin{aligned}\theta_1^*-\theta_0^* &= \frac{\upsilon^2}{\upsilon^2+\sigma^2}\left(\theta_1-\theta_0+\epsilon_1-\epsilon_0\right)+\frac{\sigma^2}{\upsilon^2+\sigma^2}\left(\mu_1-\mu_0\right)\\ &\Leftrightarrow\ \theta_1-\theta_0=\frac{\upsilon^2+\sigma^2}{\upsilon^2}\left(\theta_1^*-\theta_0^*\right)-\left(\epsilon_1-\epsilon_0\right)+\frac{\sigma^2}{\upsilon^2}\left(\mu_1-\mu_0\right),\end{aligned}$$

we have

$$\frac{\upsilon^2+\sigma^2}{\upsilon^2}\left(\theta_1^*-\theta_0^*\right)\sim\mathcal{N}\left(\theta_1-\theta_0,2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right).$$

Expectations with respect to $\mathbb{I}_1'(i)$ are given as follows:

$$\begin{aligned}\mathbb{E}[\theta_1|\mathbb{I}_1'(i)] &= \mathbb{E}[\theta_0|\mathbb{I}_1'(i)]+\mathbb{E}[\theta_1-\theta_0|\ \mathbb{I}_1'(i)]\\ &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i)+\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0+\frac{\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^*-\theta_0^*)\\ &\quad+\frac{\zeta^{-2}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}0,\end{aligned}$$

$$\begin{aligned}\mathbb{V}[\theta_1|\mathbb{I}_1'(i)] &= \left(\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\right)^2\tau^2+\left(\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\right)^2\delta^2\\ &\quad+\left(\frac{\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right)^2\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)\\ &\quad+\left(\frac{\zeta^{-2}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\right)^2\zeta^2\\ &= \frac{1}{\tau^{-2}+\delta^{-2}}+\frac{1}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}.\end{aligned}$$

Therefore, expectations with respect to $\mathbb{I}_1(i)$ are given by

$$\begin{aligned}
\mathbb{E}[\theta_1|\mathbb{I}_1(i)] &= \kappa\mathbb{E}[\theta_1|\mathbb{I}_1'(i)] + (1-\kappa)x_1(i) \\
&= \kappa\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \kappa\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 \\
&\quad +\kappa\frac{\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}{\zeta^{-2}+\left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right)^{-1}}\frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^*-\theta_0^*) + (1-\kappa)x_1(i),
\end{aligned}$$

where

$$\kappa = \frac{\mathbb{V}[\theta_1|\mathbb{I}_1'(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}_1'(i)]^{-1}+\tau^{-2}}.\square$$

C.5 Proof of Proposition 5

Proof of (i). Suppose $\zeta \to \infty$. Then, from proposition 1, we have

$$\widetilde{\gamma}_1 = \frac{\sigma^{-2}}{\tau^{-2}+\sigma^{-2}}, \widetilde{\gamma}_2 = 0, \widetilde{\gamma}_3 = \frac{\tau^{-2}}{\tau^{-2}+\sigma^{-2}}, \widetilde{\gamma}_4 = 0, \widetilde{\gamma}_5 = 0.$$

Therefore, $\hat{\alpha} = \widetilde{\gamma}_5 = 0$, and thus $\partial\hat{\alpha}/\partial\tau = 0$. Next, $\hat{\beta}$ is given by,

$$\hat{\beta} \equiv \left(\frac{\tau^{-2}}{\tau^{-2}+\sigma^{-2}}\right)^2\tau^2 = \frac{\tau^{-2}}{(\tau^{-2}+\sigma^{-2})^2},$$

and thus,

$$\begin{aligned}
\frac{\partial\hat{\beta}}{\partial\tau^{-2}} &= \frac{\sigma^{-2}-\tau^{-2}}{(\tau^{-2}+\sigma^{-2})^2} \Leftrightarrow \frac{\partial\hat{\beta}}{\partial\tau^{-2}} = \begin{cases} <0 \text{ for } \tau<\sigma \\ =0 \text{ for } \tau=\sigma \\ >0 \text{ for } \tau>\sigma \end{cases}. \\
&\Leftrightarrow \frac{\partial\beta}{\partial\tau^2} = \begin{cases} >0 \text{ for } \tau<\sigma \\ =0 \text{ for } \tau=\sigma \\ <0 \text{ for } \tau>\sigma \end{cases}.
\end{aligned}$$

Proof of (ii). Suppose $\zeta \to 0$. Then, from proposition 1, we have,

$$\begin{aligned}
\widetilde{\gamma}_1 &\equiv \frac{\sigma^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}, \widetilde{\gamma}_2 \equiv \frac{\sigma^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}, \\
\widetilde{\gamma}_3 &\equiv \frac{\tau^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}, \widetilde{\gamma}_4 \equiv \frac{\tau^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}, \\
\widetilde{\gamma}_5 &\equiv \frac{\delta^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}.
\end{aligned}$$

$\hat{\alpha}$ is given as

$$\hat{\alpha} = \widetilde{\gamma}_5 = \frac{\delta^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}},$$

which is decreasing in τ^{-2} and thus $\partial\hat{\alpha}/\partial\tau > 0$.

$\hat{\beta}$ is given as

$$\hat{\beta} = 2\left(\frac{\tau^{-2}}{\delta^{-2}+2\tau^{-2}+2\sigma^{-2}}\right)^2 \tau^2,$$

and thus the condition that $\partial\hat{\beta}/\partial\tau > 0$ is

$$\begin{aligned}\frac{\partial\hat{\beta}}{\partial\tau^{-2}} &= 2\frac{\delta^{-2}+2\sigma^{-2}-2\tau^{-2}}{(\delta^{-2}+2\tau^{-2}+2\sigma^{-2})^3} < 0 \\ &\Leftrightarrow \tau^2 < \frac{2}{\delta^{-2}+2\sigma^{-2}} = \sigma^2\frac{2\delta^2}{\sigma^2+2\delta^2} < \sigma^2.\end{aligned}$$

Proof of (iii). Suppose $\zeta \in (0,\infty)$. The responsiveness $\hat{\alpha}$ is given by,

$$\begin{aligned}\hat{\alpha} &= \widetilde{\gamma}_5 = \frac{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}}{\left(\frac{1}{\tau^{-2}+\delta^{-2}+\sigma^{-2}}+\zeta^2\right)^{-1}+\tau^{-2}+\sigma^{-2}}\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}} \\ &= \frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\sigma^{-2}+(1+(\tau^{-2}+\delta^{-2}+\sigma^{-2})\zeta^2)(\tau^{-2}+\sigma^{-2})},\end{aligned}$$

which is decreasing in τ^{-2} and thus increasing in τ^2.

Heterogeneity $\hat{\beta}$ for $\zeta \in (0,\infty)$ is given as

$$\begin{aligned}\hat{\beta} &= \left(\widetilde{\gamma}_3^2+\widetilde{\gamma}_4^2\right)\tau^2 \\ &= \frac{\tau^{-2}}{(\tau^{-2}+\sigma^{-2})^2}\frac{[(1+(\tau^{-2}+\delta^{-2}+\sigma^{-2})\zeta^2)^2+1]}{[(1+(\tau^{-2}+\delta^{-2}+\sigma^{-2})\zeta^2)+(\tau^{-2}+\delta^{-2}+\sigma^{-2})/(\tau^{-2}+\sigma^{-2})]^2}.\end{aligned}$$

The first term has the property that

$$\frac{\partial\frac{\tau^{-2}}{(\tau^{-2}+\sigma^{-2})^2}}{\partial\tau^{-2}} > 0,$$

if $\sigma^{-4} > \tau^{-4}$ and thus $\tau > \sigma$.

Next, the second term is monotonically increasing in τ^{-2}. Fix $(\tau^{-2}+\delta^{-2}+\sigma^{-2})/(\tau^{-2}+\sigma^{-2})$ and define

$$\frac{[(1+(\tau^{-2}+\delta^{-2}+\sigma^{-2})\zeta^2)^2+1]}{[(1+(\tau^{-2}+\delta^{-2}+\sigma^{-2})\zeta^2)+(\tau^{-2}+\delta^{-2}+\sigma^{-2})/(\tau^{-2}+\sigma^{-2})]^2} \equiv \frac{X^2+1}{X^2+2XY+Y^2}.$$

Then, the following inequality holds:

$$\frac{\partial\frac{X^2+1}{X^2+2XY+Y^2}}{X} > 0 \leftrightarrow 2(XY-1)(X+Y) > 0.$$

Here $X > 1$ and $Y > 1$, and thus $XY > 1$ holds. Therefore, the second terms is increasing in X even for fixed Y. In addition, Y is decreasing in X. Thus, the second term is increasing in X(i.e. τ^{-2}). Together with all the inequalities above, we obtain the conclusion that $\hat{\beta}$ is increasing in τ^{-2} and thus decreasing in τ for $\tau \geq \sigma$. □

C.6 Proof of Proposition 6

Proof of (i). If $\zeta \to 0$, then

$$\begin{aligned}\mathbb{E}[\theta_1|\mathbb{I}_1'(i)] &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0,\\ \mathbb{V}[\theta_1|\mathbb{I}_1'(i)] &= \frac{1}{\tau^{-2}+\delta^{-2}},\end{aligned}$$

and thus,

$$\begin{aligned}\mathbb{E}[\theta_1|\mathbb{I}_1(i)] &= \kappa\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \kappa\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + (1-\kappa)x_1(i)\\ &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\tau^{-2}}x_0(i) + \frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\tau^{-2}}\pi_0 + \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\tau^{-2}}x_1(i).\end{aligned}$$

Therefore,

$$\kappa = \frac{\mathbb{V}[\theta_1|\mathbb{I}_1'(i)]^{-1}}{\mathbb{V}[\theta_1|\mathbb{I}_1'(i)]^{-1}+\tau^{-2}}.$$

The responsiveness is

$$\frac{\partial\alpha}{\partial\tau^{-2}} = \frac{\partial}{\partial\tau^{-2}}\left[\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}+\tau^{-2}}\right] < 0 \Leftrightarrow \frac{\partial\alpha}{\partial\tau} > 0.$$

The heterogeneity is,

$$\beta = \left(\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}+\tau^{-2}}\right)^2\tau^2,$$

and thus, from

$$\frac{\partial\beta}{\partial\tau^{-2}} = \frac{\delta^{-2}-\tau^{-2}}{(\delta^{-2}+2\tau^{-2})^2} \gtreqless 0 \Leftrightarrow \delta^{-2} \gtreqless \tau^{-2} \Leftrightarrow \tau^2 \gtreqless \delta^2,$$

we have

$$\frac{\partial\beta}{\partial\tau}\begin{cases} >0 \text{ for } \tau<\delta\\ =0 \text{ for } \tau=\delta\\ <0 \text{ for } \tau>\delta\end{cases}.$$

Thus, for $\delta \to \infty$, $\partial\beta/\partial\tau > 0$.

Proof of (ii). If $\zeta \to \infty$, then

$$\begin{aligned}\mathbb{E}[\theta_1|\mathbb{I}_1'(i)] &= \frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + \frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^*-\theta_0^*),\\ \mathbb{V}[\theta_1|\mathbb{I}_1'(i)] &= \left(\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}\right)^2\tau^2 + \left(\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\right)^2\delta^2 + \left(2\sigma^2+\left(\frac{\sigma^2}{\upsilon^2}\right)^2\varphi^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[\theta_1|\mathbb{I}_1(i)] &= \kappa\mathbb{E}[\theta_1|\mathbb{I}_1'(i)] + (1-\kappa)x_1(i)\\ &= \kappa\frac{\tau^{-2}}{\tau^{-2}+\delta^{-2}}x_0(i) + \kappa\frac{\delta^{-2}}{\tau^{-2}+\delta^{-2}}\pi_0 + \kappa\frac{\upsilon^2+\sigma^2}{\upsilon^2}(\theta_1^*-\theta_0^*) + (1-\kappa)x_1(i),\end{aligned}$$

where

$$\kappa = \frac{\left[\frac{1}{\tau^{-2}+\delta^{-2}} + \left(2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right)\right]^{-1}}{\left[\frac{1}{\tau^{-2}+\delta^{-2}} + \left(2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right)\right]^{-1} + \tau^{-2}} = \frac{1}{1 + \tau^{-2}\left[\frac{1}{\tau^{-2}+\delta^{-2}} + \left(2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right)\right]}.$$

The responsiveness is

$$\begin{aligned} \frac{\partial \alpha}{\partial \tau^{-2}} &= \frac{\partial}{\partial \tau^{-2}} \left[\frac{\delta^{-2}}{\tau^{-2} + \delta^{-2} + \left[1 + (\tau^{-2} + \delta^{-2})\left(2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right)\right]\tau^{-2}} \right] \\ &< 0 \Leftrightarrow \frac{\partial \alpha}{\partial \tau} > 0. \end{aligned}$$

The heterogeneity is

$$\beta = \left[\kappa^2 \left(\frac{\tau^{-2}}{\tau^{-2} + \delta^{-2}}\right)^2 + (1-\kappa)^2\right]\tau^2,$$

where

$$\kappa = \frac{1}{1 + \left[\frac{1}{\tau^{-2}+\delta^{-2}} + \left(2\sigma^2 + \left(\frac{\sigma^2}{\upsilon^2}\right)^2 \varphi^2\right)\right]\tau^{-2}}.$$

For $\delta \to \infty$ and $\varphi \to \infty$, $\kappa \to 0$ and thus $\beta \to \tau^2$, which is monotonically increasing in τ.□