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Abstract

We present a state-dependent pricing model that generates inflation fluctuations from idiosyncratic shocks to the cost of price changes of individual firms. A firm's nominal price increase, lowers other firms' relative prices, thereby inducing further nominal price increases. This snow-ball effect of repricing causes fluctuations to the aggregate price level without exogenous aggregate shocks. The fluctuations caused by this mechanism are more volatile when the density of firms at the repricing threshold is high, and the density at the threshold is high when the trend inflation level is high. Thus, the model implies that higher trend inflation produces larger volatility of short-term inflation rates. Analytical and numerical analyses show that the model can account for the positive relationship between inflation level and volatility that has been observed empirically.

Keywords: Trend inflation; Inflation volatility; State-dependent pricing; Aggregate fluctuations; Self-organized criticality; Power law
JEL classification: E31

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1 Introduction

Monthly aggregate prices exhibit chronic fluctuations, but the aggregate shocks that drive these fluctuations are often elusive. Macroeconomic models often use stochastic macro-level shocks such as technology shocks or monetary policy shocks to produce these aggregate fluctuations. However, empirical counterparts of those macro-level shocks have been identified largely as residuals. Direct evidences for exogenous shocks are usually found at more disaggregated levels.

Bak et al. [5] and Scheinkman and Woodford [30] proposed a model in which aggregate fluctuations are generated by idiosyncratic shocks and endogenous correlations of firm's actions. They considered a lattice network of supply chains. Each firm has a fixed cost of producing and uses inventory as a buffer to sales shocks. When the inventory hits bottom, the firm produces goods and replenishes its inventory. They show that the inventory profile converges to a distribution at which the correlation function of firms' productions with respect to the supply-chain distance exhibits polynomial decline. Thus, the economy *self-organizes* to an inventory profile at which the aggregate fluctuations emerge endogenously. This mechanism was extended to a general equilibrium model by Nirei [27, 28].

In this paper, we apply the self-organization model for inflation fluctuations that are generated endogenously from firm-level interactions. Firms set their prices depending on their competitors' pricing and incur price-adjustment costs. A firm's upward repricing reduces all the competitors' relative prices, inducing some of them to reprice upward. Thus firms' pricing behavior in equilibrium exhibits complementarity. The strength of this propagation depends on the average number of firms that are induced to change their prices when a firm adjusts. The average number of affected firms is increasing in the trend inflation rate because the stationary density at the threshold is increasing in the stationary inflation rate. We show that this higher density results in greater complementarity and higher short-run volatility. Thus our model generates a positive correlation between the long-run level of inflation and the volatility of inflation.

First observed by Okun [29], this positive correlation has been confirmed repeatedly, *e.g.*, by Judson and Orphanides [21] or Dincer and Eichengreen [15] who controlled for central bank independence and trade openness. The observation of the positive association between volatility and trend inflation is used to partially justify central banks' choice of low inflation targets. In addition to the trend-volatility correlation, our model generates other testable implications such as that the inflation volatility positively depends on the elasticity of substitution between goods and negatively depends on the economy size.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 builds a tractable state-dependent pricing model with linear production and solves for the pricing policy and the stationary distribution of relative prices when the number of firms tends to infinity. We show the existence of a unique stationary equilibrium for each long-run inflation level and produce key comparative statics results. Section 4 demonstrates that, in the case of a finite number of firms, the model generates substantial fluctuations of inflation rates around the long-run level. Key to the analysis is the degree of complementarity between firms' repricing behavior, which generates a power-law distribution of the propagation size and slows down the canceling effects from the law of large numbers. Using the comparative-statics result that an increase in long-run inflation leads to a higher degree of complementarity, we establish that higher inflation causes higher volatility in inflation. In Section 5 we calibrate model parameters and investigate the magnitude of the positive association between inflation level and volatility for a wide range of inflation levels. Section 6 concludes.

2 Related Literature

A standard argument is that the law of large numbers precludes aggregate fluctuations as a result of direct aggregation of the micro-level independent shocks. However, a recent literature demonstrates that the micro shocks may produce macro shocks in an economy in which some firms or industries are very large. Gabaix [18], an early contributor, dubbed this mechanism the "granular hypothesis." Acemoglu et al. [1] show that input-output relationships can produce industries that have a very large influence in the overall economy. The granular hypothesis implies that the origin of aggregate shocks should be identifiable as a set of micro shocks on large firms or key industries.[1] Nonetheless, estimates by Gabaix [18] or Carvalho and Grassi [13] leave the majority of aggregate shocks still unexplained.

There is an extensive literature on inflation volatility. Common wisdom is to target the annual inflation rate at about 2%, and even researchers who advocate higher inflation targets do not suggest inflation rates higher than 4% (e.g., Ball [8]), because the higher inflation rate would bring high social costs. Along with the redistribution effects of inflation, greater dispersion of relative prices and greater volatility or uncertainty of inflation rates are commonly cited as such costs (Fischer and Modigliani [17]). The connection between the relative price dispersion and inflation is well established and its welfare implications are extensively discussed (see, e.g., Golosov and Lucas [20]; Burstein and Hellwig [9]; Nakamura and Steinsson [26]). In contrast, the positive asso-

[1]In a recent paper, Moran and Bouchaud [24] examine the effect of replacing the Cobb-Douglas production function in [1] by a CES with "small" degree of substitutability, while not assuming the existence of "key industries." They show that near critical parameter values these networks generate power-tail size distributions of firms and that small idiosyncratic shocks cause aggregate output losses that have a power law distribution.

ciation between inflation level and volatility has been little explored in state-dependent pricing models.[2] A New Keynesian explanation was provided by Ball, Mankiw and Romer [7]. In their model, a high level of inflation induces frequent repricing and more flexible aggregate prices, under which exogenous demand shocks generate large volatility of aggregate price and small volatility of output. This particular consequence of high inflation would have a positive impact on social welfare.

The impact of trend inflation has been investigated mostly in a time-dependent pricing model (see Ascari and Sbordone [4] for a survey). However, the lack of microfoundation for repricing frequency in the time-dependent models has been a serious limitation for policy analysis. The state-dependent pricing model, which provides a microfoundation, has been beyond analytical analysis. The model loses tractability quickly, unless one employs linear-quadratic approximation or exogenous real wage. Our state-dependent pricing model is similar to the model in Dotsey, King and Wolman [16] and Golosov and Lucas [20] which has been analyzed mostly numerically.[3]

3 Model

3.1 Setup

This section presents a dynamic general equilibrium model with menu costs. The model is a continuous-time, continuous-state version of Dotsey et al. [16] with linear

[2]Earlier work has focused on the possibility that monetary authority chooses the combination of high-level, high-volatility inflation (see Ball [6] and references therein).

[3]The Golosov-Lucas model spawned empirical analyses by Midrigan [23] and Gagnon [19]. These papers document that repricing occurs when the current price deviates sufficiently from the desired price. Gagnon [19] also documents that a higher level of inflation is associated with more firms located at the threshold—a key mechanism of our study of endogeneous price fluctuations.

production. The stationary equilibrium of the model allows for comparative statics analytically under an infinite number of firms and without aggregate shocks. In particular, equilibrium real wage turns out to be decreasing in trend inflation rate, reflecting the misallocation effect of the relative price dispersion.

Consider that there are n firms, each of which supplies a differentiated intermediate good monopolistically. The intermediate good i is produced using labor $l_{i,t}$ linearly,

$$y_{i,t} = l_{i,t}. \tag{1}$$

A competitive sector produces composite goods using the intermediate goods as

$$Y_t = \left(\sum_{i=1}^{n} y_{i,t}^{(\eta-1)/\eta} n^{-1/\eta} \right)^{\eta/(\eta-1)}$$

where $\eta > 1$ is the elasticity of substitution. Let P_t denote the price of the final goods. Then, the optimization condition for the production of final goods leads to a demand function for intermediate goods as $y_{i,t} = p_{i,t}^{-\eta} Y_t/n$, where $p_{i,t} := P_{i,t}/P_t$ denotes the relative price of good i. The price of the final goods is $P_t = \left(\sum_{i=1}^{n} P_{i,t}^{1-\eta}/n\right)^{1/(1-\eta)}$. Thus, the inflation rate of consumer price index in this model is $\pi_t = (dP_t/dt)/P_t$.

Let W_t denote the nominal wage rate and $w_t := W_t/P_t$ the real wage that firms face. Firm i's real net revenue from production is

$$Z(p_{i,t}, w_t, Y_t) = (p_{i,t}^{1-\eta} - w_t p_{i,t}^{-\eta}) Y_t/n. \tag{2}$$

Firm i chooses its price $p_{i,t}$ to maximize the discounted value of the firm's future profit stream. There is no capital in this economy, but the prices of intermediate goods are sticky. We assume that the firm can adjust the price of its good by bearing adjustment costs $\delta Y_t/n$. We choose this specification of the adjustment cost so that the firms' pricing solution is explicitly derived. Let $\lambda_{t,t+dt}$ denote a fraction of firms that incur repricing costs in time interval $[t, t + dt)$.

In addition, we assume that the firms face a homogeneous Poisson point process with an intensity μ. At each t in which the point process jumps the firm can adjust price without paying adjustment costs. The Poisson processes are independent across firms. Thus, this model includes both elements of the state-dependent pricing and the Calvo pricing models, as in Stokey [32]. The Calvo shock is the only exogenous shock in this model. Thus, the Calvo shock represents all idiosyncratic disturbances on pricing conditions such as adjustment costs, marginal costs and productivity.

Composite good C_t is consumed by representative households whose preference is given by $\int_0^\infty e^{-\rho t} U(C_t, N_t) dt$, where ρ denotes the time discount rate and N_t labor supply. The labor market is competitive. Combining the goods demand function and the labor demand function $p_{i,t}^{-\eta}/n = y_{i,t}/Y_t = l_{i,t}/Y_t$, we obtain

$$Y_t = N_t / (\sum_{i=1}^{n} p_{i,t}^{-\eta}/n) \tag{3}$$

at the labor market equilibrium $\sum_{i=1}^{n} l_{i,t} = N_t$.

We consider the stationary equilibrium where the aggregate price P_t grows at a stationary inflation rate π for any t. We focus on the case of positive inflation rate $\pi > 0$. The deflationary case $\pi < 0$ can be symmetrically analyzed in our framework. In the stationary equilibrium firms incur repricing costs at a rate λ per unit of time. Also, in the stationary equilibrium the risk-free real rate is ρ. Since n is large and the Calvo shocks are independent, we assume that firms discount future real profits at the risk-free rate.

The intra-temporal optimality condition for households is $U_N = -U_C w$. The market clearing condition for final goods is

$$Y = C + \lambda \delta Y, \tag{4}$$

since $\lambda \delta Y$ is spent on price adjustments per-unit of time. Given w and the stationary distribution of $p_{i,t}$, the steady-state output Y is determined by these conditions. We need $\lambda\delta < 1$ so that $C > 0$. We will show later (Proposition 3) that this is indeed satisfied for large π, since $\lambda\delta$ asymptotes to $1/\eta > 1$ as $\pi \to \infty$.

Consider a stationary equilibrium and a firm at $t = 0$. Since the maximization problems are identical for all i we drop the subscript i on prices. Let T' denote the (stopping) time when the firm reprices by incurring adjustment cost $\delta Y/n$ and τ the time when the firm draws a Calvo shock. Let p' denote the relative price to which the firm reverts when it adjusts the price. The firm's value function satisfies:

$$V(p_0, w, Y) = \sup_{\{p', T'\}} E_\tau \left\{ \int_0^{T' \wedge \tau} e^{-\rho t} Z(p_t, w, Y) dt + e^{-\rho(T' \wedge \tau)} (V(p', w, Y) - \mathbf{1}_{T' < \tau} \delta Y/n) \right\}.$$

The homogeneity of the Poisson process guarantees that conditional on p_0 the optimal stopping time T is independent of the past realizations of the Poisson process. Since in a stationary equilibrium, until the firm draws a Calvo shock, the evolution of p_t is deterministic, the optimal T^* is also deterministic and the optimal p^* to which the firm reverts is independent of past-histories of Calvo shocks and prices. Thus we may optimize over deterministic stopping times and reverting-prices that are independent of past-histories of Calvo shocks and prices.

Since adjustment cost and profits (2) are linear on Y the value function is also linear on Y. Writing

$$\begin{aligned} v &:= \frac{V(p, w, Y)}{Y/n}, \\ z &:= \frac{Z(p, w, Y)}{Y/n} = p^{1-\eta} - w p^{-\eta}, \end{aligned}$$

using the exponential distribution of τ, and omitting the arguments (Y, w) we have:

$$
\begin{aligned}
v(p_0) = \sup_{\{p', T'\}} \Bigg\{ & e^{-\mu T'} \left(\int_0^{T'} e^{-\rho t} z(p_t) dt + e^{-\rho T'} (v(p') - \delta) \right) \\
& + \int_0^{T'} \mu e^{-\mu \tau} \left(\int_0^{\tau} e^{-\rho t} z(p_t) dt + e^{-\rho \tau} v(p') \right) d\tau \Bigg\} .
\end{aligned} \tag{5}
$$

Thus, the optimal pricing policy (p^*, T^*) is not affected by Y.

In a stationary equilibrium firms' repricing plan is given by a stopping time characterized by a threshold rule, because they incur fixed costs $\delta Y/n$ when adjusting prices (see, *e.g.*, Ahlin and Shintani [2]). Under $\pi > 0$, the optimal pricing policy takes the form of a one-sided regulator, in which a firm reprices upward to a price p^* when it reaches a lower threshold, denoted by $\underline{p}$. An interval $(\underline{p}, p^*]$ forms a firm's inaction region where it is optimal for the firm not to adjust the price unless the firm draws a Calvo shock. The lower threshold $\underline{p}$ must satisfy

$$
z(\underline{p}) = \rho v(p^*) - (\rho + \mu)\delta. \tag{6}
$$

Condition (6) follows because by marginally increasing $\underline{p}$, a firm loses an instantaneous profit $z(\underline{p})dt$ and a chance of not paying adjustment costs $\delta \mu dt$, on the one hand. On the other hand, the firm gains the discounted value of adjusting, $(v(p^*) - \delta)\rho dt$. Thus, the condition (6) assures that a marginal change in $\underline{p}$ does not increase the value of firm.

A stationary equilibrium when n tends to infinity is defined as aggregate variables $(\pi, w, Y, C, N, \lambda)$, the distributions of $(p_{i,t}, y_{i,t}, l_{i,t})_{i,t}$, the value function v and pricing policy $\{\underline{p}, p^*\}$, and the path of aggregate price P_t that satisfy the following conditions: (i) the allocation maximizes the household's utility given prices, (ii) the value function, pricing policy and allocation solve the firms' dynamic optimization problem given prices, (iii) markets clear for the final good, intermediate goods and labor, (iv) the distribution of relative prices $p_{i,t}$ is stationary, and (v) P_t grows at a constant rate π.

3.2 Comparative statics of stationary equilibrium

First, we derive firms' optimality conditions at the stationary equilibrium. For p_0 outside of inaction region $(\underline{p}, p^*)$, $T = 0$ holds, and thus $v(p_0) = v(p^*) - \delta$. Inside inaction region $p_0 \in (\underline{p}, p^*)$, the relative price evolves deterministically as $dp_t/dt = -\pi p_t$. Thus, inside the inaction region for a short time horizon dt, Equation (5) can be written as

$$v(p_t) = \sup_{p'} \int_t^{t+dt} e^{-\rho s} z(p_s) ds + \int_t^{t+dt} \mu e^{-(\mu+\rho)\tau} v(p') d\tau + e^{-(\mu+\rho)dt} v(p_{t+dt}).$$

For a small interval dt, the above equation is written as

$$\begin{aligned} v(p_t) =& \sup_{p'} \{(1 - \rho dt) z(p_t) dt + \mu(1 - (\mu + \rho)dt) v(p') dt \\ & +(1 - (\mu + \rho)dt)(v(p_t) + v'(p_t)(dp_t/dt)dt)\} + o(dt) \\ =& \sup_{p'} \{z(p_t)dt + \mu v(p')dt + (1 - (\mu + \rho)dt)v(p_t) - v'(p_t)p_t \pi dt\} + o(dt). \end{aligned}$$

Hence, the Hamilton-Jacobi-Bellman equation is obtained as:

$$(\rho + \mu)v(p) = \sup_{p'} [z(p) + \mu v(p') - \pi p v'(p)] = z(p) + \mu v(p^*) - \pi p v'(p), \tag{7}$$

where p^* maximizes v. The HJB equation makes it clear that the repricing point p^* does not depend on the current relative price p. Differentiating with respect to p one obtains: $(\rho + \mu)v'(p) = z'(p) - \pi v'(p) - \pi p v''(p)$ and $z'(p^*) = \pi p^* v''(p^*)$.

Given p^*, Equation (7) yields a first-order linear ODE up to an unknown constant $\mu v(p^*) = z(p^*)\mu/\rho$, which has a class of solution,

$$v(p) = c_0 p^{-(\rho+\mu)/\pi} + \frac{p^{1-\eta}}{\rho + \mu - \pi(\eta - 1)} - \frac{w p^{-\eta}}{\rho + \mu - \pi\eta} + \frac{z(p^*)\mu/\rho}{\rho + \mu}, \tag{8}$$

with unknown coefficient c_0.

The firm's optimal choice for p^* and $\underline{p}$ leads to the value matching condition

$$v(p^*) - \delta = v(\underline{p}) \tag{9}$$

and p^* must satisfy the first order condition

$$v'(p^*) = 0. \tag{10}$$

Conditions (6,8,9,10) must be satisfied by a $\{\underline{p}, p^*, c_0, v(p)\}$, which solves the firms' maximization problem given w. Let $q := p^*/\underline{p} > 1$ denote the size of price adjustment for the firm hitting $\underline{p}$. Combining (6,10) and $\rho v(p^*) = z(p^*)$, we obtain $z(p^*) - z(\underline{p}) = (\rho + \mu)\delta$, which may be rewritten as

$$(\rho + \mu)\delta = (q^{1-\eta} - 1)\underline{p}^{1-\eta} - w(q^{-\eta} - 1)\underline{p}^{-\eta} \qquad \text{[Value Matching]}. \tag{11}$$

Moreover, the optimal markup over the marginal labor cost satisfies (see the appendix for derivation):

$$p^* = q\underline{p} = \frac{\varphi(q, \eta - \frac{\rho+\mu}{\pi})}{\varphi(q, \eta - 1 - \frac{\rho+\mu}{\pi})} \frac{\eta}{\eta - 1} w \qquad \text{[Optimal Markup]} \tag{12}$$

where $\varphi(q, x) := (q^x - 1)/x$ is an integral of q^{x-1} with respect to q. Using this equation, and the fact that $\varphi(\cdot, \cdot)$ is strictly increasing in the second argument as shown in the appendix, one can check that $p^* > \hat{p} := (\eta/(\eta - 1))w$. Note that $\hat{p}$ maximizes the static profit: $z'(\hat{p}) = 0$. Thus, $z'(p) = -\eta p^{-\eta-1}(((\eta - 1)/\eta)p - w)$ is strictly negative for $p > \hat{p}$. Therefore if $v'(p) = 0$ and $p > \hat{p}$, $v''(p) = z'(p)/(\pi p) < 0$. Thus, there is at most one $p > \hat{p}$ such that $v'(p) = 0$ and this p must achieve the maximum of $v(p)$.

We now derive the stationary distribution of relative prices. Since P_t grows at $\pi > 0$, the firm's relative price $p_{i,t}$ deterministically declines at rate π unless the firm draws a

Calvo shock, while it is adjusted to p^* when $p_{i,t}$ reaches $\underline{p}$ or a Calvo shock arrives at rate μ. Define a state variable $s_{i,t} := (\log p_{i,t} - \log \underline{p})/\log q$, which denotes a firm's distance from the threshold relative price, normalized by the repricing size $\log q$. Let $f(s,t)$ denote the density of s_t over its support $(0,1]$. The state s declines by $\pi/\log q$ due to inflation, while it is reset to 1 when it reaches 0 or randomly at rate μ. Thus, $f(s,t)$ evolves according to Kolmogorov forward equation $\partial f(s,t)/\partial t = f(s+(\pi/\log q)dt, t) - f(s,t) - f(s,t)\mu dt$. The stationary distribution $f(s)$ solves $\partial f(s,t)/\partial t = 0$. Dividing both sides of the equation by dt and taking $dt \to 0$, we obtain an ordinary differential equation for the stationary distribution, $0 = f'(s)\pi/\log q - f(s)\mu$, whose solution has an exponential form. Since $f(s)$ must integrate to 1, we obtain the solution as

$$f(s) = f_o q^{s\mu/\pi}, \tag{13}$$

where $f_o := \log q/\varphi(q, \mu/\pi)$ denotes the stationary density of firms at the repricing threshold $s = 0$.

From $p_{i,t} = P_{i,t}/P_t$ and demand functions, an aggregation condition must hold:

$$\sum_{i=1}^{n} p_{i,t}^{1-\eta}/n = 1.$$

As $n \to \infty$, the left-hand side tends to $E[p_{i,t}^{1-\eta}]$ where the expectation is evaluated using the stationary distribution $f(s)$ and $p = \underline{p}(p^*/\underline{p})^s$. Applying the stationary distribution, the aggregation condition leads to

$$\underline{p}^{\eta-1} = \frac{\varphi(q, 1-\eta+\frac{\mu}{\pi})}{\varphi(q, \frac{\mu}{\pi})} \qquad \text{[Price Aggregation]}. \tag{14}$$

Hence, given w, the pricing policy $(q, \underline{p})$ is determined by (11,12), and the aggregation condition (14) puts a restriction on the policy $(q, \underline{p})$, determining q, $\underline{p}$, and w. Note that $\underline{p}$ is linear in w in (12). Substituting w out and using (11,12,14), we obtain an equation to determine q:

$$\frac{\eta-1}{(\rho+\mu)\delta}\left[\frac{\varphi(q,-\eta)}{\varphi(q,1-\eta)}\frac{\varphi(q,1-\eta+\frac{\rho+\mu}{\pi})}{\varphi(q,-\eta+\frac{\rho+\mu}{\pi})}-1\right]=\frac{\varphi(q,1-\eta+\frac{\mu}{\pi})}{\varphi(q,\frac{\mu}{\pi})\varphi(q,1-\eta)}. \tag{15}$$

The left-hand side is equal to $\underline{p}^{\eta-1}/\varphi(q,1-\eta)$, indicating that the difference in profits between p^* and $\underline{p}$ is determined by the repricing cost δ. An increase in q raises $\underline{p}$. The right-hand side expresses that the relative prices must aggregate to 1. Thus, increasing q decreases $\underline{p}$. This establishes:

Proposition 1. *There exists a unique stationary equilibrium for each $\pi > 0$.*

All proofs are deferred to the appendix unless stated otherwise. This proposition establishes the existence of unique stationary equilibrium for a state-dependent pricing model with endogenous real wage and without using quadratic approximations.[4] In this model, the real wage is affected by the inflation rate through optimal markup behavior, even though the marginal product of labor is constant due to the linear production technology. We can show that the stationary equilibrium satisfy the following properties:

Proposition 2.
1. *The repricing size $\log q$ grows asymptotically linearly in π with coefficient $(\log(1+\mu\delta\eta))/\mu$.*
2. *The target price p^* increases unboundedly as π increases.*

[4]In the tradition of state-dependent pricing models, analytical results are obtained by taking real wage or real cost of production exogenous (Sheshinski and Weiss [31]; Caplin and Spulber [12]; Caballero and Engel [10]; Caplin and Leahy [11]; Ahlin and Shintani [2]; Stokey [32]; Alvarez and Lippi [3]) and the general equilibrium results are obtained numerically (Dotsey et al. [16]; Golosov and Lucas [20]). An exception is Danziger [14] which analytically establishes the existence of Markov perfect equilibrium under $\eta = 2$.

3. The real wage w decreases as π increases for sufficiently large π.

The repricing size increases in π as in Sheshinski and Weiss [31], and the intuition is clear: a higher inflation would cause more frequent repricing, inducing a firm to adjust its price by a greater intensive margin in order to gain more time until the next repricing. (1) establishes the sharper result that the increase is asymptotically linear on inflation in our model. (2) implies that the target price to which firms adjust their prices is increasing in the inflation π. Combined with (3), this implies that a higher inflation induces firms to choose a higher markup when they revert to p^*, because otherwise the firms would spend longer time with low markups. (3) states that, when the inflation rate is sufficiently high, the real wage decreases as inflation increases. When the inflation is higher, the relative price dispersion and the resulting inefficiency loss in the production sector are larger. Thus, in an equilibrium of the good and labor markets, as the inflation rate increases, the real wage decreases. To our knowledge, this is a novel result in a literature where most analytical results are obtained with an exogenous real wage or using quadratic approximations, whereas our result is obtained asymptotically for large π.

The positive relationship between π and q implies that high inflation increases the intensive margin of aggregate price adjustments, which generates a positive correlation between the inflation level and inflation volatility. However, the main source of correlation between the level of inflation and inflation volatility in our model—in a causal sense as well as in terms of quantitative significance— is the extensive margin, that is, the number of firms that reprice simultaneously, as we show in Section 4.

3.3 Complementarity of repricing at the extensive margin

In a stationary equilibrium, the fraction of firms per unit of time that reprice at the lower threshold of the inaction band, λ, is the extensive margin of aggregate price adjustments. The value of λ is determined by the complementarity of repricing behavior across firms. We define, the degree of complementarity in (stationary) equilibrium as the mean number of firms that are induced to reprice as a result of a firm's repricing at the threshold. We next derive an $O(n^{-1})$ approximation for the mean number of firms that are induced to reprice as a result of a firm's repricing at the threshold that we denote by θ.

Suppose that firm i hits the threshold $\underline{p}$ and reprices its log price $\log P_{i,t}$ by $\log q$. The price change by firm i increases the aggregate good price P_t, which in turn decreases the relative price of other firms $p_{j,t}$, $j \neq i$. The impact of $P_{i,t}$ on P_t may be computed as follows. Note that $P_t = \left(\sum_{i=1}^{n} e^{(1-\eta)\log P_{i,t}}/n\right)^{1/(1-\eta)}$. Now $\log P_{i,t}$ is increased by $\Delta \log P_{i,t}$. Using a Maclaurin's series expansion around the given initial price $\log P_{i,t}$, we obtain

$$
\begin{aligned}
\Delta \log P_t &= \frac{1}{n}\frac{p_{i,t}^{1-\eta}}{1-\eta}\sum_{k=1}^{\infty}\frac{((1-\eta)\Delta \log P_{i,t})^k}{k!} + O(n^{-2}) \\
&= \frac{p_{i,t}^{1-\eta}}{n}\frac{e^{(1-\eta)\Delta \log P_{i,t}} - 1}{1-\eta} + O(n^{-2}) \\
&= \frac{\underline{p}^{1-\eta}}{n}\varphi(q, 1-\eta) + O(n^{-2})
\end{aligned} \tag{16}
$$

The increase in aggregate price reduces $s_{j,t}$ by $\Delta \log P_t / \log q$. Firms with $s_{j,t}$ in the interval $(0, \Delta \log P_t / \log q]$ are induced to reprice when firm i reprices. Since the stationary density at the threshold is f_o, the mean number of firms that are induced to reprice by firm i's repricing is $f_o\varphi(q,1-\eta)\underline{p}^{1-\eta}/\log q$. Using (13) and (14), we obtain

$$\theta = \frac{\varphi(q,1-\eta)}{\varphi(q,1-\eta+\mu/\pi)}. \tag{17}$$

The firm j's repricing further increases the aggregate price by $\Delta \log P_t$, decreases other firms' states by $\Delta \log P_t/\log q$, and induces some of them to reprice. The mean number of these firms is θ^2. Those firms further induce some other firms to reprice. Thus, the total effect is $1+\theta+\theta^2+\cdots = 1/(1-\theta)$, which is well defined since (17) implies $\theta < 1$ for any π. Hence, the degree of complementarity θ determines the extent of multiplier effects on the extensive margin λ.

However, the mean number of firms induced to reprice due to a firm that draws a Calvo shock is different from θ, because the repricing size for the firm i drawing a Calvo shock is not $\log q$ but $\log p^* - \log p_{i,t}$. Substituting this for $\Delta \log P_{i,t}$ in the second line of (16), we obtain that the impact on aggregate price is $\Delta \log P_t = (p^{*1-\eta} - p_{i,t}^{1-\eta})/(n(1-\eta)) + O(n^{-2})$. The first term has an expected value $(p^{*1-\eta}-1)/(n(1-\eta)) = \varphi(p^*,1-\eta)/n$ when $p_{i,t}$ is drawn from a stationary distribution. Let m_0 denote the number of firms in the interval $(0, \varphi(p^*,1-\eta)/(n\log q)]$. Then, the mean of m_0 is $f_o\varphi(p^*,1-\eta)/\log q$, which is rewritten as $\varphi(p^*,1-\eta)/\varphi(q,\mu/\pi)$ using (14) and (17). The number of firms that are induced to reprice due to m_0 is $m_0/(1-\theta)$. Now, μdt fraction of firms incur Calvo shocks in a small time interval dt. Thus, the stationary extensive margin is determined as

$$\lambda = \frac{\mu\varphi(p^*,1-\eta)}{\varphi(q,\mu/\pi)(1-\theta)}. \tag{18}$$

The extensive margin λ and the degree of complementarity θ turns out to be increasing as the inflation rate rises.

Proposition 3. *1. The extensive margin λ and the degree of complementarity θ converge to $1/(\delta\eta)$ and 1, respectively, as $\pi \to \infty$.*

2. *λ is increasing in π for sufficiently large π.*

3. *θ is increasing in π for sufficiently large π.*

This proposition shows that a higher inflation results in a greater multiplier effects in pricing behavior. As the trend inflation rate π increases, the stationary distribution of relative prices skews to the left. This leads to an increase in the density of firms near the repricing threshold (f_o increases towards $(\log(1+\mu\delta\eta))/(\mu\delta\eta)$.) In this situation, a repricing action by a triggering firm causes a larger size of avalanche of repricing by other firms. Hence, a higher inflation rate generates a larger multiplier effect and results in a larger fraction of repricing firms λ.

Finally, the proposition states that the extensive margin λ is finite even when π diverges. In particular, this implies that the aggregate adjustment cost $\lambda\delta$ converges to a constant less than 1 since $\eta > 1$. Hence, the resource constraint is satisfied for any π.

In this section, we showed that the degree of complementarity θ leads to a multiplier effect on the mean repricing behavior. In the next section, we argue that complementarity generates not only the mean multiplier effect but also *volatility* in aggregate price when n is large but finite.

4 Fluctuations in a finite economy

In this section, we show that the model economy exhibits quantitatively significant fluctuations in inflation rates around the long-run level π when the number of firms n is large but finite. When n is finite, idiosyncratic shocks (Calvo events) generate some aggregate fluctuations in principle, but the variance of the aggregate fluctuation vanishes quickly as n tends to infinity according to the law of large numbers if the repricing behavior is independent. Whether the sum of idiosyncratic shocks matters for the aggregates depends on the degree of complementarity θ of repricing behavior. This section shows that complementarity generates a fat-tailed distribution of the number of firms that reprice simultaneously, leading to (i) short-term inflation rates that exhibit non-negligible fluctuations and (ii) a high level of long-term inflation causes high volatility of short-term inflation rates.

4.1 Stochastic number of firms that reprice simultaneously

In an economy with finite n, any realized moment of the cross-section distribution of $s_{i,t}$ is generically not equal to the population moment derived from the stationary distribution $f(s)$. The aggregate price in the finite model can deviate from its stationary counterpart in the model with infinitely many firms. Because firms' pricing decisions are positively correlated, a firm's discrete adjustment in price may have substantial avalanche effects on adjustments by other firms. In this section we characterize the asymptotic property of this avalanche effect.

To keep the analysis simple with a finite number of firms and fluctuating aggregate prices, we assume that the monetary authority is capable of implementing the average inflation rate at its target π, by increasing the money supply M_t at the rate π, while also accommodating shocks by changing the nominal rate i_t to maintain a constant real interest rate $r_t := i_t - \pi$ at the household's time-discount rate, that is, $r_t = \rho$.[5] Households hold the monetary base M_t which yields the nominal interest rate i_t, and households must satisfy a cash-in-advance constraint $M_t = P_t C_t$. We also assume

[5]Although it is beyond the scope of this paper, we conjecture that the fluctuation on real variables is smaller than the fluctuation in realized inflation, so that one could relax the assumption on the real economic variables.

that households own the firms in the economy and are paid the total nominal firms' profits as a dividend D_t. The household's flow budget constraint is thus $P_tC_t + \dot{M}_t = W_tN_t + D_t + i_tM_t$.

When a firm reprices due to a Calvo shock, it reduces the relative prices of all other firms, which may induce some firms near the threshold to reprice. The number of firms that reprice after the initial firm depends on the distribution of $s_{j,t}$. Let L_t denote the total number of firms that reprice after a firm adjusts due to a Calvo shock at t. The ratio of those firms to all firms, L_t/n, forms the stochastic extensive margin of aggregate price adjustments in a finite economy. The aggregate price level $\log P_t$ is thus a compound Poisson process with arrival rate μ: the aggregate price level does not move when no firms draw Calvo shocks, whereas it jumps if a Calvo shock arrives at a firm, and the jump size of $\log P_t$ is determined by the extensive margin L_t/n and the intensive margin $\log q$.

We make an approximation that, in the finite economy, the real macroeconomic variables r, w, and Y stay at the stationary level and firms follow the stationary policy rule $(\underline{p}, p^*)$. This environment holds asymptotically as n tends to infinity, because the real aggregate variables are dependent only on relative prices, and the moments of relative prices $(p_{i,t})$ obey the law of large numbers even though aggregate price inflation rates fluctuate. Thus, the deviation of relative prices from the stationary distribution has only vanishingly small impacts on real aggregate variables as $n \to \infty$. The real wage rate asymptotically stays at the stationary level, since the nominal wage rate adjusts flexibly in the model. The monetary authority is able to accommodate a jump in P_t by adjusting monetary base M_t instantly.

Suppose that a state profile $(s_{i,t})_{i=1}^{n}$ is randomly drawn from the joint stationary density function $f^n(s)$. If firm i draws a Calvo shock in t, the repricing by i reduces

$(s_{j,t})_{j \neq i}$ and induces some firms near the threshold to reprice. Let m_0 denote the number of these repricing firms. The repricing of these m_0 firms further reduce the relative prices of other firms, possibly inducing some other m_1 firms to reprice. This process continues until there is no firm induced to reprice. Thus, the total number of adjusting firms within t (other than firm i) is $L = \sum_{u=0}^{U} m_u$, where U is the smallest integer satisfying $m_U = 0$. All these L firms reprice at the extensive margin, so their repricing size is always $\log q$. In the stationary case, we characterized the mean number of firms θ that are induced to reprice by a firm repricing by $\log q$ in (17). Hence, in this adjustment process, each repricing firm gives rise to a random number of repricing firms, which follows a Poisson distribution which has mean θ asymptotically as $n \to \infty$. Thus, m_u for $u = 0, 1, \ldots, U$ is embedded in a so-called Poisson branching process with mean θ. This allows for the following characterization of the fluctuation of the sum L.

Proposition 4. *The mean and variance of L conditional on $m_0 = 1$ converge as $n \to \infty$ to $1/(1-\theta)$ and $\theta/(1-\theta)^3$, respectively. Furthermore, as $n \to \infty$,*

$$\Pr(L = \ell \mid m_0) \to \frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell-m_0}}{(\ell-m_0)!} \tag{19}$$

for $\ell = m_0, m_0+1, m_0+2, \ldots$. The tail of the above probability function satisfies

$$\frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell-m_0}}{(\ell-m_0)!} \propto e^{-(\theta-1-\log\theta)\ell} \ell^{-1.5} \qquad \text{as } \ell \to \infty. \tag{20}$$

The implication of this proposition is three-fold. First, it shows that the mean of L conditional on $m_0 = 1$ converges to $1/(1-\theta)$. Since each firm draws a Calvo event at rate μ, there are $n\mu dt$ firms on average that draw a Calvo event in a short time horizon dt. Hence, the average fraction of firms that adjust within dt is $\mu dt/(1-\theta)$. Note that, as $n \to \infty$, this converges to the fraction of firms that adjust in the stationary equilibrium. This means that our choice of L in a finite economy is consistent with the economy with infinitely many firms.

Second, this proposition shows that the asymptotic variance of L is increasing in the degree of complementarity θ. The increase is non-linear and rather rapid when θ is close to 1. Moreover, θ is increasing in π by Proposition 3. Hence, we obtain that the variance of the extensive margin of aggregate price adjustments increases as the long-run inflation level π increases.

In the previous section, we showed that the degree of complementarity θ determines a multiplier effect of repricing. A firm's reprice induces θ firms to reprice on average, leading to the mean number of subsequently repricing firms to be $1/(1-\theta)$. Proposition 4 now shows that the multiplier effect is stochastic under finite n, and the variance of the multiplier effect is increasing in θ as $\theta/(1-\theta)^3$. When θ approaches 1 as π increases unboundedly, the variance of L diverges. Therefore, Proposition 4 points to the possibility that the fluctuations of aggregate prices are obtained for any large number of firms n when π is sufficiently high.

Thirdly, Proposition 4 also indicates that the number of firms that reprice simultaneously has a power-law distribution with exponential truncation as shown in (20). Therefore, the multiplier effect caused by the complementarity of repricing is not only stochastic under finite n, but also exhibits a fat right tail, signified by the power-law distribution, up to the exponential truncation point determined by θ. When θ reaches 1, the truncation point diverges, implying that the entire tail is characterized by a power-law distribution.

The power law of the multiplier effect L is the key to generate inflation fluctuations. The power-law tailed distribution for the avalanche of simultaneously repricing firms is reminiscent of the self-organized criticality model of inventories proposed by Bak et al. [5] and Scheinkman and Woodford [30]. In their model, the configuration of agents' states (inventory profile) globally converges to the criticality point of pairwise correlation of actions, at which the power-law distribution of simultaneous actions emerges, resulting in non-trivial aggregate fluctuations arising from micro-level interactions. In our model, the relevant configuration is the profile of relative prices. If a relatively large number of firms have their relative prices near the repricing threshold, a few small shocks cause many large avalanches of repricing behavior, leading to a quick decrease in the number of firms near the threshold. In contrast, if a relatively small number of firms are located near the threshold, only small-sized avalanches occur, and the number of firms near the threshold gradually rises. In either case, the relative price distribution converges toward the stationary distribution, at which the complementarity of repricing is θ. Therefore, when θ is close to 1, the relative prices globally converge to the point at which substantial fluctuations of the number of repricing firms emerge.

4.2 Volatile short-run inflation under high long-run inflation

Proposition 4 shows that the number of repricing firms exhibits higher volatility when the trend inflation is higher. In Proposition 2, we observed that the repricing size q increases as the trend inflation increases. Therefore, both the extensive and intensive margins of aggregate price adjustments contribute to the higher volatility of inflation under higher trend inflation. This leads to the following main result:

Proposition 5. *For sufficiently large n and in a range of sufficiently large π, the variance of inflation $d \log P_t$ is increasing in π.*

In this section, we showed that for an economy with a finite number of firms, the inflation rate fluctuates due to the complementarity of firms' repricing behavior and the volatility of inflation increases as the trend inflation level increases. To conclude this section, we discuss how this fluctuation sustains even when n is very large. In conventional analysis of state-dependent pricing, a continuum of firms is often assumed. As a direct consequence of the assumption, there is no aggregate fluctuation arising from idiosyncratic shocks. If the distribution of the number of firms that reprice simultaneously, L, had an exponential tail, then all moment would be finite, and aggregate fluctuations would be absent in the infinite limit. In the present model, this prediction holds for any finite π, since the distribution of L has an exponential tail as in (20): $\Pr(L = \ell) \propto e^{-(\theta-1-\log\theta)}\ell^{-1.5}$. As seen in this function, the exponential decline takes effect when L has the order of magnitude $1/(\theta - 1 - \log\theta)$. This value diverges as $\theta \nearrow 1$. At the limit $\theta = 1$, the distribution of L becomes a pure power law. This transition can be also seen in that the variance of L is determined by an inverse of $1 - \theta$. In this extreme case, L has no first moment and the aggregate fluctuations survive even as $n \to \infty$. Note that θ is increasing toward 1 as π increases. Thus, for any finite n, there is a finite level of π for which the model economy exhibits sizable aggregate price fluctuations. In the next section, we investigate if the model economy exhibits meaningful fluctuations under reasonable parameter values.

5 Numerical analysis

In this section, we investigate the model quantitatively within a realistic range of long-run inflation rates and a finite number of firms. The purpose of this exercise is to extend the analytical results obtained asymptotically so far to the environment of low inflation rates. Thus, the quantification here should be regarded as a proof of concept rather than serious estimation of the model. We calibrate the model to match some key empirical patterns reported by Nakamura and Steinsson [25]. They reported that the median size of price increases is 7.3 percent (all sectors). In the periods they studied

(1988-2005), the U.S. monthly CPI experienced 0.1% inflation (1.17% annual rate) with a standard deviation of 0.23%. Thus, we set δ at 0.0023 so that q is equal to 1.07 and calibrate μ at 0.1 so that the inflation volatility is 0.23%, when the average annual inflation rate π is 1.17%. Note that μ is the arrival rate of exogenous repricing. Since there are other firms that reprice by paying menu costs, the total fraction of repricing firms in the model is always greater than μ. The elasticity of substitution η is chosen to be 3 so that the labor share is two-thirds of the total value added. The time discount rate ρ is set to the long-term real interest rate 0.02.

We first show comparative statics results. The left panel of Figure 1 plots the stationary real wage w on the left axis for each π. The plot confirms that the real wage is decreasing in π, extending our comparative statics (Proposition 2) to the low inflation range under the calibrated parameter set. The same plot shows the monthly repricing probability for a firm on the right axis. The right panel of Figure 1 plots the degree of complementarity θ and the standard deviation of the stochastic multiplier effect $\sqrt{\theta/(1-\theta)^3}$ for each π. In the plot, θ increases monotonically with π. This result extends the asymptotic analytical result of Proposition 3 to the range of low inflation rates. θ approaches to the critical value 1 quickly even in a low inflation range 2%–5%. As a result, the stochastic multiplier effect increases almost linearly in π, as can be seen in the plot.

The left panel of Figure 2 plots the stationary distribution $f(s)$ for each level of π. Observe that the exponential stationary distributions converge to the uniform distribution as π increases. This implies that the increase in the degree of complementarity θ toward 1 along with π is accompanied by the increase in f_o, the density of firms at the threshold $s = 0$, toward 1. The right panel of Figure 2 plots repricing size q and threshold $\underline{p}$ for each π. Both numerical results show the quantitative extent of the

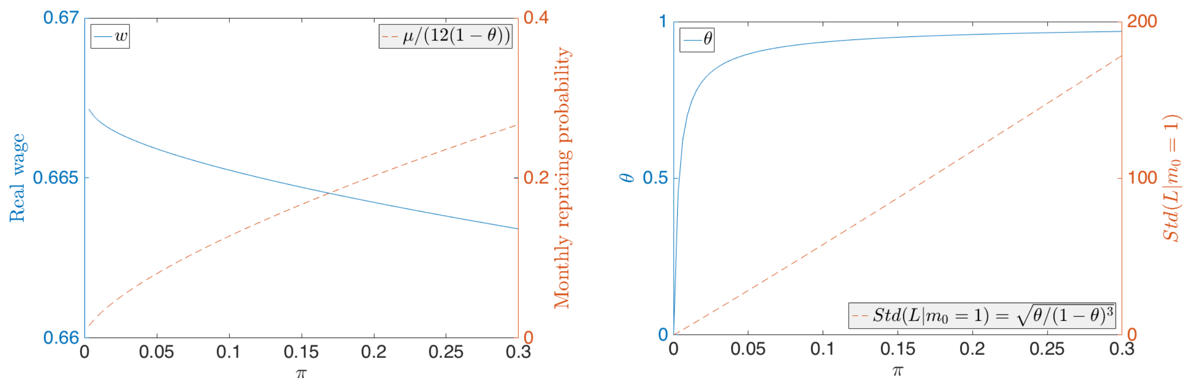

Figure 1: Comparative statics for various inflation level π. *Left:* Real wage rate (left axis) and repricing frequency (right axis) for various inflation levels π. *Right:* Degree of complementarity θ (left axis) and the standard deviation of L conditional on $m_0 = 1$ (right axis).

analytical results obtained in Proposition 2.

Finally, Figure 3 plots the simulated standard deviation of monthly inflation rates for each π in three cases $n = 10000, 30000, 100000$. For each Monte Carlo trial, a state profile $(s_1, s_2, \ldots, s_n)$ is randomly drawn from a stationary joint density function $f^n(s)$. Next, a firm i is selected randomly to receive a Calvo shock. Then, profile s is updated as in Section 3, and the final profile s and the number of firms L that reprice following firm i are computed. With these, the increase in aggregate price $d\log P$ is computed. This procedure is repeated for ten thousand times to compute the standard deviation of $d\log P$. The plotted result agrees with Proposition 5, for the standard deviation of inflation is increasing in the long-run inflation level π.

The simulated results are compared to the empirical observation between inflation level and volatility. Figure 5 plots the short-term volatility against the long-term level of inflation across countries or periods. A casual observation confirms the positive

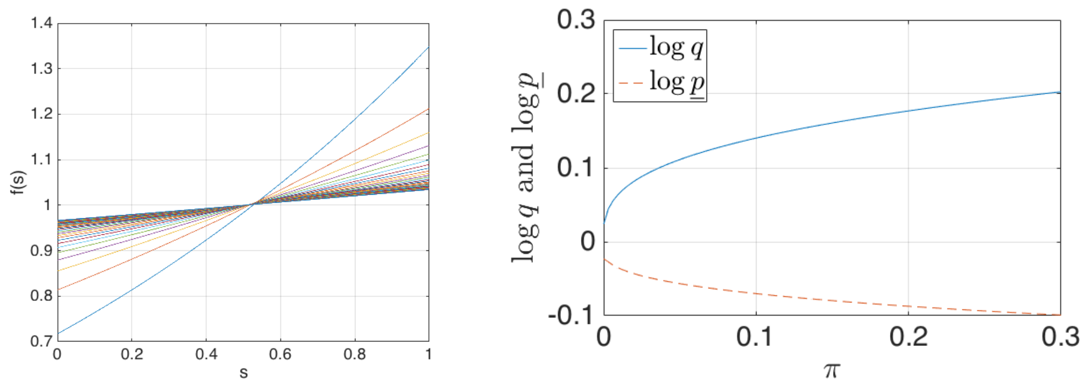

Figure 2: *Left:* Stationary distributions of s. The steepest slope (blue) corresponds to the inflation rate π at 1%, the second steepest (pink) at 2%, and the flattest at 30%. *Right:* Repricing size q and repricing threshold relative price $\underline{p}$

association between the level and volatility in a high inflation range, whereas this relationship seems somewhat attenuated in a low inflation range less than about 5% at annual rate. The dotted line in Figure 3 plots the same line as in the right panel of Figure 5 except that the standard deviation is reduced by the standard deviation observed at $\pi = 0$. Here, we interpret the standard deviation observed at $\pi = 0$ as the price volatility caused by aggregate shocks which are not incorporated in the model. The plot shows that the observed relation is consistent with the case of $n = 30000$. This implies that the number of firms that affect a firm's pricing is a relevant parameter to determine the sensitivity of aggregate price volatility to the inflation level.

The standard deviation of inflation rates is increased by more than 10 times for the increase of π from 5% to 30% in Figure 3 (from 0.2% to 2% for $n = 10000$; from 0.02% to 0.6% for $n = 100000$). This makes contrast with repricing size $\log q$ in the right panel of Figure 2, which increases less than twice (from 11% to 20%) when π

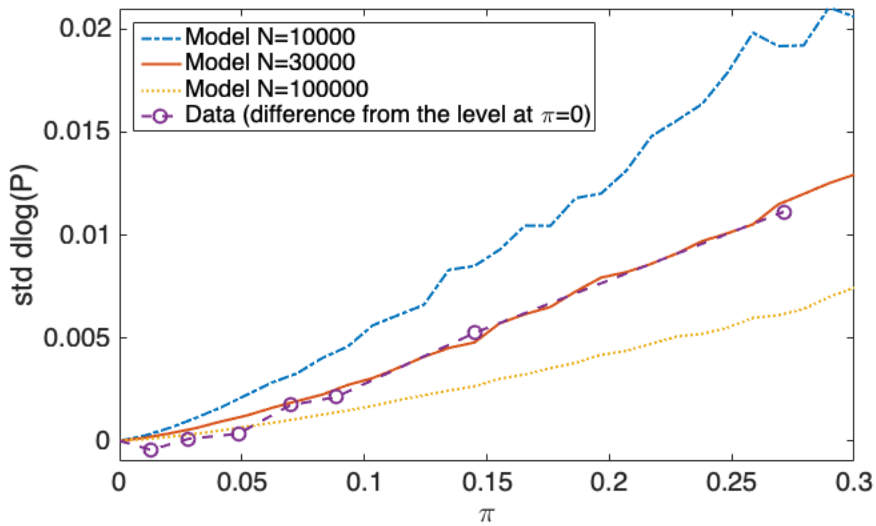

Figure 3: Levels of annual inflation rates and the standard deviations of monthly inflation rates in the model simulations and the data.

increases by the same degree. This observation implies that the extensive margin (L) rather than the intensive margin ($\log q$) accounts for the dominant portion of inflation volatility for a high inflation range. In the model, the extensive margin *causes* the aggregate fluctuations, since without the complementarity of repricing behavior at the extensive margin there would be only negligible aggregate fluctuations. The numerical analysis shows that the extensive margin is important not only in the causal sense but also in quantitative terms for generating the positive association between the inflation level and volatility.

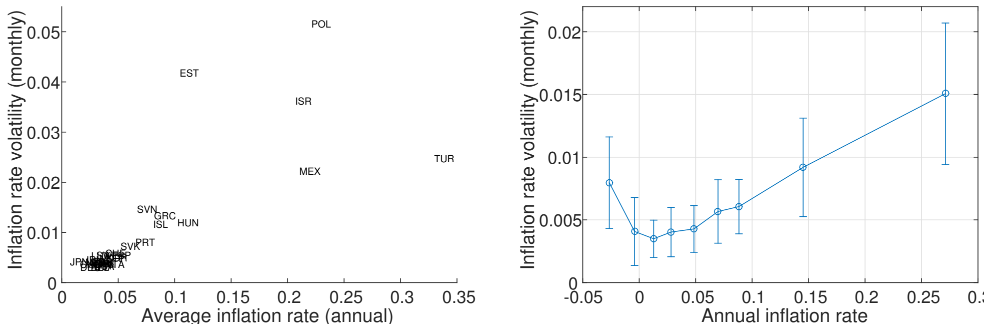

Figure 4: Inflation level and volatility. *Left:* Country-wise scatter plot of inflation level and volatility for 32 OECD countries for all the periods available in OECD database. The inflation rate is defined by a difference of logarithm of CPI. *Right:* Stratified pooled data. The horizontal axis shows annual inflation rates in two-year periods, while the vertical axis shows the volatility of monthly inflation rates during the two-year period. The error bar shows ±1 standard errors. The sample consists of 32 OECD countries during 1980–2015.

6 Conclusion

This paper provides an explanation for the positive association between inflation level and inflation volatility. Since a firm's relative price is determined by other firms' pricing behavior, a firm's price increase necessarily leads to a decrease in other firms' relative prices. Thus, repricing behavior exhibits complementarity across firms. The complementarity leads to a possibility of stochastic synchronization of repricing behavior, in which the aggregate price exhibits fluctuations due to the varying number of firms that reprice simultaneously.

Analytical results demonstrate that the stochastic number of repricing firms exhibits a fat tail when the trend inflation is high. When the trend inflation is high, firms' relative prices are quickly drifted away from optimum, causing a relatively higher density of firms located at the repricing threshold. Hence, there is a higher probability for a firm's repricing to cause another firm's repricing. By this mechanism, we analytically show that a sufficiently high long-run inflation rate causes the high volatility in short-run inflation rates.

We calibrate the model with reasonable parameter values to investigate the level-volatility nexus in a relatively low inflation range. Numerical analyses show that the considerable magnitude of inflation fluctuation and the positive association between level and volatility are generated by this mechanism. This suggests that there is another possible source of welfare costs of high inflation: a high inflation causes a high volatility in aggregate prices, leading to welfare loss due to intertemporal misallocation of resources. Further exploration of the welfare loss is left for future research.

Appendix

Derivation of Equation (12)

From the value matching condition (9), $z(p^*) = \rho v(p^*)$, and thus $v(p^*) - (\mu/(\rho + \mu))v(p^*) = z(p^*)/(\rho+\mu)$. The smooth pasting condition (6), $z(\underline{p}) = \rho v(p^*) - (\rho+\mu)\delta$, yields $z(\underline{p}) = \rho v(\underline{p}) - \mu\delta$. Combining these expressions, we have

$$\begin{aligned} v(\underline{p}) - \frac{\mu}{\rho+\mu} v(p^*) &= \frac{z(\underline{p}) + \mu\delta}{\rho} - \frac{z(\underline{p}) + (\rho+\mu)\delta}{\rho} \frac{\mu}{\rho+\mu} \\ &= \frac{z(\underline{p})}{\rho+\mu}. \end{aligned}$$

Using the general solution (8), we obtain, for both $p = p^*$ and $p = \underline{p}$,

$$\frac{z(p)}{\rho+\mu} = c_0 p^{-\frac{\rho+\mu}{\pi}} + \frac{p^{1-\eta}}{\rho+\mu-\pi(\eta-1)} - \frac{wp^{-\eta}}{\rho+\mu-\pi\eta}.$$

Expanding the function $z(p) = p^{1-\eta} - wp^{-\eta}$, we have, for both $p = p^*$ and $p = \underline{p}$,

$$-\frac{c_0(\rho+\mu)}{\pi(\eta-1)}p^{-\frac{\rho+\mu}{\pi}} = \frac{p^{1-\eta}}{\rho+\mu-\pi(\eta-1)} - \frac{(\eta/(\eta-1))wp^{-\eta}}{\rho+\mu-\pi\eta}.$$

Applying this equation at p^* and $\underline{p}$ we obtain

$$q^{-\frac{\rho+\mu}{\pi}} = \frac{p^{*1-\eta} - c_1 w p^{*-\eta}}{\underline{p}^{1-\eta} - c_1 w \underline{p}^{-\eta}}$$

where

$$c_1 := \frac{\eta}{\eta-1}\frac{\eta-1-\frac{\rho+\mu}{\pi}}{\eta-\frac{\rho+\mu}{\pi}}.$$

Since $p^* = q\underline{p}$ we establish (12) as follows.

$$\begin{aligned}
& q^{-\frac{\rho+\mu}{\pi}}(\underline{p} - c_1 w)\underline{p}^{-\eta} = (q^{1-\eta}\underline{p} - c_1 w q^{-\eta})\underline{p}^{-\eta} \\
\Leftrightarrow\quad & \underline{p}(q^{\eta-1-\frac{\rho+\mu}{\pi}} - 1) = c_1 w (q^{\eta-1-\frac{\rho+\mu}{\pi}} - q^{-1}) \\
\Leftrightarrow\quad & \underline{p}\varphi(q, \eta-1-(\rho+\mu)/\pi) = \frac{\eta}{\eta-1}\frac{w}{q}\varphi(q, \eta-(\rho+\mu)/\pi)
\end{aligned}$$

Properties of $\varphi(q,x)$

We summarize properties of the function $\varphi(q,x)$ in the following lemma. Here $\varphi_q(q,x)$ and $\varphi_x(q,x)$ denote the partial derivatives of $\varphi(q,x)$.

Lemma 1. *For $q > 1$, $\varphi(q,x) := (q^x - 1)/x$ satisfies the following properties.*

(i) $\varphi(q,x) > 0$, $\varphi_q(q,x) > 0$, *and* $\varphi(q,-x) = \varphi(q,x)/q^x = \varphi(q,x)/(\varphi_q(q,x)q)$

(ii) $\log\varphi(q,x)$ *is strictly increasing and convex in x,*

$$\begin{aligned}
\frac{\varphi_x}{\varphi}(q,x) &= \frac{\log q}{1-q^{-x}} - \frac{1}{x} > 0, \\
\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi}\right)(q,x) &= \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x-1)^2}\right) \geq 0 \quad \text{with equality holding at } x = 0,
\end{aligned}$$

and $(\partial/\partial x)(\varphi_x/\varphi)(q,x) \leq 1/x^2$.

(iii) $(\partial/\partial q)(q^x\varphi(q,y)/\varphi(q,x+y)) > 0$ *if and only if* $x > 0$

(iv) $(\partial/\partial q)(\varphi(q,x)\varphi(q,y)/\varphi(q,x+y)) > 0$ *if* $x > 0$

(v)

$$\frac{\varphi_x}{\varphi^2}(q,x) = \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x) = \frac{q^x\log(q^x) - q^x + 1}{(q^x-1)^2}$$

is strictly positive, continuous and decreasing in x.

Proof.

(i) $\varphi(q,x) = (q^x-1)/x > 0$ for $q > 1$ and $x \neq 0$, and $\lim_{x\to 0}\varphi(q,x) = \log q > 0$ for $q > 1$. Also, $\varphi_q(q,x) = q^{x-1} > 0$. Furthermore, $\varphi(q,-x)q^x = (1-q^x)/(-x) = \varphi(q,x)$.

(ii)

$$\varphi_x(q,x) = \frac{\partial}{\partial x}\frac{q^x-1}{x} = \frac{1}{x^2}\left[q^x\log(q^x) - (q^x-1)\right].$$

Since $y\log y - (y-1) > 0$ for $y > 0$ and $y \neq 1$, $\varphi_x(q,x)$ is strictly positive for $x \neq 0$. For $x = 0$, l'Hôpital's rule yields $\lim_{x\to 0}\varphi_x(q,x) = (\log q)^2/2$. Hence, $\varphi_x(q,x) > 0$ for any x. Moreover,

$$\frac{\varphi_x}{\varphi}(q,x) = \frac{\partial}{\partial x}\log\frac{q^x-1}{x} = \frac{x}{q^x-1}\frac{(q^x\log q)x - (q^x-1)}{x^2} = \frac{\log q}{1-q^{-x}} - \frac{1}{x}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi}\right)(q,x) = -\frac{q^{-x}(\log q)^2}{(1-q^{-x})^2} + \frac{1}{x^2} = \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x-1)^2}\right).$$

Set $h(y) := (y-1)^2 - y(\log y)^2$ for $y > 0$. Note that $h'(y) = 2(y-1) - 2\log y - (\log y)^2$ and $h(1) = h'(1) = 0$. Also, $h''(y) = (2/y)(y-1-\log y)$ is positive for $y > 0$. Thus, h' is increasing in y, implying that $h'(y) < 0$ for $0 < y < 1$ and $h'(y) > 0$ for

$y > 1$. Hence, $h(y)$ achieves a minimum 0 at $y = 1$ and $h(y) > 0$ for $y \neq 1$. This leads to $1 \geq y(\log y)^2/(y-1)^2$ for $y > 0$ with equality holding at $y = 1$. Thus, $(\partial^2/\partial x^2)\log\varphi(q,x) \geq 0$ with equality holding at $x = 0$.

(iii) Using $\varphi(q,-x) = \varphi(q,x)/q^x$ from Lemma 1(i), we have

$$\frac{q^x\varphi(q,y)}{\varphi(q,x+y)} = \frac{\varphi(q,y)}{q^y}\frac{q^{x+y}}{\varphi(q,x+y)} = \frac{\varphi(q,-y)}{\varphi(q,-x-y)}.$$

Thus, the derivative with respect to q is

$$\begin{aligned}\frac{\partial}{\partial q}\frac{\varphi(q,-y)}{\varphi(q,-x-y)} &= \frac{1}{\varphi^2(q,-x-y)}\left[q^{-y-1}\varphi(q,-x-y) - \varphi(q,-y)q^{-x-y-1}\right] \\ &= \frac{q^{-x-2y-1}}{\varphi^2(q,-x-y)}\left[\varphi(q,x+y) - \varphi(q,y)\right].\end{aligned}$$

By Lemma 1(ii), $\varphi(q,x)$ is increasing in x. Hence, the last expression has the same sign as x.

(iv) First, we obtain for $y \neq 0$:

$$\begin{aligned}&\frac{\partial}{\partial q}\frac{q^x\varphi(q,y)}{\varphi(q,x+y)} = \frac{\partial}{\partial q}\frac{q^x(q^y-1)}{q^{x+y}-1}\frac{x+y}{y} \\ &= \frac{x+y}{y(q^{x+y}-1)^2}\left[(q^{x+y}-1)((x+y)q^{x+y-1} - xq^{x-1}) - (x+y)q^{x+y-1}(q^{x+y}-q^x)\right] \\ &= \frac{x+y}{y(q^{x+y}-1)^2}\left(yq^{2x+y-1} - (x+y)q^{x+y-1} + xq^{x-1}\right) \\ &= \frac{x(x+y)q^{x+y-1}}{(q^{x+y}-1)^2}\left(\frac{q^x-1}{x} - \frac{q^{-y}-1}{-y}\right) \\ &= x\left[\frac{(x+y)q^{x+y-1}}{(q^{x+y}-1)^2}\left(\varphi(q,x) - \varphi(q,-y)\right)\right].\end{aligned}$$

By taking a limit as $y \to 0$ we can show that this same expression holds for the case $y = 0$. Since $\varphi(q,x)$ is increasing in x, $\varphi(q,x) - \varphi(q,-y)$ has the same sign as $x + y$.

Thus, the expression inside the square brackets is positive. Hence, the entire right-hand side expression has the same sign as x. Write

$$\frac{\partial}{\partial q}\frac{\varphi(q,x)\varphi(q,y)}{\varphi(q,x+y)} = \frac{\partial}{\partial q}\frac{q^x\varphi(q,y)}{x\varphi(q,x+y)} - \frac{\partial}{\partial q}\frac{\varphi(q,y)}{x\varphi(q,x+y)}.$$

Since $x > 0$, the first term is strictly positive and the second term (including the negative sign) is positive by (iii).

(v) Define a function $g(y) = (y\log y - y + 1)/(y-1)^2$. Note that $y\log y > y - 1$ for $y \neq 1$. By l'Hôpital's rule,

$$g(1) = \lim_{y\to 1}\frac{\log y}{2(y-1)} = \lim_{y\to 1}\frac{1}{2y} = 1/2.$$

Thus, $g(y)$ is continuous and $g(y) > 0$. Moreover,

$$\begin{aligned} g'(y) &= \frac{1}{(y-1)^4}\big((y-1)^2\log y - 2(y-1)(y\log y - y + 1)\big) \\ &= \frac{1}{(y-1)^3}\left(2(y-1) - (y+1)\log y\right). \end{aligned} \tag{21}$$

Note that a function $h(y) := 2(y-1) - (y+1)\log y$ satisfies $h(1) = 0$ and

$$h'(y) = 1 - 1/y - \log y \leq 0 \quad \text{with equality holding at } y = 1.$$

Hence $h(y)$ is positive for $0 < y < 1$ and negative for $y > 1$. Thus $g'(y) = h(y)/(y-1)^3$ is negative for $0 < 1 < y$ or $y > 1$, and $g(y)$ is a decreasing function for $y > 0$. Hence, $(\partial/\partial x)(-1/\varphi)(q,x) = g(q^x)$ is a decreasing function in x for $q > 1$. □

Proof of Proposition 1: There exists a unique steady state equilibrium for each $\pi > 0$.

We define δ_o, $A(q,\pi)$, and $B(q,\pi)$ by $\delta_o := \delta(\rho+\mu)/(\eta-1)$,

$$A(q,\pi) := \frac{\varphi(q,-\eta)}{\varphi(q,1-\eta)}\frac{\varphi(q,1-\eta+\frac{\rho+\mu}{\pi})}{\varphi(q,-\eta+\frac{\rho+\mu}{\pi})} \tag{22}$$

$$B(q,\pi) := \frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)\varphi(q,1-\eta)} \tag{23}$$

and rewrite Equation (15) as

$$A(q,\pi) = B(q,\pi)\delta_o + 1. \tag{24}$$

B is decreasing in q by Lemma 1(iv). Since any ratio $\varphi(q,x)/\varphi(q,y)$ converges to 1 as $q \to 1$ by l'Hôpital's rule and since $\lim_{q\to 1}\varphi(q,\mu/\pi) = 0$, B tends to ∞ as $q \to 1$ for any π. Also, B converges to 0 as $q \to \infty$, because for any $x > 0$, $\lim_{q\to\infty}\varphi(q,x) = \infty$ and $\lim_{q\to\infty}\varphi(q,-x) = 1/x$. Thus, for $q > 1$ the right-hand side of (24) is a continuously decreasing function onto $(1,\infty)$.

Next we investigate the left-hand side of (24). Setting $a := (\rho+\mu)/\pi$ and using Lemma 1(i),

$$A = \frac{\varphi(q,-\eta)}{\varphi(q,1-\eta)}\frac{\varphi(q,1-\eta+a)}{\varphi(q,-\eta+a)} = \frac{\varphi(q,\eta)}{\varphi(q,\eta-1)}\frac{\varphi(q,\eta-1-a)}{\varphi(q,\eta-a)}. \tag{25}$$

From Lemma 1(i),

$$
\begin{aligned}
A_q/A &= \frac{\partial}{\partial q}\log A \\
&= \frac{\partial}{\partial q}\left(\log\varphi(q,\eta) - \log\varphi(q,\eta-1) + \log\varphi(q,\eta-1-a) - \log\varphi(q,\eta-a)\right) \\
&= \frac{1}{q}\left(\frac{q^{\eta}}{\varphi(q,\eta)} - \frac{q^{\eta-1}}{\varphi(q,\eta-1)} + \frac{q^{\eta-1-a}}{\varphi(q,\eta-1-a)} - \frac{q^{\eta-a}}{\varphi(q,\eta-a)}\right) \\
&= \frac{1}{q}\left(\frac{1}{\varphi(q,-\eta)} - \frac{1}{\varphi(q,1-\eta)} + \frac{1}{\varphi(q,1+a-\eta)} - \frac{1}{\varphi(q,a-\eta)}\right) \\
&= \frac{1}{q}\left(\int_{-\eta}^{\min\{1-\eta,a-\eta\}} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx - \int_{\max\{1-\eta,a-\eta\}}^{1+a-\eta} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\right). \qquad (26)
\end{aligned}
$$

By Lemma 1(v), the integrand $(\partial/\partial x)(-1/\varphi)(q,x)$ in (26) is a positive-valued, decreasing function. This implies that A_q is strictly positive, since $-\eta < \min\{1-\eta, a-\eta\} \leq \max\{1-\eta, a-\eta\} < 1+a-\eta$. Hence, A is increasing in q. Moreover, since $\lim_{q\to 1} A(q,\pi) = 1$, the left-hand side of (24) takes values above 1 for $q > 1$. Hence, Equation (24) has a unique solution q in the range $q > 1$.

Proof of Proposition 2:

1. **The repricing size** $\log q$ **grows asymptotically linearly in** π **with coefficient** $(\log(1+\mu\delta\eta))/\mu$**.**

2. **The target price** p^* **increases unboundedly as** π **increases.**

3. **The real wage** w **decreases as** π **increases for sufficiently large** π**.**

Throughout this paper, $x(\pi) \sim y(\pi)$ means $\lim_{\pi\to\infty} x(\pi)/y(\pi) = 1$, and we write $y(\pi) = O(x(\pi))$ if and only if there exists a positive real number M and a real number π_o such that $|y(\pi)| \leq Mx(\pi)$ for all $\pi \geq \pi_o$.

We first show that q is increasing in π in Equation (24). By taking total derivative of (24) and rearranging, we have

$$\frac{d\log q}{d\pi} = \frac{-1}{q}\frac{A_\pi - \delta_o B_\pi}{A_q - \delta_o B_q}.$$

A_q was shown positive in the proof of the last proposition. Moreover, Lemma 1(iv) implies that $(\partial/\partial q)(1/B) > 0$, and thus $B_q < 0$. Hence, the denominator $q(A_q - \delta_o B_q)$ is positive. Using the right-hand side of Equation (25), we calculate A_π as

$$\begin{aligned}\frac{A_\pi}{A} = \frac{\partial \log A}{\partial \pi} &= \left(\frac{\varphi_x}{\varphi}(q, \eta - 1 - a) - \frac{\varphi_x}{\varphi}(q, \eta - a)\right)\frac{d(-a)}{d\pi} \\ &= \int_{\eta-1-a}^{\eta-a} \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x-1)^2}\right) dx \left(\frac{-a}{\pi}\right) \qquad (27)\end{aligned}$$

where the second line used Lemma 1(ii). Since the integrand is positive by Lemma 1(ii), we obtain $0 > A_\pi > -(a/\pi)/(\eta - 1 - a)^2$. Similarly, $B_\pi = (\partial \log B/\partial \pi)B$ is calculated as

$$B_\pi = \left[\frac{\varphi_x}{\varphi}(q, 1 - \eta + \mu/\pi) - \frac{\varphi_x}{\varphi}(q, \mu/\pi)\right]\left(\frac{-\mu}{\pi^2}\right)\frac{\varphi(q, 1 - \eta + \mu/\pi)}{\varphi(q, \mu/\pi)\varphi(q, 1-\eta)}. \qquad (28)$$

By Lemma 1(ii), $\log \varphi(q, x)$ is convex in x. Thus, the term in square brackets is negative, implying that B_π is positive. Combining with the previous result on A_π, we obtain that the numerator $-A_\pi + \delta_o B_\pi$ is strictly positive. This establishes that $d\log q/d\pi > 0$.

Next, we show that $\log q$ increases unboundedly as $\pi \to \infty$. Suppose to the contrary that $\log q$ is bounded from above. Then for any sequence $\pi_n \to \infty$, $q_n = q(\pi_n)$ is bounded and $A(q_n, \pi_n)$ converges to 1 as $\pi_n \to \infty$. Hence, Equation (24) implies that $B(q_n, \pi_n)$ converges to 0 as $\pi_n \to \infty$. The numerator of B, $\varphi(q, 1 - \eta + \mu/\pi)$, is strictly positive and increasing in q for $q > 1$. Therefore, the denominator must tend to infinity in order for B to converge to 0. However, $\varphi(q_n, 1 - \eta)$ is bounded. Thus, $\varphi(q_n, \mu/\pi_n)$

must tend to infinity. For a fixed $\log q$, we have $\lim_{\pi\to\infty}(e^{\mu(\log q)/\pi} - 1)/(\mu/\pi) = \lim_{\pi\to\infty}(e^{\mu(\log q)/\pi}\mu(\log q)/(-\pi^2))/(\mu/(-\pi^2)) = \log q$. Since $\log q_n$ is bounded by our hypothesis, this contradicts the fact that $\varphi(q_n, \mu/\pi_n)$ diverges. Hence, $\log q(\pi)$ must diverge towards infinity as $\pi \to \infty$.

Proof of Result (1).

Function $\varphi(q_0, x) = (q_0^x - 1)/x$ for fixed $q_0 > 1$ is analytic in region $x < 0$, and so is $\log \varphi(q_0, x)$. Thus, a first-order Taylor series expansion of $\log \varphi(q, x)$ around $x = -\eta$ yields

$$\log \varphi(q, -\eta + a) - \varphi(q, -\eta) = a\frac{\varphi_x}{\varphi}(q, -\eta) + O(a^2)$$

for $|a| < 1$. Similar expansion around $x = 1 - \eta$ yields

$$\log \varphi(q, 1 - \eta + a) - \varphi(q, 1 - \eta) = a\frac{\varphi_x}{\varphi}(q, 1 - \eta) + O(a^2).$$

Moreover, Lemma 1(ii) gives

$$\frac{\varphi_x}{\varphi}(q, 1 - \eta) - \frac{\varphi_x}{\varphi}(q, -\eta) = \int_{-\eta}^{1-\eta} \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x - 1)^2}\right) dx. \tag{29}$$

Using again the notation $a = (\rho + \mu)/\pi$, we obtain the first-order Taylor expansion of $\log A$ around $a = 0$ as

$$\begin{aligned}\log A(q, \pi) &= \log \varphi(q, 1 - \eta + a) - \log \varphi(q, 1 - \eta) - \log \varphi(q, -\eta + a) + \log \varphi(q, -\eta) \\ &= \left(\frac{\varphi_x}{\varphi}(q, 1 - \eta) - \frac{\varphi_x}{\varphi}(q, -\eta)\right) a + O(a^2).\end{aligned}$$

Combining with (29), we obtain

$$\pi \log A(q, \pi) = (\rho + \mu) \int_{-\eta}^{1-\eta} \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x - 1)^2}\right) dx + O(a).$$

Since $q \to \infty$ as $\pi \to \infty$, we have $\lim_{\pi\to\infty} q^x(\log q^x)^2/(q^x - 1)^2 = 0$ for $x < 0$. Also note $\int_{-\eta}^{1-\eta} 1/x^2 dx = 1/(\eta(\eta - 1))$. Thus, we obtain $\pi \log A(q, \pi) \to (\rho + \mu)/(\eta(\eta - 1))$

as $\pi \to \infty$. From Equation (22), $A(q,\pi)$ converges to 1 as $\pi \to \infty$. Thus, $\lim_{\pi\to\infty}(A - 1)/\log A = 1$ by l'Hôpital's rule. Combining with the above result yields

$$\pi(A(q,\pi) - 1) \to \frac{\rho+\mu}{\eta(\eta-1)} \quad \text{as } \pi \to \infty.$$

Moreover, using $\varphi(q, 1-\eta+\mu/\pi) = \varphi(q, 1-\eta) + O(\mu/\pi)$ in Equation (23) we have

$$B(q,\pi) = \frac{\varphi(q, 1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)\varphi(q, 1-\eta)} \sim \frac{1}{\varphi(q,\mu/\pi)}.$$

Equation (24) is written as $A(q,\pi) - 1 = B(q,\pi)\delta(\rho+\mu)/(\eta-1)$. Applying the asymptotic relations for A and B above, this equation implies

$$\frac{\varphi(q,\mu/\pi)}{\pi} \sim \delta\eta.$$

Since $\varphi(q,\mu/\pi) = (e^{\mu(\log q)/\pi} - 1)\pi/\mu$, the asymptotic relation further implies

$$\frac{\log q}{\pi} \sim \frac{\log(1+\mu\delta\eta)}{\mu}.$$

That is, $\log q$ grows asymptotically linearly in π. □

Proof of Result (2).

From Equations (12,14), we obtain

$$(p^*)^{\eta-1} = \frac{\varphi(q, 1-\eta+\mu/\pi)q^{\eta-1}}{\varphi(q,\mu/\pi)}.$$

First, $\varphi(q, 1-\eta+\mu/\pi)$ converges to a positive constant. Second, we know from Result (1) that $\log q$ grows linearly in π for large π. This implies that, for large π, $q^{\eta-1} = e^{(\eta-1)\log q}$ grows exponentially in π, while $\varphi(q,\mu/\pi) = (\pi/\mu)(e^{\mu(\log q)/\pi} - 1)$ grows only linearly in π. Therefore, the numerator dominates the denominator for large π. Thus p^* grows unboundedly. □

In order to prove Result (3), we need the following lemma.

Lemma 2.

$$\frac{d\log q}{d\pi} \sim \frac{\log q}{\pi}.$$

Proof.

We evaluate the asymptotic behavior of

$$\frac{d\log q}{d\pi} = \frac{-1}{q}\frac{A_\pi - \delta_o B_\pi}{A_q - \delta_o B_q}.$$

We start from A_q in the denominator. Using Equation (26), we have for sufficiently large π such that $(\rho+\mu)/\pi < 1$,

$$\begin{aligned} A_q &= \frac{A}{q}\left[\int_{-\eta}^{\min\{1-\eta,a-\eta\}} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx - \int_{\max\{1-\eta,a-\eta\}}^{1+a-\eta} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\right] \\ &= \frac{Aa}{q}\left[\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x_1) - \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x_2)\right] \\ &= \frac{Aa}{q}\int_{x_2}^{x_1}\frac{\partial^2}{\partial x^2}\left(\frac{-1}{\varphi}\right)(q,x)dx = \frac{Aa}{q}\int_{x_1}^{x_2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{\varphi}\right)(q,x)dx \end{aligned}$$

where $x_1 \in [-\eta, a-\eta]$ and $x_2 \in [1-\eta, 1+a-\eta]$ and $x_1 < x_2 < 0$ for large π.

From Lemma 1(v) and Equation (21), we have

$$\frac{\partial^2}{\partial x^2}\left(\frac{-1}{\varphi}\right)(q,x) = \frac{\partial}{\partial x}\frac{q^x\log(q^x) - q^x + 1}{(q^x-1)^2} = \frac{1}{(q^x-1)^2}\left(2 - \frac{q^x+1}{q^x-1}\log(q^x)\right)q^x\log q.$$

Substituting this into the equation above, we obtain

$$\begin{aligned} A_q &= \frac{Aa}{q}\left[\frac{-q^{x_3}\log q}{(q^{x_3}-1)^2}\left(2 - \frac{q^{x_3}+1}{q^{x_3}-1}\log(q^{x_3})\right)\right] \quad \text{for some } x_3 \in [x_1, x_2] \\ &= \frac{(\log q)^2}{\pi q^{1-x_3}}\frac{-x_3 A(\rho+\mu)}{(1-q^{x_3})^2}\left(\frac{2}{x_3\log q} + \frac{1+q^{x_3}}{1-q^{x_3}}\right) \\ &= \frac{(\log q)^2}{\pi q^{1-x_3}}(-x_3 A(\rho+\mu))(1 - O(1/\log q)) \end{aligned}$$

where the last equality uses the fact that q^{x_3} tends to 0 as a power function with exponent $x_3 < 0$, and hence it is dominated by $1/\log q$ for large π.

We also obtain

$$
\begin{aligned}
-B_q &= -\frac{\partial \log B}{\partial q} B = \left(\frac{1}{\varphi(q,-\mu/\pi)} + \frac{1}{\varphi(q,\eta-1)} - \frac{1}{\varphi(q,\eta-1-\mu/\pi)} \right) \frac{B}{q} \\
&= \left(\frac{1}{\varphi(q,-\mu/\pi)} + \int_{\eta-1-\mu/\pi}^{\eta-1} \frac{\partial}{\partial x} \left(\frac{1}{\varphi} \right) (q,x) dx \right) \frac{B}{q} \\
&= \left(\frac{1}{\varphi(q,-\mu/\pi)} - \frac{\mu}{\pi} \frac{\varphi_x}{\varphi^2}(q,x_4) \right) \frac{B}{q} \quad \text{where } x_4 \in [\eta-1-\mu/\pi, \eta-1] \\
&= \left(\frac{1}{\varphi(q,-\mu/\pi)} - \frac{\mu}{\pi} \frac{\varphi_x}{\varphi^2}(q,x_4) \right) \frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)\varphi(q,1-\eta)q}.
\end{aligned}
$$

By Lemma 1(v), we have $(\varphi_x/\varphi^2)(q,x_4) = O(q^{-x_4}\log q)$, which is dominated by $1/\varphi(q,-\mu/\pi)$ for $x_4 > 0$. Also, $\varphi(q,1-\eta+\mu/\pi)/\varphi(q,1-\eta) = 1 + O(1/\pi)$. Combining these results, we obtain that $\pi^2 q(A_q - \delta_o B_q)$ is equal to

$$
\frac{\pi(\log q)^2}{q^{-x_3}}(-x_3 A(\rho+\mu)) \left(1 - O\left(\frac{1}{\log q} \right) \right) + \left(\frac{\pi^2 \delta_o}{\varphi(q,-\mu/\pi)} - O\left(\frac{\pi \log q}{q^{x_4}} \right) \right) \frac{1+O(1/\pi)}{\varphi(q,\mu/\pi)}.
$$

Since q grows asymptotically exponentially as $\pi \to \infty$, both q^{-x_3} and q^{x_4} (with $x_3 < 0$ and $x_4 > 0$) grow exponentially. Also, we have shown $\lim_{\pi\to\infty} \varphi(q,\mu/\pi)/\pi = \delta\eta$. This implies that $(\log q)/\pi$ converges to a positive constant and that $\kappa := \lim_{\pi\to\infty} \varphi(q,-\mu/\pi)/\pi$ also exists and non-zero. Collecting these results, we obtain

$$
\lim_{\pi\to\infty} \pi^2 q(A_q - \delta_o B_q) = \frac{\delta_o}{\delta\eta\kappa}.
$$

Next, we turn to the numerator $-A_\pi + \delta_o B_\pi$. From (27), we have

$$
\pi^2 A_\pi = -(\rho+\mu) A \int_{\eta-1-a}^{\eta-a} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x-1)^2} \right) dx.
$$

Thus, $\lim_{\pi\to\infty} \pi^2 A_\pi = -(\rho+\mu) \int_{\eta-1}^{\eta} x^{-2} dx = -(\rho+\mu)/(\eta(\eta-1))$.

From (28) and using Lemma 1(ii), we have

$$\begin{aligned}
B_\pi &= \left(\frac{\varphi_x}{\varphi}(q, 1-\eta+\mu/\pi) - \frac{\varphi_x}{\varphi}(q, \mu/\pi)\right)\left(\frac{-\mu}{\pi^2}\right)\frac{\varphi(q, 1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)\varphi(q,1-\eta)} \\
&= \left(\frac{\log q}{1-q^{\eta-1-\mu/\pi}} + \frac{1}{\eta-1-\mu/\pi} - \frac{\log q}{1-q^{-\mu/\pi}} + \frac{\pi}{\mu}\right)\left(\frac{-\mu}{\pi^2}\right)\frac{1+O(\pi^{-1})}{\varphi(q,\mu/\pi)} \\
&= \left(-O\left(\frac{\log q}{\pi q^{\eta-1}}\right) + \frac{1}{\pi(\eta-1)/\mu - 1} - \frac{\log q}{\varphi(q,-\mu/\pi)} + 1\right)\frac{-(1+O(\pi^{-1}))}{\pi\varphi(q,\mu/\pi)} \\
&= \left(\frac{\log q}{\varphi(q,-\mu/\pi)} - 1 - O(\pi^{-1})\right)\frac{1+O(\pi^{-1})}{\pi\varphi(q,\mu/\pi)}.
\end{aligned}$$

Using $\pi/\varphi(q,\mu/\pi) \sim 1/(\delta\eta)$, we have

$$\lim_{\pi\to\infty} \pi^2 B_\pi = \left(\frac{1}{\kappa}\lim_{\pi\to\infty}\frac{\log q}{\pi} - 1\right)\frac{1}{\delta\eta}.$$

Collecting the results above, we obtain

$$\lim_{\pi\to\infty}\frac{\pi^2(-A_\pi + \delta_o B_\pi)}{\pi^2 q(A_q - \delta_o B_q)} = \left(\frac{\rho+\mu}{\eta(\eta-1)} + \left(\frac{1}{\kappa}\lim_{\pi\to\infty}\frac{\log q}{\pi} - 1\right)\frac{\delta_o}{\delta\eta}\right)\frac{\delta\eta\kappa}{\delta_o} = \lim_{\pi\to\infty}\frac{\log q}{\pi}$$

where we used $\delta_o = \delta(\rho+\mu)/(\eta-1)$. Hence, we obtain the desired result: $d\log q/d\pi \sim (\log q)/\pi$. □

Proof of Result (3).

The real wage w is determined using Equations (12,14) as

$$\begin{aligned}
w &= \frac{\eta-1}{\eta}\frac{q\varphi(q,\eta-1-\frac{\rho+\mu}{\pi})}{\varphi(q,\eta-\frac{\rho+\mu}{\pi})}\left(\frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)}\right)^{1/(\eta-1)} \\
&= \frac{\eta-1}{\eta}\frac{\varphi(q,1-\eta+\frac{\rho+\mu}{\pi})}{\varphi(q,-\eta+\frac{\rho+\mu}{\pi})}\left(\frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)}\right)^{1/(\eta-1)}.
\end{aligned} \tag{30}$$

Note that $\varphi(q,\mu/\pi)$ tends to infinity as $\pi\to\infty$ and all the other terms with φ are bounded for $q>1$. Thus, w converges to 0 as π increases.

Rewrite (30) using $a := (\rho+\mu)/\pi$ as

$$\frac{\eta-1}{\eta(\eta-1-\mu/\pi)^{1/(\eta-1)}}\frac{\eta-a}{\eta-1-a}\frac{(1-q^{1-\eta+a})(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}}{(1-q^{-\eta+a})\varphi(q,\mu/\pi)^{1/(\eta-1)}}. \tag{31}$$

The first two fractions are monotonically decreasing in π. We focus on the third fraction. We have

$$\frac{d}{d\pi}\left(1-q^{1-\eta+\mu/\pi}\right)=-q^{1-\eta+\mu/\pi}\left((1-\eta+\mu/\pi)\frac{d\log q}{d\pi}-\frac{\mu\log q}{\pi^2}\right).$$

By Proposition 2(1) and Lemma 2, we have $d\log q/d\pi \sim (\log q)/\pi \sim (\log(1+\mu\delta\eta))/\mu$. Hence, for any small $\epsilon > 0$ there exists π_o such that for all $\pi > \pi_o$, $|d\log q/d\pi - (\log q)/\pi| < \epsilon$, $|d\log q/d\pi - (\log(1+\mu\delta\eta))/\mu| < \epsilon$, and $|(\log q)/\pi - (\log(1+\mu\delta\eta))/\mu| < \epsilon$ hold. Hence, we have

$$\frac{d}{d\pi}\left(1-q^{1-\eta+\mu/\pi}\right)<\frac{1}{q^{\eta-1-\mu/\pi}}\left[(\eta-1)\left(\frac{\log(1+\mu\delta\eta)}{\mu}+\epsilon\right)+\frac{\epsilon\mu}{\pi}\right].$$

Since q asymptotically grows exponentially in π, the left-hand side is bounded by a function exponentially decreasing to 0. The same analysis holds true for functions $(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}$, $1-q^{1-\eta+a}$, and $1-q^{-\eta+a}$. Since all of these functions are bounded above by 1 and bounded below by positive constants for sufficiently large π, the logarithms of these functions also have derivatives exponentially decreasing in π for large π.

Next, we examine the derivative of $\log\varphi(q,\mu/\pi)$. For $\pi > \pi_o$ we have the following inequality:

$$\begin{aligned}\frac{d\log\varphi(q,\mu/\pi)}{d\pi} &= \frac{1}{\varphi(q,\mu/\pi)}\left[\frac{1}{\mu}(q^{\mu/\pi}-1)+q^{\mu/\pi}\left(\frac{d\log q}{d\pi}-\frac{\log q}{\pi}\right)\right]\\ &= \frac{1}{\pi}+\frac{\mu}{\pi(1-e^{-\mu(\log q)/\pi})}\left(\frac{d\log q}{d\pi}-\frac{\log q}{\pi}\right)\\ &> \frac{1}{\pi}\left(1-\frac{\mu\epsilon}{1-\frac{e^{\mu\epsilon}}{1+\mu\delta\eta}}\right).\end{aligned}$$

Thus, the left-hand side is bounded from below by a function that declines as $1/\pi$.

Combining the results, the derivative of the logarithm of the third fraction of (31),

$$\frac{d}{d\pi}\left(\log\frac{(1-q^{1-\eta+a})(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}}{(1-q^{-\eta+a})\varphi(q,\mu/\pi)^{1/(\eta-1)}}\right),$$

consists of three terms that are bounded by exponentially declining functions and one term, with negative sign, which is bounded below by a function declining as $1/\pi$. Thus, the negative term dominates the other terms for large π. Hence, the third fraction is a decreasing function in π.

Since all fractions in (31) are decreasing in π for large π, we obtain that w is asymptotically decreasing in π. $\square$

Proof of Proposition 3:

1. **The degree of complementarity $\theta \to 1$ and the extensive margin $\lambda \to 1/(\delta\eta)$ as $\pi \to \infty$.**

2. **λ is increasing in π for sufficiently large π.**

3. **θ is increasing in π for sufficiently large π.**

Proof of Result (1).

From Equation (17), we obtain

$$\theta = \frac{\varphi(q, 1-\eta)}{\varphi(q, 1-\eta+\mu/\pi)} = \frac{q^{1-\eta}-1}{1-\eta} \frac{1-\eta+\mu/\pi}{q^{1-\eta}e^{\mu(\log q)/\pi}-1}.$$

By Proposition 2(1), $(\log q)/\pi$ converges to a positive constant as $\pi \to \infty$. Since $q^{1-\eta} \to 0$, we obtain $\theta \to 1$ as $\pi \to \infty$.

From Equation (18), we have $\lambda = \mu\varphi(p^*, 1-\eta)/(\varphi(q, \mu/\pi)(1-\theta))$. We note that $\varphi(p^*, 1-\eta) \sim 1/(\eta-1)$, since $p^* \nearrow \infty$. Using Lemma 1(ii), we have

$$\log\theta = \log\varphi(q, 1-\eta) - \log\varphi(q, 1-\eta+\mu/\pi) = \int_{1-\eta+\mu/\pi}^{1-\eta} \frac{\varphi_x}{\varphi}(q, x)dx$$
$$= \int_{1-\eta+\mu/\pi}^{1-\eta} \frac{\log q}{1-q^{-x}} - \frac{1}{x}dx \sim \left(\frac{\log q}{1-q^{\eta-1}} - \frac{1}{1-\eta}\right)\frac{-\mu}{\pi} \sim \frac{-\mu}{\eta-1}\frac{1}{\pi}.$$

Since $\theta \to 1$ as $\pi \to \infty$, we have $(1-\theta)/\log\theta \to -1$ by l'Hôpital's rule. Thus $1-\theta \to \mu/(\pi(\eta-1))$ as $\pi \to \infty$. Collecting these results, we obtain

$$\lambda \sim \frac{\pi}{\varphi(q,\mu/\pi)} \sim \frac{1}{\delta\eta}.$$

Since $\eta > 1$, the condition $\delta\lambda < 1$ holds asymptotically as $\pi \to \infty$. □

Proof of Result (3).

Since $\varphi(q,x)$ is increasing in x by Lemma 1(ii), we have $\theta < 1$. By taking total derivative of the logarithm of (17) and using Lemma 1(i), we obtain

$$\frac{d\log\theta}{d\pi} = \left(\frac{1}{\varphi(q,\eta-1)} - \frac{1}{\varphi(q,\eta-1-\mu/\pi)}\right)\frac{d\log q}{d\pi} + \frac{\varphi_x}{\varphi}(q,1-\eta+\mu/\pi)\frac{\mu}{\pi^2}, \quad (32)$$

where, by Lemma 1(ii), the second term in the right-hand side is equal to

$$\left(\frac{\log q}{1-q^{\eta-1-\mu/\pi}} + \frac{1}{\eta-1-\mu/\pi}\right)\frac{\mu}{\pi^2}.$$

The term $\log q/(1-q^{\eta-1-\mu/\pi})$ converges to 0 as $\pi \to \infty$, because $\eta > 1$ and $(\log q)/\pi$ converges to a constant. Thus, the second term in the right-hand side of (32) asymptotes to $(\mu/(\eta-1))\pi^{-2}$.

The first term in the right-hand side of (32) is

$$\left(\frac{1}{\varphi(q,\eta-1)} - \frac{1}{\varphi(q,\eta-1-\mu/\pi)}\right)\frac{d\log q}{d\pi} = -\int_{\eta-1-\mu/\pi}^{\eta-1}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\frac{d\log q}{d\pi}.$$

The right-hand side is negative by Lemma 1(v). Now, for some $x_5 \in [\eta-1-\mu/\pi, \eta-1]$,

$$\int_{\eta-1-\mu/\pi}^{\eta-1}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx = \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x_5)\frac{\mu}{\pi}.$$

Hence, using Lemma 1(v) and the asymptotic result for $d\log q/d\pi$, we obtain

$$-\int_{\eta-1-\mu/\pi}^{\eta-1}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\frac{d\log q}{d\pi} \sim -\frac{q^{x_5}\log(q^{x_5})-(q^{x_5}-1)}{(q^{x_5}-1)^2}\frac{\log(1+\mu\delta\eta)}{\pi}.$$

Since $x_5 > 0$, this term declines as $e^{-x_5 \pi}$ for large π. Thus, this negative first term in the right-hand side of (32) is dominated by the positive second term for large π. Hence $d \log \theta / d\pi > 0$ for sufficiently large π. □

Proof of Result (2).

From (18), we have $\lambda = \mu\varphi(p^*, 1-\eta)/(\varphi(q, \mu/\pi)(1-\theta))$. Using (12,14) we have

$$\frac{\varphi(p^*, 1-\eta)}{\varphi(q, \mu/\pi)} = \frac{\left(\frac{\varphi(q, 1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)}\right)^{-1} q^{1-\eta} - 1}{(1-\eta)\varphi(q, \mu/\pi)} = \frac{1}{\eta - 1}\left[\frac{1}{\varphi(q, \mu/\pi)} - \frac{q^{1-\eta}}{\varphi(q, 1-\eta+\mu/\pi)}\right].$$

Since $\varphi(q, 1-\eta+\mu/\pi)$ is bounded and since $(\log q)/\pi$ converges to a constant, the second term in the square brackets declines to zero exponentially as $\pi \to \infty$, whereas the first term declines only as $1/\pi$. Thus, the sum of two terms is increasing for sufficiently large π. We have also shown previously that $1/(1-\theta)$ is asymptotically increasing in π. Hence, λ is increasing in π for sufficiently large π. □

Proof of Proposition 4: The mean and variance of L conditional on $m_0 = 1$ converge as $n \to \infty$ to $1/(1-\theta)$ and $\theta/(1-\theta)^3$, respectively. Furthermore, as $n \to \infty$,

$$\Pr(L = \ell \mid m_0) \to \frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell - m_0}}{(\ell - m_0)!}$$

for $\ell = m_0, m_0+1, m_0+2, \ldots$. The tail of the above probability function satisfies

$$\frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell - m_0}}{(\ell - m_0)!} \propto e^{-(\theta - 1 - \log\theta)\ell} \ell^{-1.5} \qquad \textbf{as } \ell \to \infty.$$

The probability generating function $\Psi(z)$ of the sum L of a branching process with initial value 1 has a recursive form as $\Psi(z) = z\Phi(\Psi(z))$, where Φ is a probability generating function of the number of children born from a parent. In our case, Φ

follows a Poisson distribution with mean θ asymptotically as $n \to \infty$. By the property of a probability generating function, $\Phi(1) = \Psi(1) = 1$. Moreover, $\Phi'(1)$ is equal to the mean and $\Phi''(1) + \Phi'(1) - (\Phi'(1))^2$ is equal to the variance. Thus, $\Phi'(1) = \theta$ and $\Phi''(1) + \Phi'(1) - (\Phi'(1))^2 = \theta$. Hence, $\Phi''(1) = \theta^2$.

Using the recursive relationship, we obtain $\Psi'(z) = \Phi(\Psi(z)) + z\Phi'(\Psi(z))\Psi'(z)$ and $\Psi''(z) = 2\Phi'(\Psi(z))\Psi'(z) + z\Phi''(\Psi(z))(\Psi'(z))^2 + z\Phi'(\Psi(z))\Psi''(z)$. Evaluating at $z = 1$, we obtain the mean of L as $\Psi'(1) = 1/(1-\theta)$. For $\Psi''(1)$, we have

$$\Psi''(1) = \frac{\theta(2-\theta)}{(1-\theta)^3}.$$

Hence, the variance of L is

$$\Psi''(1) + \Psi'(1) - (\Psi'(1))^2 = \theta/(1-\theta)^3.$$

The sum L conditional on m_0 of a Poisson branching process is known to follow the Borel-Tanner distribution (19) (see Kingman [22]; Nirei [27, 28]). Applying the Stirling's formula for the factorial $(\ell - m_0)!$ in (19), we obtain (20).

Preparation for Proof of Proposition 5

To fully characterize $d \log P_t$, we need to pin down the distribution of m_0 and the unconditional distribution of L. To do so, first investigate the direct effect of the firm that draws a Calvo shock. Its effect differs from the effect of a firm which reprices at the extensive margin. First, its repricing size is not $\log q$. This difference can be safely ignored when n is large. However, the different repricing size leads to the second difference: its impact on the extensive margin which defines the distribution of m_0. This point can be formally analyzed as follows.

Suppose firm i is hit by the Calvo event and reprices from $P_{i,t}$ to $P^*_{i,t}$. The size of price jump is $\Delta \log P_{i,t} = \log p^* - \log p_{i,t}$. Thus, we obtain that the decline of $s_{j,t}$ for

$j \neq i$ caused by the increase in $\log P_{i,t}$ is equal to $\epsilon_o(s_i)/n$, where

$$\epsilon_o(s_i) := \frac{1}{\log q} \frac{p^{*1-\eta} - p_{i,t}^{1-\eta}}{1-\eta}.$$

This implies that firm i's price adjustment has an impact on the state of other firms with an order of magnitude $1/n$. This shift in $s_{j,t}$ causes firm j to adjust its price if $s_{j,t}$ is located close enough to the adjustment threshold. Namely, firms in $(0, \epsilon_o(s_i)/n]$ will adjust their prices as an optimal response to the price change by i. The probability for firm j to adjust is $F(\epsilon_o(s_i)/n)$, where $F(\cdot)$ denotes the cumulative distribution function of s. Since there are $n-1$ firms that are affected by the initial price change by i, the number of firms that adjust due to the price change by i follows a binomial distribution with population $n-1$ and probability $F(\epsilon_o(s_i)/n)$. This is the distribution of m_0.

When n tends to infinity, the binomial distribution of m_0 asymptotes to a Poisson distribution. Since ϵ_o has an order of magnitude $1/n$, $nF(\epsilon_o(s_i)/n)$ tends to

$$\theta_{\epsilon_o(s_i)} := \frac{f_o}{\log q} \frac{p^{*1-\eta} - p_{i,t}^{1-\eta}}{1-\eta}$$

where f_o denotes the density of firms at the repricing threshold. Then, when n tends to infinity and given $\epsilon_o(s_i)$, m_0 asymptotically follows a Poisson distribution with mean $\theta_{\epsilon_o(s_i)}$. Given the distribution of m_0, the distribution of L conditional on the state s_i of the firm that draws a Calvo shock is obtained as follows.

Lemma 3. *As $n \to \infty$, the probability function of L conditional on s_i converges as*

$$\Pr(L = \ell \mid s_i) \to \frac{\theta_{\epsilon_o(s_i)}}{\ell!} e^{-(\theta\ell + \theta_{\epsilon_o(s_i)})} (\theta\ell + \theta_{\epsilon_o(s_i)})^{\ell-1} \tag{33}$$

for $\ell = 1, 2, \ldots$.

Proof.

A profile of state $(s_{i,t})_{i=1}^n$ is drawn randomly from a joint density function f^n. $L_t > 0$ holds only when a firm draws a Calvo event in t. The firm's price adjustment affects the profile s, and the other firms' repricing continues until $s_{i,t} \in (0, 1]$ is achieved. This new profile constitutes a momentary equilibrium in t. L_t is the number of firms that are involved in the repricing within the instance t. We compute L_t by a best response dynamics proposed by Nirei [28] as follows.

Suppose that i is hit by a Calvo event, and all the other firms' $s_{j,t}$ are reduced by $\epsilon_o(s_i)/n$. If there are no firms with $s_{j,t}$ in the range $(0, \epsilon_o/n]$, the adjustment process stops and the equilibrium price distribution is obtained. If there are some firms in the range, each of them adjusts the price from $p_{j,t}$ to p_t^*. The impact of the price change by j on other firms is calculated similarly to the case of firm i, and denoted by ϵ_1/n. Noting that $p_{j,t} \in (\underline{p}, \underline{p} + (\log p^* - \log \underline{p})\epsilon_o/n)$, $p_{j,t}$ converges to $\underline{p}$ as $n \to \infty$. Thus, when n is large,

$$\epsilon_1 = \frac{p^{*1-\eta} - p_{j,t}^{1-\eta}}{(1-\eta)\log q} \underset{n\to\infty}{\to} \frac{\varphi(q, 1-\eta)\underline{p}^{1-\eta}}{\log q}.$$

Let m_1 denote the number of firms with $s_{j,t} \in (\epsilon_o/n, (\epsilon_o + \epsilon_1)/n]$. m_1 follows a binomial distribution with population $n - m_1$ and probability $F((\epsilon_o + \epsilon_1)/n) - F(\epsilon_o/n)$.

The shift of $s_{k,t}$ by the price adjustments of m_1 firms may cause further adjustments. The adjustment process is formulated as a best response dynamics m_u for $u = 0, 1, \ldots, U$ as defined in Section 4.1. The sum $L = \sum_{u=0}^{U} m_u$ conditional on m_0 has been already obtained in (19). We combine (19) with the fact that m_0 follows a

Poisson distribution with mean θ_{ϵ_o}.

$$\Pr(L=\ell) = \sum_{m_0=0}^{\ell} \frac{\theta_{\epsilon_o}^{m_0} e^{-\theta_{\epsilon_o}}}{m_0!} \frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell-m_0}}{(\ell-m_0)!}$$
$$= \frac{\theta_{\epsilon_o} e^{-\theta\ell-\theta_{\epsilon_o}}}{\ell} \sum_{m_0=1}^{\ell} \frac{(\theta\ell)^{\ell-m_0}\theta_{\epsilon_o}^{m_0-1}}{(\ell-m_0)!(m_0-1)!}$$
$$= \frac{\theta_{\epsilon_o}}{\ell!} e^{-(\theta\ell+\theta_{\epsilon_o})}(\theta\ell+\theta_{\epsilon_o})^{\ell-1}.$$

□

Proof of Proposition 5: For sufficiently large n and in a range of sufficiently large π, the variance of inflation $d\log P_t$ is increasing in π.

Using Lemma 3, we derive the variance of $d\log P$. The main determinants of the variance, L and $\log q$, have already been established. What remains to show is that the quantitative role played by the Calvo shock directly is negligible.

In each instance, there may be a firm that reprices due to the Calvo shock. Conditional on a firm repricing due to the Calvo shock, m_0 denotes the number of firms that reprice due to the decreased relative price caused by the firm that reprices due to the Calvo shock. Asymptotically as $n \to \infty$, m_0 follows a Poisson distribution with mean θ_0. The probability generating function of the total number of firms that adjust at the instance of the exogenous repricing event is $\Psi_0(z) := E[\Psi^{m_0}(z)]$. Note that m_0 follows a Poisson distribution with mean $\theta_{\epsilon_o(s)}$. This notation makes explicit the dependence of ϵ_o on s which follows a distribution function $f(s)$. Then, the probability generating function is solved as $\Psi_0(z) = \int_0^1 e^{\theta_{\epsilon_o(s)}(\Psi(z)-1)} f(s)ds$. The derivative with respect to z

yields:

$$\Psi_0'(z) = \int_0^1 \Psi'(z)e^{\theta_{\epsilon_o(s)}(\Psi(z)-1)}\theta_{\epsilon_o(s)}f(s)ds$$
$$\Psi_0''(z) = \int_0^1 \left(\Psi''(z)e^{\theta_{\epsilon_o(s)}(\Psi(z)-1)}\theta_{\epsilon_o(s)} + e^{\theta_{\epsilon_o(s)}(\Psi(z)-1)}(\Psi'(z)\theta_{\epsilon_o(s)})^2\right) f(s)ds.$$

Thus, we obtain that $\Psi_0'(1) = E[\theta_{\epsilon_o(s)}]/(1-\theta)$ and $\Psi_0''(1) = E[\theta_{\epsilon_o(s)}]\theta(2-\theta)/(1-\theta)^3 + E[\theta^2_{\epsilon_o(s)}]/(1-\theta)^2$.

The aggregate price $\log P_t$ follows a compound Poisson process with intensity rate μ and a jump size which is the multiple of individual repricing size $\log q$ and the total number of the firms that adjust in the branching process L_t.

The number of firms that draw such events in a unit time follows a Poisson distribution with mean μn. Let X denote the total number of firms repricing in a unit time and Y a random variable following the Poisson distribution with mean μn. Note that L follows the same distribution as X conditional on $Y = 1$. Then, $V(X) = E[V(X \mid Y)] + V(E[X \mid Y]) = E[Y]V(L) + V(Y)E[L]^2 = \mu n E[L^2]$. Thus, the variance of $d \log P_t$ in a unit time is $(\log q)^2 \mu E[L^2]/n$. Note

$$E[L^2] = E[L(L-1)] + E[L] = \Psi_0''(1) + \Psi_0'(1) = \frac{\theta_0}{(1-\theta)^3} + \frac{E[\theta^2_{\epsilon_o(s)}]}{(1-\theta)^2}$$

where $\theta_0 := E[\theta_{\epsilon_o(s)}]$. Using $E[p_{i,t}^{1-\eta}] = 1$, we have

$$\theta_0 = \frac{E[p^{*1-\eta} - p_{i,t}^{1-\eta}]}{(1-\eta)\log q} \frac{\log q}{\varphi(q, \mu/\pi)} = \frac{\varphi(p^*, 1-\eta)}{\varphi(q, \mu/\pi)}.$$

In the proof of Proposition 3, we showed that $\varphi(p^*, 1-\eta)/\varphi(q, \mu/\pi) = O(1/\pi)$ and $1-\theta \sim \mu/(\pi(\eta-1))$. Therefore, $\theta_0/(1-\theta)$ is convergent to a finite value. Moreover, $1/(1-\theta)^2 \nearrow \infty$ as $\pi \to \infty$. Hence, $\theta_o/(1-\theta)^3$ is asymptotically increasing in π.

Next, we show that $E[\theta^2_{\epsilon_o}]/(1-\theta)^2$ is also asymptotically increasing. First we derive

$E[q^{s_{i,t}x}]$ for an arbitrary parameter x.

$$\int_0^1 q^{s_{i,t}x}\frac{q^{s_{i,t}\mu/\pi}(\mu/\pi)\log q}{q^{\mu/\pi}-1}ds_{i,t} = \frac{\log q}{\varphi(q,\mu/\pi)}\frac{q^{s_{i,t}(x+\mu/\pi)}|_0^1}{(x+\mu/\pi)\log q} = \frac{\varphi(q,x+\mu/\pi)}{\varphi(q,\mu/\pi)}.$$

Note that

$$\begin{aligned}\theta_{\epsilon_o} &= \frac{f_o}{\log q}\frac{p^{*1-\eta}-p_{i,t}^{1-\eta}}{1-\eta} = \frac{(\underline{p}q)^{1-\eta}}{\varphi(q,\mu/\pi)}\frac{1-q^{(s_{i,t}-1)(1-\eta)}}{1-\eta}\\ &= \frac{q^{1-\eta}}{\varphi(q,1-\eta+\mu/\pi)}\frac{1-q^{(s_{i,t}-1)(1-\eta)}}{1-\eta}.\end{aligned}$$

Then, $E[\theta^2_{\epsilon_o(s_{i,t})}]$ can be calculated as

$$\begin{aligned}&\left(\frac{q^{1-\eta}}{(1-\eta)\varphi(q,1-\eta+\mu/\pi)}\right)^2\left(1-2q^{\eta-1}E[q^{s_{i,t}(1-\eta)}]+q^{2(\eta-1)}E[q^{2(1-\eta)s_{i,t}}]\right)\\ &=\left(\frac{q^{1-\eta}}{(1-\eta)\varphi(q,1-\eta+\frac{\mu}{\pi})}\right)^2\left(1-2q^{\eta-1}\frac{\varphi(q,1-\eta+\frac{\mu}{\pi})}{\varphi(q,\frac{\mu}{\pi})}+q^{2(\eta-1)}\frac{\varphi(q,2(1-\eta)+\frac{\mu}{\pi})}{\varphi(q,\frac{\mu}{\pi})}\right)\\ &\sim q^{2(1-\eta)}-\frac{2q^{1-\eta}}{(\eta-1)\varphi(q,\mu/\pi)}+\frac{\varphi(q,2(1-\eta)+\mu/\pi)}{\varphi(q,\mu/\pi)},\end{aligned}$$

where we used $\varphi(q,1-\eta+\mu/\pi)\sim\eta-1$. Note that the first two terms decline exponentially in π, while the last term declines as $1/\pi$. Also using $\varphi(q,2(1-\eta)+\mu/\pi)\sim 1/(2(\eta-1))$, we obtain $E[\theta^2_{\epsilon_o(s_{i,t})}]\sim 1/(2(\eta-1)\varphi(q,\mu/\pi))$. Thus,

$$\frac{E[\theta^2_{\epsilon_o(s_{i,t})}]}{(1-\theta)^2}\sim\frac{1}{2(\eta-1)\varphi(q,\mu/\pi)}\left(\frac{1}{\mu/((\eta-1)\pi)}\right)^2=\frac{(\eta-1)\pi/\mu}{2(q^{\mu/\pi}-1)}.$$

Since $(\log q)/\pi$ converges to a finite value, $q^{\mu/\pi}$ in the denominator is bounded, whereas the numerator increases as π. Hence, $E[\theta^2_{\epsilon_o(s_{i,t})}]/(1-\theta)^2$ is asymptotically increasing in π.

Combining these results, $E[\theta_{\epsilon_o(s)}]/(1-\theta)^3+E[\theta^2_{\epsilon_o(s)}]/(1-\theta)^2$ increases as π increases for sufficiently large π.

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