IMES DISCUSSION PAPER SERIES

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Discussion Paper No. 2015-E-11

IMES

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

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JAPAN

IMES Discussion Paper Series 2015-E-11
July 2015

Variation of Wrong-Way Risk Management and Its Impact on Security Price Changes

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Abstract

Wrong-way risk arises when an unexpected adverse change in interdependency among financial products or financial variables (such as interest rates, equities, exchange rates, credits, and commodities) triggers huge losses in portfolios. During the global financial crisis of 2007–09, financial institutions suffered huge losses due to the materialization of wrong-way risk. While its importance has been recognized by market participants, however, they have not reached a consensus on how to model and measure it. This paper proposes a method to model wrong-way risk in pricing and risk management and investigate the mechanism in which it can generate booms and busts in security prices. This paper assumes that there exist two types of investors with differing views on the management of the wrong-way risk and that they trade a derivative security with two underlying assets. The prudent (imprudent) investors are supposed to have a heavy (thin) tail structure in the joint distributions of risky assets in their models. This assumption implies that the reservation value on the security held by imprudent investors is higher than that by prudent investors. In this setup, a numerical analysis shows that (1) as time passes from the latest materialization of wrong-way risk and many investors tend to be imprudent, the market price is bidden up; and (2) once the wrong-way risk materializes, many imprudent investors realize the necessity for prudent management of wrong-way risk and thus the price drops suddenly to the lowest level.

Keywords: Wrong-way risk; Systemic risk; Jump-diffusion process; Asset pricing; Market microstructure

JEL Classification: G12, G13, G32

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The authors would like to thank Katsumasa Nishide, Peng Xu, and the staff of the Institute for Monetary and Economic Studies (IMES), the Bank of Japan, for their useful comments and discussion. The views expressed in this paper are those of the authors and do not necessarily reflect the official views of the affiliated institutions of the authors. All the remaining errors are our own.

I. Introduction

It has been widely recognized that credit default swaps (CDSs) were one of the major driving forces behind the malfunctioning in the financial system during the global financial crises of 2007–09. During the crisis, American International Group (AIG) suffered a huge amount of losses from its CDS position as a protection seller and its business condition deteriorated drastically.[1] For protection buyers of AIG, this meant that the counterparty risk and the amount of exposure to AIG soared unexpectedly at the same time and they were forced to record huge mark-to-market losses caused by the inflated credit valuation adjustment (CVA).[2]

The CDS problem just stated can be considered as a kind of materialization of wrong-way risk (WWR). WWR arises when an unexpected adverse change in interdependency among financial products or financial variables (such as interest rates, equities, exchange rates, credits, and commodities) triggers huge losses in portfolios.[3] If market participants do not consider WWR, financial products tend to be overvalued compared with their true value incorporating WWR. Under favorable conditions without any adverse events, an optimistic view prevails and the degree of overvaluation continues to widen. As a result, once an unexpected adverse shock occurs, a huge loss is realized in these products. During the crises, many events were observed in which WWR eventuated and the financial system became destabilized. In this regard, Senior Supervisors Group (2009) states that the materialization of systemic risk during the financial crisis is deeply related to WWR.

While the importance of WWR has been recognized by market participants, they have not reached a consensus on how to model and measure it. Current financial regulations require financial institutions to measure and monitor their portfolio risks up to the second moments (volatility and correlation) as a minimum requirement. Against this background, there is no need to consider the tail structure of the joint distribution among multiple financial products or variables, which are the source of WWR, and therefore only a limited number of financial institutions try to model and measure WWR. A contribution of this paper is to show that even if all investors implement prudent risk management up to the marginal distribution with linear correlation across risk categories, it is insufficient to prevent the market from suffering huge losses caused by the materialization of WWR.

This paper proposes a method to model WWR in pricing and risk management and

[1] During the crisis, AIG lost US$99.2 billion from its CDS position as a protection seller in fiscal 2008. By the end of March 2008, AIG had accumulated a CDS position of up to US$475 billion (principal amount).

[2] According to the Basel Committee on Banking Supervision (BCBS), two-thirds of losses related to counterparty credit risk during the financial crisis were generated due to mark-to-market losses via CVA.

[3] The International Swaps and Derivatives Association (ISDA) defines WWR as a risk that occurs when "exposure to a counterparty is adversely correlated with the credit quality of that counterparty." In this paper, we do not limit the scope of WWR to the correlation between the counterparty credit risk and the derivative exposure, but we cover all the events in which an unexpected adverse change in interdependency among financial products or financial variables (such as interest rates, equities, exchange rates, credits, and commodities) amplifies the loss. In addition, we place more emphasis on the tail property of WWR.

investigates the mechanism by which WWR can generate booms and busts in security prices. We consider an economy with two risky assets and use the bivariate jump-diffusion process with a common jump component to express an unexpected adverse change in interdependency. There is a market for a derivative security with two underlying assets, and there are a number of investors in the market trading the security with arm's-length transactions.[4] The security is assumed to be exposed to WWR. Then, we assume that although all the investors recognize an identical marginal distribution for each asset and the same covariance among assets, there exist two kinds of investors with differing techniques for considering the tail structure of the joint distribution: prudent (or conservative) and imprudent (or optimistic) investors. Prudent (imprudent) investors suppose a heavy (thin) tail structure and require a high (low) risk premium on WWR. As favorable conditions continue and many investors tend to be imprudent, security prices are bidden up. Once a large co-movement of underlying assets breaks out, however, the market price drops suddenly. This mechanism generates price trajectories with cyclical patterns of boom and bust.

The remainder of the paper is structured as follows. Section 2 gives a brief review of the existing literature related to our study. Section 3 proposes a modeling approach for WWR in the pricing of derivative securities. We also introduce two types of investors with differing techniques for managing WWR. Section 4 provides the trading mechanism for our market and shows how the market price is formed. We prove that under certain conditions the market price converges to an overestimation price that does not reflect WWR. Section 5 demonstrates a numerical analysis of our model and shows how the market price fluctuates due to the differing approaches for modeling WWR across investors. Section 6 summarizes the paper.

II. Literature Review

Our study is related to three topics: (1) modeling of WWR; (2) the jump process and its application to risk modeling; and (3) asset pricing with heterogeneous agents. In this section, we briefly explain prior research on these issues.

A. Modeling of WWR

Most research on modeling WWR focuses on the relationship between counterparty credit risk and the derivative exposure to the counterparty. To the best of our knowledge, the first article dealing with WWR in derivative pricing is Duffie and Huang (1996). They examine the effect of correlation between the credit quality of the counterparty and the London Interbank Offered Rate (LIBOR) on the pricing of interest rate swaps. The research on WWR has seen many advances recently, especially since the financial crises of 2007–09. The fifth chapter of Brigo, Morini, and

[4] A market in which financial products are traded with arm's-length transactions among investors is defined as an over-the-counter (OTC) market.

Pallavicini (2013) evaluates WWR embedded in CVA in the pricing of interest rate derivatives and avoids the undervaluation of derivatives indicated by Morini (2011) by introducing a jump component into the default intensity process of the counterparty. Brigo, Morini, and Pallavicini (2013) argue that it is indispensable to consider WWR properly in the valuation of derivative securities because the price impact of WWR is generally not trivial. Pykhtin and Sokol (2013) construct a model of the exposure for systemically important counterparties (SICs) and show that a default by SICs strongly affects market factors, which triggers the materialization of WWR and thus adversely affects the exposure of derivative transactions. There are many other studies dealing with WWR in CVA such as Redon (2006), Lipton and Sepp (2009), Caesari *et al.* (2009), Gregory (2012), Hull and White (2012), and it is not too much to say that the consideration of WWR in the valuation of OTC derivatives has become common practice in global financial markets. In addition to the derivative pricing related to WWR, a number of studies focus on the correlation structure among several assets from a statistical viewpoint. These studies emphasize that the amount of risk estimated in any metric depends largely on the tail structure of the joint distribution and that asset prices might be underestimated unless full information is available on the joint distribution. For example, Embrechts, Wang, and Wang (2014) compare the maximum and minimum of both value at risk (VaR) and expected shortfall (ES) on asset portfolios without using a specific model of the tail structure for the joint distribution (e.g., copulas) and they argue that the improper modeling of the tail structure causes a serious underestimation of risk regardless of the risk metrics.

B. The Jump Process and Its Application to Risk Modeling

Jump models, which are central to modeling WWR in this paper, have been utilized in many areas of financial studies. For example, in the pricing of credit derivatives, it is often observed that the credit spread suddenly widens along with the devaluation of the reference name's creditworthiness (e.g., a downgrade), and the second chapter of Morini (2011) indicates that the undervaluation of derivative prices is deeply related to the modeling skill of this "jump" phenomenon. He concludes that, by comparing the default intensity processes with and without jumps, the introduction of a jump component in the intensity process is essential to evaluate the credit derivative appropriately. Pan (2002) reports that approximately 40 percent of the risk premium estimated from S&P500 option prices is attributed to the jump risk embedded in the underlying process and argues that the modeling approach to jump risk largely affects the estimation of the risk premium. Das and Uppal (2004) examine the optimal portfolio allocation for several risky assets, whose prices follow the jump diffusion process with systemic jumps (i.e., simultaneous jumps), and they conclude that the existence of systemic jumps worsens the effectiveness of portfolio diversification and causes serious losses to highly levered portfolios. Duffie and Garleanu (2001) introduce three types of jump components in the default intensity process—the common jump, the sector jump, and the idiosyncratic jump—to express the complex correlation structure of defaults among reference names in their pricing model for collateralized debt obligations (CDOs).

C. Asset Pricing with Heterogeneous Agents

Research on asset pricing with heterogeneous agents, which we introduce to generate booms and busts in security prices, has developed rapidly together with progress in market microstructure studies. Studies in this area clarify the determination mechanism for the transaction price in a variety of markets (stocks, interest rates, commodities, foreign exchange rates (FX), etc.). Among them, Duffie, Garleanu, and Pedersen (2005) investigate the impact of both the price-bargaining behavior and the counterparty-searching behavior of market participants on the transaction price under heterogeneous agents with arm's-length transactions. Their model combines the search model and the bargaining model and uses the two-step auction procedure in which an investor first searches for his/her trading counterparty and then negotiates with him/her to determine an individual transaction price. Kijima and Uchida (2005) extend the model of Duffie, Garleanu, and Pedersen (2005) using a Markov chain model. They assume that there are a finite number of investors, and they explore the effects of supply-demand fluctuations on transaction prices for an instrument. Their contribution is that they express the supply-demand balance across different types of investors as a distribution but not as a point estimation which is examined in Duffie, Garleanu, and Pedersen (2005).

III. Modeling of WWR

In this paper, we define WWR as the hazard of suffering huge losses caused by an unexpected adverse change in interdependency among risky assets. Specifically, we assume that the management of WWR is related to how to model the tail part of the joint distribution and use the bivariate jump-diffusion process with a common jump component to express the unexpected adverse change in interdependency between two risky assets. To examine the impact of WWR on the market price of a derivative security, we construct several structured products that are exposed to WWR. We also introduce two types of investors with differing views on the management of WWR. One takes WWR into account in the pricing and risk management of the security, while the other does not. The former views WWR as highly probable (type H), but the latter views it as less probable (type L). The reservation value of the security of type H investors is lower than that of type L investors, because the former takes WWR into account in the valuation of the security, while the latter does not.

A. Stochastic Environment

We introduce a filtered probability space $(\Omega,\ \mathcal{F},\ \mathbb{F},\ \mathbb{P})$ where $\mathcal{F} := \mathcal{F}_T, T \leq +\infty$, $\mathbb{F} \coloneqq \{\mathcal{F}_t\colon t \geq 0,\ \mathcal{F}_t \subset \mathcal{F}\}$ is a complete, right-continuous, increasing filtration and $\mathbb{P}$ is a (physical) probability measure. Adapted to this filtration are two correlated standard Brownian motions $\{B_{1,t}, B_{2,t}\}_{t\geq 0}$ and four Poisson jump processes $\{J_I^1(t), J_I^2(t), J_C^1(t), J_C^2(t)\}_{t\geq 0}$ where $J_l^i(t) \coloneqq \int_0^t \int_{\mathbb{Z}} z_{li} dN_l^i(z_{li}, s)$ ($z_{li} \in \mathbb{Z} \coloneqq (-1, +\infty)$, $i \in \{1,2\}$ and $l \in \{C, I\}$). We set a special assumption on jump components that jumps occur simultaneously between $i = 1$ and 2 on $\{J_C^1(t), J_C^2(t)\}_{t\geq 0}$ while independently on $\{J_I^1(t), J_I^2(t)\}_{t\geq 0}$. Hence, we define $\{J_C^1(t), J_C^2(t)\}_{t\geq 0}$ as the "common jump" processes and $\{J_I^1(t), J_I^2(t)\}_{t\geq 0}$ as the "independent jump" processes. Jumps are assumed to occur independently over time. $N_l^i(z_{li}, t)$ represents a Poisson random measure that counts the number of jumps occurred up to time t, and its compensated version is given by $\tilde{N}_l^i(z_{li}, t) = N_l^i(z_{li}, t) - \mathbb{E}[N_l^i(z_{li}, t)] = N_l^i(z_{li}, t) - v_l^i(z_{li}, t)$ where $v_l^i(z_{li}, t)$ is a compensator with $v_l^i(\mathbb{Z} \times [0, \mathrm{t}]) < +\infty$ a.s. $(\forall t \geq 0)$. Here we specify the compensator $v_l^i(z_{li}, t)$ $(\forall (i, l))$ such that $v_l^i(z_i, t) \coloneqq \lambda_{li} t \varphi_l^i(z_i)$ where $\varphi_l^i(z_i)$ is a marginal distribution function with respect to jump size z_{li} and $\lambda_{li} (\geq 0)$ is a constant Poisson intensity. As for common jump intensities, we set $\lambda_{C1} = \lambda_{C2} \coloneqq \lambda_C$. We also assume that the jump size z_{li} follows the log-normal distribution satisfying $u_{li} \coloneqq \ln(1 + z_{li}) \sim N_l^i(\zeta_{li}, \delta_{li}^2)$ $(\forall (i, l))$ where u_{C1} and u_{C2} are positively correlated with correlation coefficient $\rho_{12}^J \geq 0$ while u_{I1} and u_{I2} are independently distributed. For simplicity, we assume that the asset price in the market is observable at the end of any date.

There are one riskless bond and two risky assets in a market. Their respective price processes $\{P_t\}_{t\geq 0}$ and $\{S_{1,t}, S_{2,t}\}_{t\geq 0}$ evolve according to the following equations:

$$dP_t = rP_t dt,$$

$$\begin{aligned} \frac{dS_{i,t}}{S_{i,t}} &= \mu_i dt + \sigma_i dB_{it} + \int_{\mathbb{Z}} z_{Ii}\, d\tilde{N}_I^i(z_{Ii}, t) + \int_{\mathbb{Z}} z_{Ci}\, d\tilde{N}_C^i(z_{Ci}, t), \quad \forall i \in \{1,2\}, \\ &= (\mu_i - \lambda_{Ii}\kappa_{Ii} - \lambda_{Ci}\kappa_{Ci})dt + \sigma_i dB_{i,t} + \int_{\mathbb{Z}} z_{Ii}\, dN_I^i(z_{Ii}, t) + \int_{\mathbb{Z}} z_{Ci}\, dN_C^i(z_{Ci}, t), \end{aligned} \tag{1}$$

where

$$\kappa_{Ii} = \int_{\mathbb{Z}} z_{Ii} \varphi_I^i(dz_{Ii}) = \exp\left(\zeta_{Ii} + \frac{1}{2}\delta_{Ii}^2\right) - 1,$$

$$\kappa_{Ci} = \int_{\mathbb{Z}} z_{Ci} \varphi_C^i(dz_{Ci}) = \exp\left(\zeta_{Ci} + \frac{1}{2}\delta_{Ci}^2\right) - 1,$$

$$dB_{2,t} = \rho_{12} dB_{1,t} + \sqrt{1 - \rho_{12}^2}\, dB_{0,t},$$

where $r \geq 0$ is a constant risk-free rate, $\{B_{0,t}\}_{t\geq 0}$ is a standard Brownian motion independent of all other variables and $\rho_{12} \in \mathbf{R}$ is a correlation coefficient for the diffusion part of the processes $(\rho_{12} dt = d\langle B_{1,t}, B_{2,t}\rangle)$. We assume that the price dynamics of two risky assets are given by the bivariate jump-diffusion process in which the jump component consists of the common jump and the independent jump.[5] These dynamics are the common knowledge of all investors in the market.

B. Investors with Differing Views on the Management of WWR

There are $N(< +\infty)$ investors in the security market with two types of differing modeling approaches to WWR on the stochastic processes of two risky assets. "Type H" investors are considered to have a prudent view on managing WWR, while "type L" investors have an imprudent view. In this paper, we impose a rather unprecedented assumption that type H investors focus only on the common jump in the modeling of WWR, while type L investors focus only on the independent jumps, even though the dynamics of risky assets expressed in equation (1) are the common knowledge of all the investors. This assumption implies that, compared with equation (1), type H investors assess WWR more conservatively by imposing a heavier probability on the WWR scenarios, while type L investors recognize WWR more optimistically by imposing a thinner probability on the WWR events.

Here, we should note that we do not consider equation (1) to be a "correct" model, but on the contrary both the common jump model and the independent jump model could be "incorrect" models. No investor is assumed to have a correct knowledge of WWR, and thus all the models could be categorized as incorrect ones. Therefore, all the investors are exposed to *model uncertainty*[6] with respect to the modeling of WWR. Below, we modify equation (1) and generate two nested versions of jump processes so that each can be fitted to the WWR management view of their respective types. We note that both types are assumed to have the same filtered space $(\Omega,\ \mathcal{F},\ \mathbb{F})$ but different probability assessments with respect to future events (scenarios).

The price processes of risky assets recognized by type H are given by the bivariate jump-diffusion process with the jump component consisting only of common jumps:

$$\frac{dS_{i,t}}{S_{i,t}} = (\mu_i - \lambda_{Ci}\kappa_{Ci})dt + \sigma_i dB_{i,t} + \int_{\mathbb{Z}} z_{Ci}\, dN_C^i(z_{Ci}, t), \qquad \forall i \in \{1,2\}, \tag{2}$$

where

$$\kappa_{Ci} = \mathbb{E}^H[z_{Ci}] = \exp\left(\zeta_{Ci} + \frac{1}{2}\delta_{Ci}^2\right) - 1,$$

$$dB_{2,t} = \rho_{12}^H dB_{1,t} + \sqrt{1 - (\rho_{12}^H)^2}\ dB_{0,t}.$$

[5] In this paper, the common jump is assumed to have a systematic impact on the entire market and this kind of jump risk cannot be diversified away. On the other hand, the independent jump is assumed to occur for either a firm-specific or sector-specific reason and does not have a systematic impact, because its risk can be diversified away.

[6] "Model uncertainty" is related to the uncertainty on the choice of the model itself. Unlike "risk," which is the uncertainty on outcomes for which the probabilities are known, "model uncertainty" is recognized when several specifications are possible for such probabilities (Cont [2006]). In our setting, although each investor faces the model uncertainty, he/she will adopt either the common jump model or the independent jump model dependent on his/her type.

This modification can be completed with the independent jump intensities in equation (1) converging to zero, that is, $\lambda_{Ii} \to 0, \forall i \in \{1,2\}$. The probability space is modified as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^H)$ where $\mathbb{P}^H$ is a subjective probability measure set by type H investors. We note that the measure $\mathbb{P}^H$ has heavier weights on the WWR events than the measure $\mathbb{P}$ in equation (1). We define the conditional expectation under $\mathbb{P}^H$ as $\mathbb{E}_t^H[\cdot] \coloneqq \mathbb{E}^H[\cdot\,|\mathcal{F}_t]$.

On the other hand, the price processes of type L have no common jumps, and are given by the bivariate jump-diffusion process with the jump component consisting only of independent jumps:

$$\frac{dS_{i,t}}{S_{i,t}} = (\mu_i - \lambda_{Ii}\kappa_{Ii})dt + \sigma_i dB_{i,t} + \int_{\mathbb{Z}} z_{Ii}\, dN_I^i(z_{Ii}, t), \qquad \forall i \in \{1,2\}, \tag{3}$$

where

$$\kappa_{Ii} = \mathbb{E}^L[z_{Ii}] = \exp\left(\zeta_{Ii} + \frac{1}{2}\delta_{Ii}^2\right) - 1,$$

$$dB_{2,t} = \rho_{12}^L dB_{1,t} + \sqrt{1 - (\rho_{12}^L)^2}\, dB_{0,t}.$$

This modification can be completed with the common jump intensity in equation (1) converging to zero, that is, $\lambda_C \to 0$. The probability space is modified as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^L)$ where $\mathbb{P}^L$ is a subjective probability measure set by the type L investor. We note that the measure $\mathbb{P}^L$ has thinner weights on the WWR events than the measure $\mathbb{P}$ in equation (1). We define the conditional expectation under $\mathbb{P}^L$ as $\mathbb{E}_t^L[\cdot] \coloneqq \mathbb{E}^L[\cdot\,|\mathcal{F}_t]$.

Both types are assumed to have the same parameter values on asset price processes (i.e., drifts $\{\mu_1, \mu_2\}$, diffusion coefficients $\{\sigma_1, \sigma_2\}$, jump intensity $\{\lambda_{l1}, \lambda_{l2}\}$ $(\forall l \in \{C, I\})$, and mean and volatility of the jump size $\{(\zeta_{l1}, \delta_{l1}), (\zeta_{l2}, \delta_{l2})\}$ $(\forall l \in \{C, I\})$ are all identical between types) except for the diffusion correlation coefficient $\rho_{12} \in \boldsymbol{R}$ (ρ_{12}^H for type H , ρ_{12}^L for type L and $\rho_{12}^H \neq \rho_{12}^L$) and the jump-size correlation coefficient $\rho_{12}^J \in \boldsymbol{R}_+$ in the common jump process ($\rho_{12}^J = 0$ in the independent jump process). Every parameter value is constant, and the Poisson intensities are identical across assets and across types, that is, $\lambda_{I1} = \lambda_{I2} = \lambda_C$.

C. Moment Restrictions and Tail Structure

Given two distinctive bivariate price dynamics described above, we impose several constraints on means, variances, and correlations of risky assets so that both types have an identical marginal distribution for each asset and the same covariance between assets. These moment restrictions are given by

$$\mathbb{E}^H\left[\frac{dS_{i,t}}{S_{i,t}}\right] = \mu_i = \mathbb{E}^L\left[\frac{dS_{i,t}}{S_{i,t}}\right], \quad \forall i \in \{1,2\},$$

$$Var_H\left[\frac{dS_{i,t}}{S_{i,t}}\right] = (\sigma_i^2 + \lambda\mathbb{E}^m[z_i^2])dt = Var_L\left[\frac{dS_{i,t}}{S_{i,t}}\right], \quad \forall i \in \{1,2\}, \forall m \in \{H,L\},$$

$$Corr_H\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right) = Corr_L\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right),$$

where

$$Corr_H\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right) = \frac{\rho_{12}^H\sigma_1\sigma_2 + \lambda\mathbb{E}^H[z_1z_2]}{\sqrt{\sigma_1^2 + \lambda\mathbb{E}^H[z_1^2]}\sqrt{\sigma_2^2 + \lambda\mathbb{E}^H[z_2^2]}},$$

$$Corr_L\left(\frac{dS_{1,t}}{S_{1,t}}, \frac{dS_{2,t}}{S_{2,t}}\right) = \frac{\rho_{12}^L\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \lambda\mathbb{E}^L[z_1^2]}\sqrt{\sigma_2^2 + \lambda\mathbb{E}^L[z_2^2]}},$$

$$\mathbb{E}^H[z_i^2] = \mathbb{E}^L[z_i^2] = (\kappa_i + 1)\left(\exp\left(\zeta_i + \frac{3}{2}\delta_i^2\right) - 2\right) + 1,$$

$$\mathbb{E}^H[z_1z_2] = (\kappa_1 + 1)(\kappa_2 + 1)\exp\left(\rho_{12}^J\delta_1\delta_2\right) - (\kappa_1 + 1) - (\kappa_2 + 1) + 1.$$

Here, we generate two sample joint distributions for the common jump model (type H) of equation (2) and the independent jump model (type L) of equation (3). We use parameter values satisfying the moment restrictions stated above. These are listed in Table 1. We fix the jump intensity parameter (λ) at $\lambda = 0.05$ so that a jump occurs on average every 20 years.

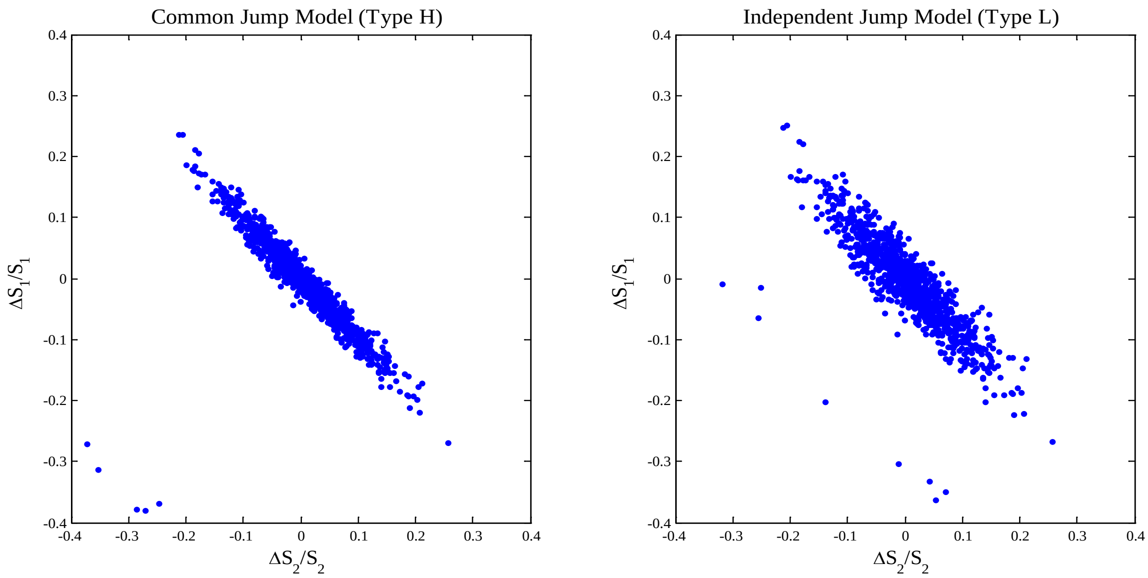
Table 1 Trial Parameter Set

Common Jump Model (Type H)							
$\mu_1(=\mu_2)$	$\sigma_1(=\sigma_2)$	ρ_{12}^H	λ	$\zeta_1(=\zeta_2)$	$\delta_1(=\delta_2)$	ρ_{12}^J	$Corr_H(dS_1/S_1, dS_2/S_2)$
0.00	0.25	–0.9750	0.05	–0.325	0.01	0.999	–0.4384
Independent Jump Model (Type L)							
$\mu_1(=\mu_2)$	$\sigma_1(=\sigma_2)$	ρ_{12}^L	λ	$\zeta_1(=\zeta_2)$	$\delta_1(=\delta_2)$	ρ_{12}^J	$Corr_L(dS_1/S_1, dS_2/S_2)$
0.00	0.25	–0.9134	0.05	–0.325	0.01	—	–0.4384

Figure 1 plots the joint distributions with monthly frequency of 1,000 draws from both models. Both figures look alike, because they have the same statistical properties up to the second moments, but have completely different negative tail structures. On the left-hand side (the common jump model), the interdependency on the negative tail portion is contrary to our expectation that is formed by the negative correlation on the body portion. If the realization of events on the negative tail portion has an adverse impact on our portfolios and if we ignore it, we will suffer a huge loss as a result. This is the materialization of WWR consistent with our definition. On the right-hand side (the independent jump model), there is no event of WWR. Here, we have shown that even if the statistical properties are the same up to the second moments, we can construct two joint distributions with completely different tail structures.

Each model could generate a completely different risk amount due to the different structures on the negative tail portions of the joint distributions. For example, using the sample distributions generated above, we compute the VaR at a confidence level of 99.9 percent (denoted by $VaR_{99.9}$) for a portfolio of $S_1 + S_2$ given conditions that the holding period is one month and the initial value of each asset is set to 1.0. Then the common jump model gives an estimate of $VaR_{99.9} = -0.675$, while the independent jump model generates $VaR_{99.9} = -0.312$. This result suggests that the risk amount of a portfolio would be entirely underestimated if we do not take WWR into account in the modeling of joint distribution.

Figure 1 Sampling of Joint Distributions



In the following analyses, we show that the difference in modeling tail portions of the joint distribution could have the large impact on the valuation of financial instruments in the presence of WWR.

D. Pricing Kernels for Two Types of Investors

In the context of pricing derivative securities, models are specified under the appropriate probability measure. In this paper, we focus on two candidate pricing kernels for the respective jump models, each of which generates price dynamics under the martingale measure equivalent to its physical measure ($\mathbb{P}^H$ or $\mathbb{P}^L$). The pricing kernels proposed in this study are related to those derived in several studies such as Naik and Lee (1990), Bates (2000), Liu and Pan (2003), and Liu, Pan, and Wang (2005), in which the pricing kernel is derived from the expected utility maximization in equilibrium contexts. Our parametric forms can be regarded as an extension of their models with a single risky asset to those with two risky assets. We first propose two parametric forms of pricing kernel processes for respective types, that is, $\{\xi_t^H, \xi_t^L\}_{t\geq 0}$, and then verify that these are valid pricing kernels which exclude arbitrage opportunities by showing that the deflated prices under respective physical measures ($\mathbb{P}^H$ or $\mathbb{P}^L$) are (local) martingale. Detailed derivations are presented in Appendix 1.

1. Common jump process (type H)

The parametric form of pricing kernel process $\{\xi_t^H\}_{t\geq 0}$ for type H investors is given by

$$\frac{d\xi_t^H}{\xi_t^H} = -rdt - (\boldsymbol{\eta}_t^H)^T d\boldsymbol{B}_t - \int_{\mathbb{Z}} (1-\psi)\, d\tilde{N}_C(\boldsymbol{z}, t), \tag{4}$$

where

$$\boldsymbol{B}_t = \left(B_{1,t}, B_{2,t}\right)^T,\ \boldsymbol{z} = (z_1, z_2)^T,\ \boldsymbol{u} \coloneqq (\ln(1+z_1), \ln(1+z_2))^T,\ \boldsymbol{u} \sim N\left(\boldsymbol{\zeta}_J, \boldsymbol{\Sigma}_J\right),$$

$$\boldsymbol{\zeta}_J = (\zeta_1, \zeta_2)^T,\ \boldsymbol{\Sigma}_J = \begin{bmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{bmatrix},\ \delta_{11} := \delta_1^2,\ \delta_{22} := \delta_2^2,\ \delta_{12} = \rho_{12}^J \delta_1 \delta_2,$$

$$\psi := \exp\left\{\nu_1 + \boldsymbol{\nu}_2^T \boldsymbol{u} - \boldsymbol{\nu}_2^T \boldsymbol{\zeta}_J - \tfrac{1}{2}\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2\right\},\ \nu_1 \text{: scalar},\ \boldsymbol{\nu}_2 = (\nu_{21}, \nu_{22})^T,$$

$\boldsymbol{\eta}_t^H = \left(\eta_{1,t}^H, \eta_{2,t}^H\right)^T$ is a vector of the market price of diffusive risk, ψ represents the market price of common jump risk, and $\tilde{N}_C(\boldsymbol{z}, t) \coloneqq N_C(\boldsymbol{z}, t) - \lambda t \varphi_C(\boldsymbol{z})$ represents the compensated version of Poisson random measure for common jumps with a joint distribution function $\varphi_C(\boldsymbol{z})$ for two jump sizes $\boldsymbol{z} = (z_{C1}, z_{C2})^T$.

From the specifications above, we can derive the bivariate jump diffusion process for type H under a martingale measure $\mathbb{Q}^H$ that is equivalent to its physical measure $\mathbb{P}^H$.

$$\frac{dS_{i,t}}{S_{i,t}} = \left(r - \lambda^{\mathbb{Q}^H} \kappa_i^{\mathbb{Q}^H}\right) dt + \sigma_i dB_{i,t}^{\mathbb{Q}^H} + \int_{\mathbb{Z}} z_i^*\, dN_C^{\mathbb{Q}^H i}(z_i^*, t), \qquad \forall i \in \{1,2\}, \tag{5}$$

where

$$\lambda^{\mathbb{Q}^H} := \lambda e^{\nu_1},\ \kappa_i^{\mathbb{Q}^H} := \mathbb{E}^{\mathbb{Q}^H}[z_i^*] = (\kappa_i + 1)\exp\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1,$$

$$z_i^* := \exp(u_i^*) - 1, \quad \boldsymbol{u}^* \coloneqq (u_1^*,\ u_2^*)^T \sim N\left(\boldsymbol{\zeta}^*, \boldsymbol{\Sigma}_J\right), \quad \boldsymbol{\zeta}^* \coloneqq \left(\zeta_1 + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_1, \zeta_2 + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_2\right)^T,$$

$\boldsymbol{B}_t^{\mathbb{Q}^H} = \left(B_{1,t}^{\mathbb{Q}^H}, B_{2,t}^{\mathbb{Q}^H}\right)^T$ represents a vector of standard Brownian motions under $\mathbb{Q}^H$,

$\int_0^t \int_{\mathbb{Z}} z_i^*\, dN_C^{\mathbb{Q}^H i}(z_i^*, t)$ is a jump process under $\mathbb{Q}^H$ and $\boldsymbol{\varepsilon}_i \coloneqq \begin{cases}(1,0)^T & \text{if } i = 1\\ (0,1)^T & \text{if } i = 2\end{cases}$.

2. Independent jump process (type L)

The parametric form of pricing kernel process $\{\xi_t^L\}_{t\geq 0}$ for type L investors is given by

$$\frac{d\xi_t^L}{\xi_t^L} = -rdt - (\boldsymbol{\eta}_t^L)^T d\boldsymbol{B}_t, \tag{6}$$

where $\boldsymbol{\eta}_t^L = \left(\eta_{1,t}^L, \eta_{2,t}^L\right)^T$ is a vector of market price of diffusive risks.

We note that there is no jump component in the pricing kernel defined above because we assume that the independent jump is purely idiosyncratic and that its risk can be diversified away.[7] Hence, no investor requests any reward for bearing the independent jump risk. Then, we derive the bivariate jump diffusion process for type L under the martingale measure $\mathbb{Q}^L$ equivalent to its physical measure $\mathbb{P}^L$.

$$\frac{dS_{i,t}}{S_{i,t}} = (r - \lambda\kappa_i)dt + \sigma_i dB_{i,t}^{\mathbb{Q}^L} + \int_{\mathbb{Z}} z_i \, dN_I^i(z_i, t), \qquad \forall i \in \{1,2\}, \tag{7}$$

where $\boldsymbol{B}_t^{\mathbb{Q}^L} = \left(B_{1,t}^{\mathbb{Q}^L}, B_{2,t}^{\mathbb{Q}^L}\right)^T$ is a vector of standard Brownian motions under $\mathbb{Q}^L$.

E. Valuation of Financial Instruments with WWR

To clarify the impact of different modeling approaches of WWR on the market price of a financial instrument, we introduce several fictitious but realistic structured products that typically incorporate WWR in their payoff structure.[8] We construct these instruments so that most of the premiums reflected in the future payoffs are attributed mostly to the common jump risk. Therefore, if an investor uses the independent jump model to evaluate the instrument, the resulting value will be overvalued relative to that evaluated with the common jump model. We define the "WWR premium" as the difference between the value computed with the independent jump model and that computed with the common jump model.

Here, we introduce several versions of asset-linked structured notes that have two underlying assets whose prices follow the bivariate jump-diffusion process, as already discussed. We characterize the materialization of WWR as the occurrence of a common jump on underlying assets within a short time interval (e.g., one month), causing great damage to either the coupon

[7] Without affecting the results of this study, the independent jumps can be assumed to be exposed to the systematic risk and thus have the risk premium. In this case, the pricing kernel for the independent jump process has the jump component.

[8] Credit derivatives in OTC markets are representative products incorporating WWR. For example, CDS includes WWR via the adverse change of interconnectedness between the credit quality of its reference name and that of the counterparty. Securitized products such as CDOs, collateralized loan obligations (CLOs), and commercial mortgage-backed securities (CMBSs) are exposed to WWR through either the simultaneous defaults of reference names or the simultaneous jumps in credit spreads. Many OTC derivatives with multiple risk factors such as interest rates (domestic and foreign) and FX, such as power reverse dual-currency notes (PRDCs), would be exposed to WWR through the adverse change in interdependency among risk factors. Furthermore, asset portfolios such as equities, bonds, commodities, FX, and their compositions—and even traditional loan portfolios—are on the edge of WWR via a simultaneous price plunge or default.

cash flow or the principal amount of the note. The common setup of the structured notes is described as follows.

The coupon is paid discretely at each time grid t_k $(\forall t_k \in (0,T], k \in \{1,\cdots,N\}, t_0 := 0, t_N := T)$ with a constant interval $\Delta \coloneqq t_k - t_{k-1}$ $(\forall k \in \{1,\cdots,N\})$ where T is the terminal date of the notes. Here, Δ expresses a year fraction of the interval such as $\Delta = 1/4$ (quarterly) and $\Delta = 1/2$ (semiannually). In this paper, we set $\Delta = 1/12$ (monthly). The principal amount of one unit of the note is standardized as $X_0 = 1.0$ and the redemption amount X_T at T depends on the sample paths of two risky assets up to maturity, that is, $\{S_{1,t}, S_{2,t}\}_{0 \le t \le T}$, and the coupon rate C_{t_k} at each grid is either high coupon C_{High} or low coupon C_{Low} $(C_{High} > C_{Low} \ge 0)$, depending on the paths of two risky assets up to t_k, i.e., $\{S_{1,t}, S_{2,t}\}_{0 \le t \le t_k}$.

In this study, we consider three cases of payoff conditions that determine either the principal amount redeemed at the maturity (T) (case 1 and 2) or the coupon cash flows (all cases). We assume no credit risk (no default of the issuer of the note) and no liquidity risk in all cases.

【Case 1】

$$X_T = \begin{cases} X_0 \min\left\{\min\left(\dfrac{S_{1,T}}{S_{1,0}}, \dfrac{S_{2,T}}{S_{2,0}}\right), 1\right\} & \text{if } \exists (t_{k-1}, t_k] \in [0,T] \text{ with } \dfrac{S_{1,t_k}}{S_{1,t_{k-1}}} \le K \text{ and } \dfrac{S_{2,t_k}}{S_{2,t_{k-1}}} \le K \\ X_0 & \text{otherwise} \end{cases}$$

$$C_{t_k} = \begin{cases} C_{Low} & \text{if } \dfrac{S_{1,t_k}}{S_{1,t_{k-1}}} \le \theta \text{ and } \dfrac{S_{2,t_k}}{S_{2,t_{k-1}}} \le \theta \\ C_{High} & \text{otherwise} \end{cases}$$

where $K \in (0,1)$ is a knock-in threshold for the principal redemption amount X_T and $\theta \in (0,1)$ is a threshold for determining each grid's coupon rate C_{t_k}. We assume $K < \theta$ throughout all cases.

【Case 2】

$$X_T = \begin{cases} X_0 \min\left\{\min\left[\min\limits_{0 \le t \le T} \dfrac{S_{1,t}}{S_{1,0}}, \min\limits_{0 \le t \le T} \dfrac{S_{2,t}}{S_{2,0}}\right], 1\right\} & \text{if } \exists (t_{k-1}, t_k] \in [0,T] \text{ with } \dfrac{S_{1,t_k}}{S_{1,t_{k-1}}} \le K \text{ and } \dfrac{S_{2,t_k}}{S_{2,t_{k-1}}} \le K \\ X_0 & \text{otherwise} \end{cases}$$

The condition for determining coupon rates $\{C_{t_k}\}$ is the same as in case 1. In addition, the knock-in condition for the principal amount is identical to case 1. Compared with case 1, the potential damage on the principal amount is more severe. This is considered to be the most severe of all cases of WWR.

【Case 3】

$$C_{t_k} = \begin{cases} C_{Low} & \text{if } t_k > \min\left\{s \in \{t_0, t_1, \cdots, t_{N-1}\}: (s, s+\Delta] \in [0,T] \text{ with } \dfrac{S_{1,s+\Delta}}{S_{1,s}} \le K \text{ and } \dfrac{S_{2,s+\Delta}}{S_{2,s}} \le K\right\} \\ C_{High} & \text{otherwise} \end{cases}$$

In this case, the principal amount is fully guaranteed as $X_T = X_0$ but the coupons paid at t_k and thereafter could be fixed to the low rate C_{Low} depending on the paths of two risky assets up to t_k, that is, $\{S_{1,s}, S_{2,s}\}_{0 \le s \le t_k}$. This is considered to be the mildest case of WWR.

The value of structured note for type $m \in \{H, L\}$ at time t_0 is expressed as V_0^m and is computed by taking the expectation of the sum of discounted coupons and the redeemed principal amount X_T under the martingale measure $\mathbb{Q}^m$ $(\forall m \in \{H, L\})$.

$$V_0^m = \mathbb{E}_0^{\mathbb{Q}^m}\left[\sum_{i=1}^{N} \exp(-rt_i) C_{t_i} \Delta\, X_0 + \exp(-rT) X_T\right]. \tag{8}$$

In this study, we utilize the Monte Carlo simulation for computing the value of each instrument. We simulate the bivariate jump-diffusion process of both the common and independent jump models under the respective martingale measures ($\mathbb{Q}^H$ or $\mathbb{Q}^L$). The implementation procedure is described in Appendix 2.

Next, we prepare several parameter sets, each of which reflects different conditions of correlations, diffusion coefficients, and jump intensities. We prefix several parameter values such as $K = 0.75$, $\theta = 0.85$, $\rho_{12}^J = 1.0$, $\zeta_i = -0.3, \delta_i = 0.1$ $(i = 1,2)$ and $r = 0.01$. We also fix the low coupon rate equal to zero, that is, $C_{Low} = 0$. We specify the high coupon rate C_{High} so that the value of the structured note under the independent jump model (type L) is equal to 1.0. In addition, we examine different lengths of the maturity such that $T \in \{5,10,20\}$.

To determine the jump risk parameters for the common jump model under the martingale measure $\mathbb{Q}^H$ ($\lambda^{\mathbb{Q}^H} = \lambda e^{\nu_1}$ and $\kappa^{\mathbb{Q}^H} = (\kappa_i + 1)\exp\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1$), a knowledge of $\{\nu_1, \nu_{21}, \nu_{22}\}$ is required. In this study, we refer to Bates (2000) to explore these values.[9] The jump-related parameters in his model under a martingale measure $\mathbb{Q}$ are given by $\lambda^{\mathbb{Q}} = \lambda(1+\kappa)^{-\pi} \exp\left(-(1/2)\delta^2 \pi(1+\pi)\right)$ and $\kappa^{\mathbb{Q}} = \exp(\zeta - (1/2)\delta^2 - \pi\delta^2) - 1$, where π $(\in [0,1])$ is a portfolio weight for a risky asset whose price evolution follows a jump-diffusion process. We set $\pi = 0.5$, which implies that 50 percent of his portfolio consists of the risky asset. In addition, we impose a constraint of $\nu_{21} = \nu_{22}$, that is, the market prices of the jump-size risk for the two jumps are assumed to be symmetric. We can determine the values of $\{\nu_1, \nu_{21}, \nu_{22}\}$ by

[9] Bates (2000) derives a risk-neutral process for a jump-diffusion process in an equilibrium context in which an agent maximizes the expected utility by optimally allocating the wealth between a risky asset and a riskless bond.

equating $\left(\lambda^{\mathbb{Q}^H}, \kappa^{\mathbb{Q}^H}\right)$ with $(\lambda^{\mathbb{Q}}, \kappa^{\mathbb{Q}})$ under the conditions stated above.

We prepare nine parameter sets, each of which has distinct parameter values on the risky asset processes. These are presented in Table 2. Note that every parameter set satisfies the moment restrictions imposed in this section. We choose negative values for the overall correlation coefficient $\left(\text{Corr.} \coloneqq Corr_m\left(dS_{1,t}/S_{1,t}, dS_{2,t}/S_{2,t}\right),\ m \in \{H, L\}\right)$ so that our model can generate a typical situation in which the realization of WWR will cause a great change in the correlation structure of the two assets. The jump intensity parameter (λ) is selected with small numbers to make the materialization of WWR (the occurrence of the common jump) a rare event. For example, $\lambda = 0.1$ implies that WWR will materialize on average every 10 $(= 1/0.1)$ years.

Table 3 summarizes the valuation results for three cases of the asset-linked structured notes under the nine parameter sets presented in Table 2. Each table presents the valuation results under a different length of maturity. Recall that we manipulate the high coupon rate C_{High} so that the value of type L $(= V^L)$ is equal to 1.0.

It is obvious that the values of type H $(= V^H)$ are, in all cases and across tables, much smaller than those of type L. The difference between V^L and V^H $(= V^L - V^H)$ is represented by Diff.(IJ – CJ), which expresses the WWR premium. The maximum WWR premium is more than $5{,}000$ basis points in Table 3(C) (case 1 and 2 with $T = 20$). Even in the mildest case of WWR (case 3), the largest value of the WWR premium records more than 600 basis points in Table 3(C). These results suggest that the WWR premium generated due to the disparity in the modeling approaches of WWR (between type H and L) could be huge. If the materialization of WWR causes damage to the principal amount (case 1 and 2), which is typical for many credit derivative products, the WWR premium grows much larger.

There are three obvious tendencies on the values of type H in all cases and across maturities. First, the values under the common jump model are lower (higher) when the jump intensity is higher (lower). Since we set the average jump size under martingale measure to a large negative value (-25.92 percent) and the jump size volatility to a relatively smaller value (7.43 percent), the occurrence of common jumps will in many cases lead relative prices $\left(S_{1,t}/S_{1,t-\Delta} \text{ and } S_{2,t}/S_{2,t-\Delta}\right)$ to cross the threshold $K\ (= 0.75)$ simultaneously from above. In addition, except for case 3, once the note is knocked in, the principal redemption amount will vary according to the sample paths of the two asset prices over $(0, T]$. Hence, the higher jump intensity would cause damage to the principal amount of the note more frequently than the lower one.

Table 2 Parameter Setting for Valuation of Structured Notes

Set #	Corr.	λ	Model	$\sigma_1(=\sigma_2)$	ρ_{12}^H	ρ_{12}^L	$\lambda^{\mathbb{Q}^H}$	$\zeta^{\mathbb{Q}^H}$
1	–0.3	$\frac{1}{15}$	CJ	0.21464	–0.9750	--	0.07755	–0.3050
			IJ	0.21464	--	–0.8725	--	--
2		$\frac{1}{10}$	CJ	0.26290	–0.9750	--	0.11633	–0.3050
			IJ	0.26290	--	–0.8725	--	--
3		$\frac{1}{7}$	CJ	0.31420	–0.9750	--	0.16618	–0.3050
			IJ	0.31420	--	–0.8725	--	--
4	–0.4	$\frac{1}{15}$	CJ	0.26340	–0.9750	--	0.07755	–0.3050
			IJ	0.26340	--	–0.9069	--	--
5		$\frac{1}{10}$	CJ	0.32260	–0.9750	--	0.11633	–0.3050
			IJ	0.32260	--	–0.9069	--	--
6		$\frac{1}{7}$	CJ	0.38560	–0.9750	--	0.16618	–0.3050
			IJ	0.38560	--	–0.9069	--	--
7	–0.5	$\frac{1}{15}$	CJ	0.32015	–0.9750	--	0.07755	–0.3050
			IJ	0.32015	--	–0.9289	--	--
8		$\frac{1}{10}$	CJ	0.39212	–0.9750	--	0.11633	–0.3050
			IJ	0.39212	--	–0.9289	--	
9		$\frac{1}{7}$	CJ	0.46860	–0.9750	--	0.16618	–0.3050
			IJ	0.46860	--	–0.9289	--	--

Secondly, the overall correlation coefficient between the two assets, $Corr(dS_{1,t}/S_{1,t}, dS_{2,t}/S_{2,t})$, does not have a major impact on the valuation. The overall correlation is driven mostly by the negative diffusive correlation (ρ_{12}^m, $m \in \{H, L\}$) but the diffusion parts of the processes can hardly cause large price changes within a short time interval. We construct the payoff structures of the notes so that the payoffs are largely dependent on the occurrence of the common jumps (i.e., materialization of WWR) and hence the overall correlation coefficient does not have much impact on the value of notes.

Finally, the longer (shorter) maturity induces the lower (higher) value. Since the longer maturity would raise the odds of the occurrence of common jumps within a sub-interval (i.e., one month) to the maturity, the notes with longer maturity are more susceptible to the materialization of WWR.

Table 3 Valuation of Structured Notes

(A) $T = 5$

	Case 1			Case 2			Case 3		
Set #	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)
1	0.9235	1.0000	0.0765	0.9165	1.0000	0.0835	0.9965	1.0000	0.0035
2	0.8858	1.0000	0.1142	0.8764	1.0000	0.1236	0.9952	1.0000	0.0048
3	0.8382	1.0000	0.1618	0.8270	1.0000	0.1730	0.9938	1.0000	0.0062
4	0.9225	1.0000	0.0775	0.9158	1.0000	0.0842	0.9967	1.0000	0.0033
5	0.8853	1.0000	0.1147	0.8767	1.0000	0.1233	0.9956	1.0000	0.0044
6	0.8405	1.0000	0.1595	0.8309	1.0000	0.1691	0.9944	1.0000	0.0056
7	0.9223	1.0000	0.0777	0.9161	1.0000	0.0839	0.9970	1.0000	0.0030
8	0.8859	1.0000	0.1141	0.8786	1.0000	0.1214	0.9960	1.0000	0.0040
9	0.8435	1.0000	0.1565	0.8358	1.0000	0.1642	0.9950	1.0000	0.0050

(B) $T = 10$

	Case 1			Case 2			Case 3		
Set #	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)
1	0.8425	1.0000	0.1575	0.8286	1.0000	0.1714	0.9871	1.0000	0.0129
2	0.7683	1.0000	0.2317	0.7526	1.0000	0.2474	0.9828	1.0000	0.0172
3	0.6853	1.0000	0.3147	0.6699	1.0000	0.3301	0.9784	1.0000	0.0216
4	0.8382	1.0000	0.1618	0.8260	1.0000	0.1740	0.9881	1.0000	0.0119
5	0.7684	1.0000	0.2316	0.7558	1.0000	0.2442	0.9843	1.0000	0.0157
6	0.6859	1.0000	0.3141	0.6745	1.0000	0.3255	0.9801	1.0000	0.0199
7	0.8373	1.0000	0.1627	0.8273	1.0000	0.1727	0.9890	1.0000	0.0110
8	0.7690	1.0000	0.2310	0.7597	1.0000	0.2403	0.9855	1.0000	0.0145
9	0.6942	1.0000	0.3058	0.6868	1.0000	0.3132	0.9819	1.0000	0.0181

(C) $T = 20$

	Case 1			Case 2			Case 3		
Set #	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)	CJ (V^H)	IJ (V^L)	Diff.(IJ – CJ)
1	0.7102	1.0000	0.2898	0.6906	1.0000	0.3094	0.9564	1.0000	0.0436
2	0.5979	1.0000	0.4021	0.5813	1.0000	0.4187	0.9436	1.0000	0.0564
3	0.4891	1.0000	0.5109	0.4774	1.0000	0.5226	0.9308	1.0000	0.0692
4	0.7039	1.0000	0.2961	0.6891	1.0000	0.3109	0.9589	1.0000	0.0411
5	0.5961	1.0000	0.4039	0.5852	1.0000	0.4148	0.9475	1.0000	0.0525
6	0.4928	1.0000	0.5072	0.4863	1.0000	0.5137	0.9360	1.0000	0.0640
7	0.7046	1.0000	0.2954	0.6948	1.0000	0.3052	0.9624	1.0000	0.0376
8	0.6004	1.0000	0.3996	0.5943	1.0000	0.4057	0.9512	1.0000	0.0488
9	0.5038	1.0000	0.4962	0.5009	1.0000	0.4991	0.9407	1.0000	0.0593

IV. Market Microstructure and Market Price

In this section, we introduce a trading mechanism to determine the market price when there are a large number (but finite) of investors with different types (H and L) and they deal with a derivative security with arm's-length transactions in each trading period over trading horizon $\left[0,\overline{T}\right]$ with $\overline{T} \leq +\infty$. We derive some theoretical results on the market price, which is determined as a result of transactions among different types of investors in the market. The arm's-length trading environment introduced in this section is constructed based on Duffie, Garleanu, and Pedersen (2005) and Kijima and Uchida (2005).

A. Basic Assumptions

1. Market participants

There are N $(M < N < \infty)$ investors in the security market, trading a structured note (hereafter, security) introduced in the previous section with arm's-length transactions, that is, one-to-one transactions or OTC transactions. Each investor is assumed to be risk-neutral and infinitely lived with either type H or L. The number of type H (L) investors at time *t* is given by $N_H(t)$ $\left(N_L(t)\right)$ satisfying $N_H(t) + N_L(t) = N$. M represents the total supply of the security, and each investor has at most a unit share. Short-selling is prohibited.

2. Transition between types

Many reports and papers regard poor financial risk management as one of the major causes of the latest global financial crisis. As favorable conditions continue, many investors are tempted to think "this time is different," and their financial risk management tends to be lax. Once security prices drop, however, market participants recognize the importance of risk management. Based on this observation, we assume for simplicity that investors change their type (H and L), depending on time and cycle of price dynamics, as follows.

An investor switches his/her type randomly either from H to L or L to H. A type H investor changes his/her type randomly and independently with a constant (instantaneous) switching rate $\gamma > 0$. On the other hand, the switching rate for type L investors to be type H is given by $\lambda > 0$, which is identical to the intensity of the common jump process. We also assume that all the type L investors change their type simultaneously (not independently) when a common jump occurs.

The type-switching behavior of each type can be interpreted as follows. When a common jump occurs, all the type L investors will instantly realize the danger of WWR and will pay attention to WWR in both pricing and risk management, implying a switch of their type simultaneously to H. As time passes from the latest materialization of WWR with the market situation returning to normal, the memory of market downturn among investors becomes attenuated, while investors' attitude toward risk gets aggressive. This situation motivates type H investors to become shortsighted and seek short-term profits rather than maintain prudent risk management for WWR, and hence prompts type H investors to switch their type to L.

3. Trading strategy

The value of the security computed by type H (L) at time t is represented as $V_t^H (V_t^L)$. We know from the result of Section 3 that the value of type H is lower than that of type L, that is, $V_t^H < V_t^L$. We assume that the time to maturity (T) of the security keeps unchanged and thus the security value for each type stays constant throughout the trading horizon, that is, $V_t^m = V^m$ $(\forall t \in [0, \infty), \forall m \in \{H, L\})$. An investor is assumed to be a seller if he/she owns a security, otherwise a buyer. Each investor computes his/her own reservation value by optimally evaluating both the gain from future trades and the future type change conditional on his/her current type and endowment of the security. A buyer and a seller are assumed to meet randomly, and they agree to trade if the buyer's reservation value is lower than or equal to the seller's, that is, a trade will take place when a positive rent (including zero rent) is created for both investors. Otherwise, a transaction does not take place. Investors share the rent according to the ratio of $q: 1-q$ $(q \in (0,1))$ between type L and type H. The size of q depends on the bargaining power between two types. Note that trades occur only between type H sellers and type L buyers. The instantaneous rate (intensity) of matching between type H and type L is given by $\lambda_{HL} > 0$. We denote the transaction price of the security at time t by p_t.

4. Equilibrium price

The equilibrium is defined as a Walrasian equilibrium characterized by a price process at which the supply matches the demand at each state and time in a perfectly competitive market. In an equilibrium allocation, there are no more trades between different types. That is, in equilibrium, either of the following conditions is met: (1) every type L investor owns a unit of the security; or (2) all the securities are owned by type L investors. For the equilibrium price p_t, we suppose $p_t = q \times$ (reservation value of type L) $+(1-q) \times$ (reservation value of type H).

Based on the assumptions described above, we can derive the following results with respect to the equilibrium market price p^*. See Appendix 3 for the sketch of proof.

THEOREM

Assume $q \in (0,1)$ and $\lambda_{HL} \to \infty$. Then the equilibrium price p^* is unique. Furthermore, if $N \to \infty$ with $s := M/N$ fixed, the equilibrium price satisfies

$$p^* = \frac{q(1-s)\gamma + (1-q)s\lambda}{\frac{1}{V^L} q(1-s)\gamma + \frac{1}{V^H}(1-q)s\lambda}. \tag{9}$$

The theorem suggests that if type H and L are both in the market and are matched instantly ($\lambda_{HL} \to \infty$), then the market price will converge to the equilibrium price that lies somewhere between V^H and V^L depending on the parameter values of $\{\lambda, \gamma, q, s\}$.

If the intensity of common jump (i.e., the switching rate of type L to H) converges to zero but the switching rate of type H to type L keeps a finite positive value, the market population is gradually dominated by type L and thus the equilibrium price converges to the highest value V^L with the lowest WWR premium. This is summarized as the lemma below.

LEMMA

If $\lambda/\gamma \to 0$, then $p^* \to V^L$ regardless of q, s, and γ.

V. Numerical Analysis

We demonstrate a numerical analysis of the security price dynamics and the transition of investor types based on the pricing models and the trading mechanism we stated in the previous sections. In this section, we assume that all the investors are shortsighted in the sense that they do not evaluate the trade dynamically but recognize the utility gain only from the current trade.[10] We then assume that the reservation value for type H is V^H and that for type L is V^L $(> V^H)$, and thus a trade will be implemented if the buyer's reservation value is lower than or equal to the seller's, that is, if a positive rent is created from the transaction. We note that in this study the maturity of the security (T) is assumed to stay constant and thus the reservation price for each type remains constant throughout the trading horizon, that is, $V_t^m = V^m$ $(\forall t \in [0, \infty), \forall m \in \{H, L\})$. Therefore, there are three cases in which a trade takes place between two investors: (1) a type H seller and a type L buyer; (2) a type H seller and a type H buyer; and (3) a type L seller and a type L buyer. Other conditions are the same as stated in the previous sections.

In the following subsection, we explain the details of simulation procedure adopted in the analysis and then show the results.

A. Simulation Procedure

1. Switching behavior of type H investors

Trading horizon is expressed as $[0, \overline{T}]$ with $\overline{T} \leq +\infty$ and we divide it into small trading periods with a constant interval $\Delta \coloneqq t_k - t_{k-1}$ $(\forall k \in \{1, \cdots, \overline{N}\})$ with $t_0 \coloneqq 0$ and $t_{\overline{N}} \coloneqq \overline{T}$. We set $\overline{T} = 50$ years (600 months) and $\Delta = 1/12$ (one month). Each investor can implement at most one trade in a period. In our analysis, we assume that the survival ratio of type H at each trading period t, defined as $\alpha_t \coloneqq N_H(t)/N$, is decreased deterministically period by period given the constant switching rate of $\gamma > 0$.[11] We thus redefine α_t as $\alpha_t := \exp(-\gamma(t - \tau_t))$, where

[10] In the previous section, we assume that investors consider the possibility that they change their types in the future in calculating their reservation value. In this section, however, we suppose that they do not take it into account for the purpose of simplicity. Both settings generate a boom and bust cycle in the security prices. The only difference is the degree of price fluctuation, that is, it is smaller in the former setting than in the latter.

[11] Examining the price impact of endogenously determined type-switching behavior is of interest and might be useful for clarifying the detailed dynamics of security prices. It is, however, not essential to consider the

$\tau_t \in \boldsymbol{R}$ represents the latest time of the occurrence of common jumps just before time t. On the other hand, the individual among type H investors chosen to change his/her type is randomly determined. At initial date $t_0 = 0$, the survival ratio of type H is given by $\alpha_0\big(= \exp\big(-\gamma(t - \tau_0)\big),\ \tau_0 \leq 0\big)$. We set $\tau_0 = -2.5$, that is, at the initial date t_0, 2.5 years have already passed since the latest WWR materialization.

2. Trading protocol

We next describe the trading protocol in each trading period. We assume that the total supply of the security is fixed by $M\ (< N)$ units of shares. Here, we set $N = 10{,}000$ and $M = 5{,}000$. That is, $5{,}000$ investors out of a total of $10{,}000$ in the market own the security. At initial time t_0, M units of the security are randomly allocated to investors regardless of their types, so that each investor has at most one unit of it. If an investor is endowed with one unit of the security, he/she is assumed to be a seller, otherwise a buyer. Among the three successful combinations of transactions between two types, which we stated in the beginning of this section, we give priority to transactions with a pair of type H seller and type L buyer over other transactions, satisfying the assumption of $\lambda_{HL} \to \infty$ in the theorem. Investors are assumed to share the rent evenly $(q = 0.5)$. That is, the transaction price between the type H seller and type L buyer is given by $p_t = (V^H + V^L)/2$. If a type H seller cannot find type L buyers, he/she agrees to trade only with a type H buyer ($\text{rent} = 0$) with a transaction price of $p_t = V^H$. A type H buyer can trade only with a type H seller ($\text{rent} = 0$) with a transaction price of $p_t = V^H$. Similarly, a type L seller can trade only with a type L buyer ($\text{rent} = 0$) with a transaction price of $p_t = V^L$. We assume all the transactions generating non-negative rent will be implemented. If an investor cannot find his/her desired counterparty, he/she does not trade at the period. When all desired and available trades are implemented, each investor takes over his revised position to the next period. At the next period, some of type H investors switch to type L to be consistent with the revised (deterministic) survival ratio of type H. Investors repeat the transaction based on the revised distributions on both the endowments of the security across investors and investor types. We compute the market price at each period t, denoted by $\bar{p}_t$, by taking the average of all transaction prices of individual trades executed at the period. When a common jump occurs in a trading period, all the type L investors switch their type simultaneously to type H, and then the trades among type H investors are only implementable at the period. As a result, the market price at the period will suddenly drop to $V^H\ (< V^L)$.

We first draw a Poisson jump path over the entire trading horizon (50 years with one-month intervals) and then we implement $1{,}000$ trading simulations of arm's-length transactions with different initial distributions of endowments among investors given the Poisson jump path. In each trading period, we compute the average of market price computed in each simulation. This

endogenous switching device for the purpose of current research, which elucidates the mechanism generating price trajectories with cyclical patterns of booms and busts, and thus we leave it for future research.

represents our simulated market price for each trading period. This, we define, is a set of the Poisson trial.

Secondly, we repeat the Poisson trial for 1,000 times with the same combination of λ and γ, and compute the average of the jump sizes of the simulated market price over all Poisson trials. We define the average jump sizes as "expected jump size." We repeat the simulation procedure with different combinations of λ and γ. The jump intensity λ is chosen from $\{1/15, 1/12.5, 1/10, 1/8.5, 1/7, 1/6, 1/5\}$, in which the denominator of each element expresses the average interval (yearly) between jump arrivals. In this study, for an expositional purpose, we assume that the Poisson intensity chosen in each Poisson trial is the same as the common jump intensity of type H investors.[12] This implies that type H investors have correct knowledge about the common jump intensity. The switching rate γ of type H is selected from $\{1/1, 1/1.3, 1/2, 1/3.3, 1/5, 1/10, 1/16.7, 1/33.3, 1/50\}$, in which the denominator of each element expresses the average interval (yearly) to switch. We select the structured note of case 1 with $T = 20$ and choose the overall correlation coefficient of -0.4, and resulting values of type H (V^H) are computed as $\{0.7039, 0.6569, 0.5961, 0.5359, 0.4928, 0.4292, 0.3762\}$, each of which is consistent with the jump intensity parameter selected above. The value of type L is normalized by $V^L = 1.0$.[13]

B. Simulation Results

We illustrate the results of our numerical analysis in Figures 2 and 3. Figure 2 shows two paths of simulated market prices (the upper rows) along with the evolutions of the survival ratio of type H (the lower rows) under two different Poisson jump paths (left and right). These are generated with the same parameter values except for the switching rate of type H with $\gamma = 1/5$ and $1/16.7$. In both sample paths, several discontinuous points indicate the occurrence of common jumps, that is, the materializations of WWR. We observe that, as time passes from the materialization of WWR, the market price converges to $V^L(= 1.0)$ with the least WWR premium from $V^H(= 0.5961)$ with the largest WWR premium. This situation implies that the WWR premium included in the market price gradually decreases along with the increase in the type L population. However, once WWR materializes, all the type L investors switch to type H and thus the market price drops suddenly to V^H. The materialization of WWR convinces all the type L investors of the danger of WWR and forces them simultaneously to shift toward the prudent management of WWR, by switching their type to type H, and this aggregate behavior induces the plunge in market price. Although, in this case, the individual behavior of type L investors is rational from the viewpoint of prudent risk management, the simultaneous behavior of type L investors to protect themselves results in intensifying the decline in market price,

[12] We also have examined several experiments under an assumption that the common jump intensity of type H is different from the actual (realized) one. As a result, we found no major observations altering the main results of our normal experiments. Accordingly, we omitted the results, but they can be given to interested readers on request.

[13] Recall that we set the higher coupon rate $(C_{High} > 0)$ so that the resulting security value of type L is $V^L = 1.0$.

depressing the market as a whole. This is typically the case of a *fallacy of composition*. The size of the plunge reflects the amount of WWR premium $(= V^L - V^H)$, which has been gradually excluded from V^H since the latest materialization of WWR. This mechanism generates the price trajectories with cyclical patterns of boom and bust.

We also find that the higher γ enhances the market price to converge faster to V^L, making the magnitude of price plunges greater when WWR occurs (Figure 2[A]). On the other hand, the lower γ seems to hamper the large drop in the market price, since it delays the switching of type H to L and thus lowers the speed of the price rise (Figure 2[B]).

We observe several inflection points in the paths of the market price, each of which lies in the middle of the post-WWR periods.[14] These inflection points reflect the transition point in the predominance of the investor population from type H to L. The market is gradually dominated by type L after the inflection point and the curvature of the price paths on the way to V^L becomes gentle, because the volatility across individual transaction prices gradually decreases along with the increase in the type L population.

Figure 3(A) plots the expected jump size of the market price with respect to various combinations of common jump intensity (λ) and the switching rate of type H (γ). This three-dimensional figure suggests that the expected jump size is dependent on both the jump intensity and the switching rate of type H over all combinations of λ and γ. Roughly speaking, the higher (lower) λ and the higher (lower) γ generate the larger (smaller) expected jumps in absolute magnitude (all in a negative direction). However, when γ takes a smaller value, the expected jump size becomes insensitive to λ. More specifically, we transform the three-dimensional figure into two two-dimensional figures from different viewpoints. Figure 3(B) (3[C]) plots the expected jump sizes with respect to γ (λ) with each λ (γ) fixed. Both figures indicate that the expected jump size can be dampened when we keep γ in lower values, regardless of λ. This implies that the expected jump size of the market price depends more on γ than on λ. Therefore, even if we do not have correct knowledge of the common jump intensity that is unobservable, we can prevent the sudden drop in market price, generated due to the mispricing of WWR by type L investors, if we can control the level of γ and keep it at a sufficiently low level. This result seems to have a large implication for the design of financial regulations. If we can regulate the financial market to motivate each financial institution to adopt a prudent system for managing WWR and to remain prudent consistently over time, then the plunge in the market price due to mispricing with respect to the imprudent modeling approach of WWR could be alleviated.

[14] We define the time interval between periods of WWR-materializations as the "post-WWR period."

Figure 2 Market Price and Survival Ratio of Type H

(A) Case 1 with $T = 20$, parameter set = 5
$\lambda = 1/10$ and $\gamma = 1/5$

(B) Case 1 with $T = 20$, parameter set = 5
$\lambda = 1/10$ and $\gamma = 1/16.7$

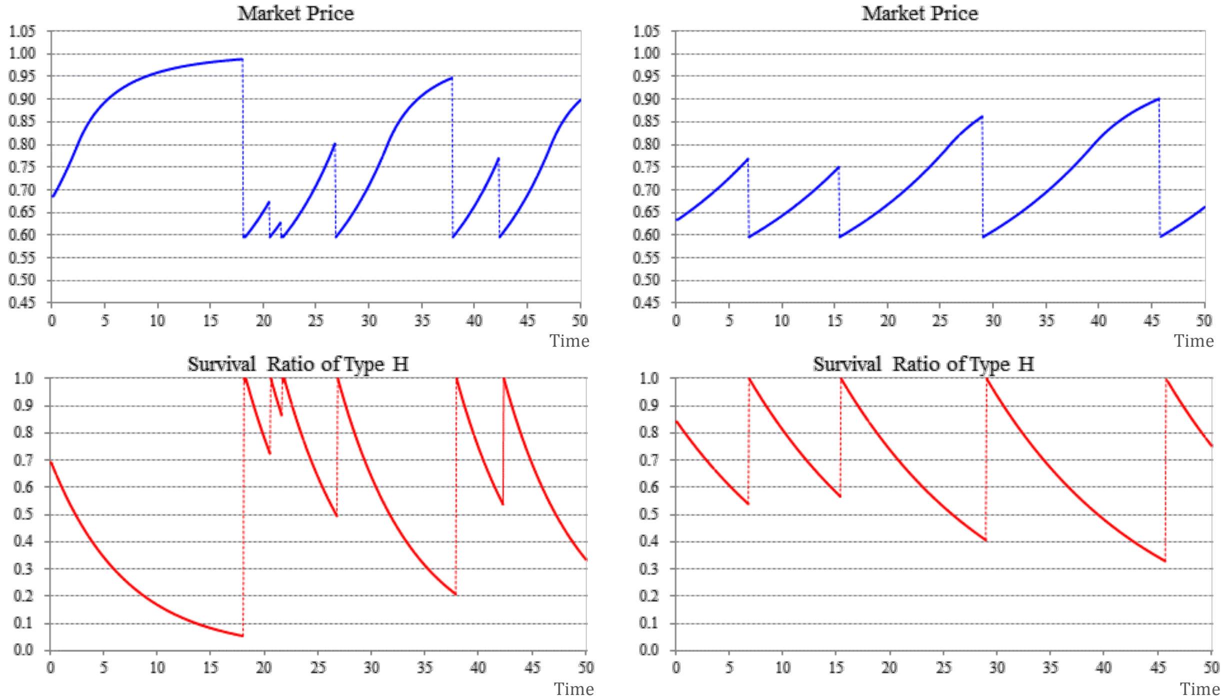
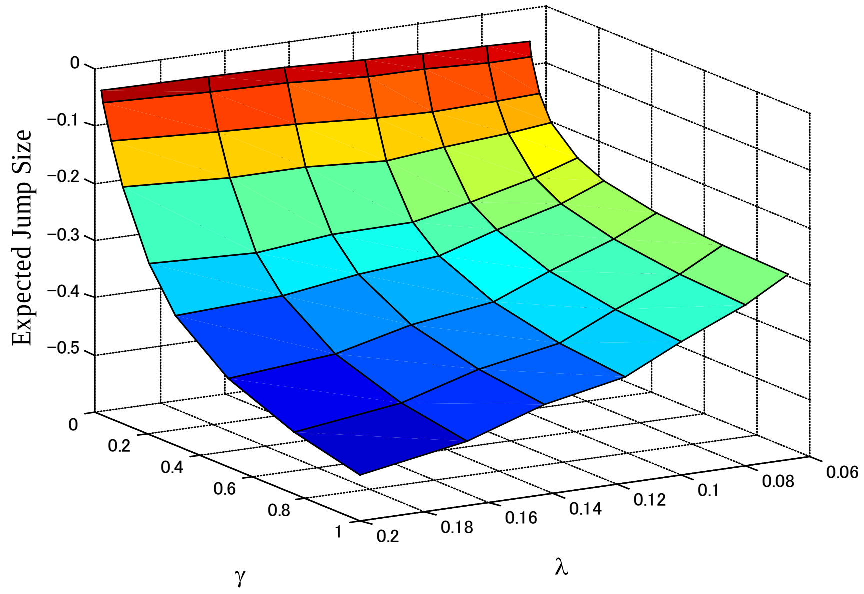
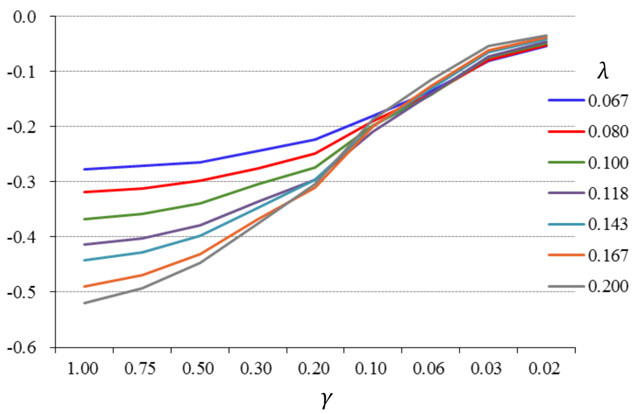


Figure 3　Expected Jump Size

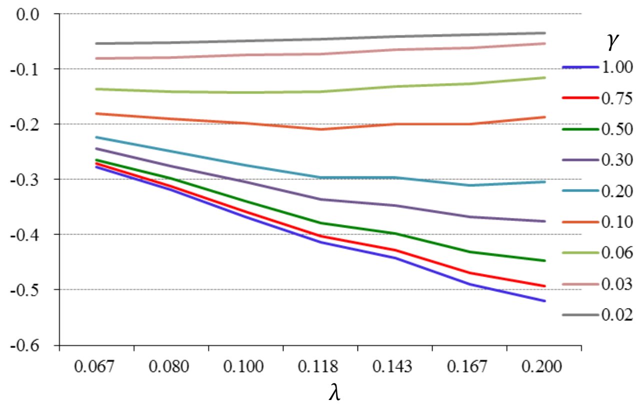
(A) Expected Jump Size (wrt λ and γ)



(B) Expected Jump Size wrt γ with fixed λs



(C) Expected Jump Size wrt λ with fixed γs



VI. Concluding Remarks

Motivated by huge losses caused by WWR during the global financial crises of 2007–09, we proposed a method to model WWR and investigated the mechanism by which WWR generates boom and bust cycles in security prices through different WWR management attitudes among investors. We show that even if all the investors implement careful risk management up to the marginal distributions with linear correlation across several risk factors, it is insufficient to prevent the market from suffering huge losses caused by the materialization of WWR.

We suppose that two types of investors, type H and type L, trade a derivative security with two underlying assets. Type H investors are assumed to monitor WWR prudently in that they regard the common jump component in the price processes of the two risky assets as the main driver behind WWR in their model. Type L investors are assumed to treat WWR imprudently in the sense that they consider only the independent jump components in their model. This assumption means that even when both types recognize the same marginal distributions and covariance for the two risky assets, they can create completely different joint distributions, especially on their tail portions. Next, we introduce several fictitious but realistic derivative securities that incorporate WWR. We show that the valuation of type L investors in the security could be much higher than that of type H investors. We find from our numerical analysis that, as time passes since the latest materialization of WWR, the market price converges to the highest value with the lowest WWR premium along with the increase in the population of imprudent investors (type L) and drops suddenly with a certain frequency to the lowest level with the highest WWR premium. The materialization of WWR convinces type L investors of the danger of WWR, and motivates them simultaneously to move toward prudent management of WWR by switching their type to type H, and this aggregate behavior induces the plunge in market price. This mechanism generates the price trajectories with cyclical patterns of boom and bust.

The results of our analyses have important implications for financial regulation. Current regulations request financial institutions to measure and monitor their portfolio risks up to the second moments (volatility and correlation) as a minimum requirement, but there is no explicit requirement on the management of higher moments on the tail portion of joint distributions. This paper demonstrates that if we can encourage investors to adopt prudent risk management of WWR, we could prevent or alleviate boom and bust cycles in security prices. Although the optimal design of financial regulation is beyond the scope of our research, an incentive-compatible method for financial institutions to enhance their motivation to adopt a prudent system for managing WWR would be essential to maintaining financial stability.

APPENDIX 1: CHANGE IN PROBABILITY MEASURES FOR TWO JUMP PROCESSES

A. Common Jump Process (Type H)

We first prove that the expectation of the pricing kernel for the common jump model under type H's subjective probability measure $\mathbb{P}^H$ is equal to the value of the zero-coupon bond matured at time t, that is, $\exp(-rt)$. By applying a generalized version of Ito's lemma (including jump components) to equation (4), we have

$$d\ln\xi_t^H = -rdt - (\boldsymbol{\eta}_t^H)^T d\boldsymbol{B}_t - \lambda \int_{\mathbb{Z}} (\psi - 1)\varphi(d\boldsymbol{z})dt - \left(\rho_{12}^H \eta_{1,t}^H \eta_{2,t}^H + \frac{1}{2}(\boldsymbol{\eta}_t^H)^T \boldsymbol{\eta}_t^H\right)dt + \int_{\mathbb{Z}} \ln\psi \, dN_C(\boldsymbol{z}, t),$$

where $\varphi_C(\boldsymbol{z})$ represents a joint distribution of $\mathbf{z} = (z_{C1}, z_{C2})^T$, and thus we derive

$$\xi_t^H = \xi_0^H \exp\left\{-\int_0^t \left(r + \rho_{12}^H \eta_{1,s}^H \eta_{2,s}^H + \frac{1}{2}(\boldsymbol{\eta}_s^H)^T \boldsymbol{\eta}_s^H + \lambda \int_{\mathbb{Z}} (\psi - 1)\varphi(d\boldsymbol{z})\right) ds - \int_0^t (\boldsymbol{\eta}_s^H)^T d\boldsymbol{B}_s + \int_0^t \int_{\mathbb{Z}} \ln\psi \, dN_C(\boldsymbol{z}, s)\right\}.$$

Here, we assume $\xi_0^H = 1$ and we obtain the following result.

$$\mathbb{E}_0^H[\xi_t^H] = \exp(-rt)\mathbb{E}_0^H\left[\exp\left(-\int_0^t (\boldsymbol{\eta}_s^H)^T d\boldsymbol{B}_s - \frac{1}{2}\int_0^t \left(2\rho_{12}^H \eta_{1,s}^H \eta_{2,s}^H + (\boldsymbol{\eta}_s^H)^T \boldsymbol{\eta}_s^H\right) ds\right)\right]$$
$$\times \mathbb{E}_0^H\left[\exp\left(-\int_0^t \int_{\mathbb{Z}} \lambda(\psi - 1)\varphi(d\boldsymbol{z})ds + \int_0^t \int_{\mathbb{Z}} \ln\psi \, dN_C(\boldsymbol{z}, s)\right)\right] = \exp(-rt),$$

where

$$\mathbb{E}_0^H\left[\exp\left(-\int_0^t (\boldsymbol{\eta}_s^H)^T d\boldsymbol{B}_s - \frac{1}{2}\int_0^t \left(2\rho_{12}^H \eta_{1,s}^H \eta_{2,s}^H + (\boldsymbol{\eta}_s^H)^T \boldsymbol{\eta}_s^H\right) ds\right)\right] = 1, \text{ and}$$

$$\begin{aligned}
&\mathbb{E}_0^H\left[\exp\left(-\int_0^t \int_{\mathbb{Z}} \lambda(\psi - 1)\varphi(d\boldsymbol{z})ds + \int_0^t \int_{\mathbb{Z}} \ln\psi \, dN_C(\boldsymbol{z}, s)\right)\right] \\
&\quad = \exp\left(-\int_0^t \int_{\mathbb{Z}} \lambda(\psi - 1)\varphi(d\boldsymbol{z})ds\right)\mathbb{E}_0^H\left[\exp\left(\int_0^t \int_{\mathbb{Z}} \ln\psi \, dN_C(\boldsymbol{z}, s)\right)\right] \\
&\quad = \exp\left(-\int_0^t \int_{\mathbb{Z}} \lambda(\psi - 1)\varphi(d\boldsymbol{z})ds\right)\mathbb{E}_0^H\left[\sum_{k=0}^{\infty} \exp\left(\sum_{h=0}^{k} \ln\psi\right)\right] \\
&\quad = \exp(-\lambda(\exp(\nu_1) - 1)t)\exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\mathbb{E}_0^H[\psi]^k \\
&\quad = \exp(-\lambda \exp(\nu_1)t)\sum_{k=0}^{\infty} \frac{(\lambda\exp(\nu_1)t)^k}{k!} = 1 \ \ (\because \mathbb{E}_0^H[\psi] = \exp(\nu_1)).
\end{aligned}$$

Next, we show $\mathbb{E}_0^H[\xi_t^H S_{i,t}] = S_{i,0}$ $(\forall i \in \{1,2\})$. To do this, we specify the risk premium of the common jump process so that we have $\mathbb{E}_0^H[d(\xi_t^H S_{i,t})] = 0$ $(\forall i \in \{1,2\})$. By applying Ito's lemma to $\xi_t^H S_{i,t}$, we have

$$\begin{aligned}
d(\xi_t^H S_{i,t}) &= S_{it} d\xi_t^H + \xi_t^H dS_{i,t} + d[\xi_t^H, S_{i,t}] + \int_{\mathbb{Z}} [\psi - 1 + z_i\psi]\xi_t^H S_{i,t} dN_C(\boldsymbol{z}, t) \\
&= \xi_t^H S_{i,t}\left[-rdt - (\boldsymbol{\eta}_t^H)^T d\boldsymbol{B}_t + \mu_i dt + \sigma_i dB_{i,t} - \lambda \int_{\mathbb{Z}} (\psi - 1)\varphi(d\boldsymbol{z})dt - \lambda \int_{Z} z_i \varphi^i(dz_i)dt\right] \\
&\quad - \xi_t^H S_{i,t}\sigma_i \eta_{i,t}^H dt + \int_{\mathbb{Z}} (\psi - 1 + z_i\psi)\xi_t^H S_{i,t} dN_C(\boldsymbol{z}, t).
\end{aligned}$$

By taking expectation under $\mathbb{P}^H$, we derive

$$\mu_i - r - \lambda\mathbb{E}_0^H[\psi - 1] - \lambda\mathbb{E}_0^H[z_i] - \sigma_i \eta_{i,t}^H + \lambda\mathbb{E}_0^H[\psi - 1 + z_i\psi] = 0.$$

Here, we derive the risk premium $\mu_i - r$ so that we have $\mathbb{E}_0^H\left[d\left(\xi_t^H S_{i,t}\right)\right] = 0$.

$$\mu_i - r = \sigma_i \eta_{i,t}^H - \left(\lambda^{\mathbb{Q}^H} \kappa_i^{\mathbb{Q}^H} - \lambda \kappa_i\right),$$

where $\lambda^{\mathbb{Q}^H} \kappa_i^{\mathbb{Q}^H} \coloneqq \lambda \mathbb{E}_0^H[z_i \psi]$ and $\mathbb{E}_0^H[z_i \psi]$ can be calculated as follows.

$$\begin{aligned}
\mathbb{E}_0^H[z_i\psi] &= \mathbb{E}_0^H\left[(e^{u_i} - 1)\exp\left(\nu_1 + \boldsymbol{\nu}_2^T \boldsymbol{u} - \boldsymbol{\nu}_2^T \boldsymbol{\zeta}_J - \frac{1}{2}\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2\right)\right] \\
&= e^{\nu_1}\left[\exp\left(\left(\boldsymbol{\nu}_2^{(i)} - \boldsymbol{\nu}_2\right)^T \boldsymbol{\zeta}_J + \frac{1}{2}\left(\boldsymbol{\nu}_2^{(i)T} \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2^{(i)} - \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2\right)\right) - 1\right] \\
&= e^{\nu_1}\left[\exp\left(\zeta_i - \frac{1}{2}\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2 + \frac{1}{2}\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2 + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_i^T \boldsymbol{\Sigma}_J \boldsymbol{\nu}_2 + \boldsymbol{\varepsilon}_i^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right)\right) - 1\right] \\
&= e^{\nu_1}\left[\exp\left(\zeta_i + \frac{1}{2}\delta_{ii} + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1\right] \\
&= e^{\nu_1}\left[(\kappa_i + 1)\exp\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1\right],
\end{aligned}$$

where $\boldsymbol{\nu}_2^{(i)} \coloneqq \boldsymbol{\nu}_2 + \boldsymbol{\varepsilon}_i$ with $\boldsymbol{\varepsilon}_i \coloneqq \begin{cases} (1,0)^T & \text{if } i = 1 \\ (0,1)^T & \text{if } i = 2 \end{cases}$.

Hence, we derive the jump-parameters $\left\{\lambda^{\mathbb{Q}^H},\ \kappa_i^{\mathbb{Q}^H}\right\}$ under $\mathbb{Q}^H$ of the form:

$$\lambda^{\mathbb{Q}^H} \kappa_i^{\mathbb{Q}^H} = \lambda e^{\nu_1}\left[(\kappa_i + 1)\exp\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1\right],$$

where we specify

$$\lambda^{\mathbb{Q}^H} := \lambda e^{\nu_1}, \kappa_i^{\mathbb{Q}^H} := \mathbb{E}^{\mathbb{Q}^H}[z_i^*] = (\kappa_i + 1)\exp\left(\boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i\right) - 1,$$

$$z_i^* := \exp(u_i^*) - 1 \text{ with } u_i^* \sim N\left(\zeta_i + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_i, \delta_i^2\right),$$

and thus $\boldsymbol{u}^* \coloneqq (u_1^*, u_2^*)^T \sim N\left(\boldsymbol{\zeta}^*, \boldsymbol{\Sigma}_J\right)$ with $\boldsymbol{\zeta}^* \coloneqq \left(\zeta_1 + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_1, \zeta_2 + \boldsymbol{\nu}_2^T \boldsymbol{\Sigma}_J \boldsymbol{\varepsilon}_2\right)^T$.

Finally, we derive the bivariate common jump diffusion process under the martingale measure $\mathbb{Q}^H$ equivalent to $\mathbb{P}^H$. By Girsanov's theorem, we have that the process

$$dB_{1,t}^{\mathbb{Q}^H} = dB_{1,t} + \eta_{1,t}^H dt$$

is a standard Brownian motion under $\mathbb{Q}^H$. On the other hand, we write $dB_{2,t}^{\mathbb{Q}^H}$ as

$$dB_{2,t}^{\mathbb{Q}^H} = \rho_{12}^H dB_{1,t}^{\mathbb{Q}^H} + \sqrt{1-(\rho_{12}^H)^2}dB_{0,t},$$

where $B_{0,t}$ remains a standard Brownian motion under $\mathbb{Q}^H$ and it is independent of $B_{1,t}^{\mathbb{Q}^H}$. We transform the process as follows.

$$\begin{aligned} dB_{2,t}^{\mathbb{Q}^H} &= \rho_{12}^H\big(dB_{1,t} + \eta_{1,t}^H dt\big) + \sqrt{1-(\rho_{12}^H)^2}dB_{0,t} \\ &= dB_{2,t} + \rho_{12}^H \eta_{1,t}^H dt \\ &= dB_{2,t} + \eta_{2,t}^H dt \quad \big(\eta_{2,t}^H \coloneqq \rho_{12}^H \eta_{1,t}^H\big). \end{aligned}$$

On the other hand, by applying Girsanov's theorem for the jump diffusion process (Oksendal and Sulem [2007]), we have that the process

$$\int_{\mathbb{Z}} z_i \, d\tilde{N}_C^{\mathbb{Q}^H}(z_i,t) = \int_{\mathbb{Z}} z_i \, dN_C^i(z_i,t) - \lambda^{\mathbb{Q}^H}\kappa_i^{\mathbb{Q}^H} dt$$

is a compensated jump process under $\mathbb{Q}^H$. Then, plugging the risk premium specification, jump process under $\mathbb{Q}^H$ and Brownian motion under $\mathbb{Q}^H$ into equation (2) generates the following process under $\mathbb{Q}^H$.

$$\begin{aligned} \frac{dS_{it}}{S_{it}} &= \big(\mu_i - \lambda_{Ci}\kappa_{Ci}\big)dt + \sigma_i dB_{i,t} + \int_{\mathbb{Z}} z_{Ci} \, dN_C^i(z_{Ci},t) \\ &= \left(r + \sigma_i \eta_{i,t}^H - \lambda^{\mathbb{Q}^H}\kappa_i^{\mathbb{Q}^H}\right)dt + \sigma_i\left(dB_{i,t}^{\mathbb{Q}^H} - \eta_{i,t}^H dt\right) + \int_{\mathbb{Z}} z_i \, d\tilde{N}_C^{\mathbb{Q}^H i}(z_i,t) + \lambda^{\mathbb{Q}^H}\kappa_i^{\mathbb{Q}^H} dt \\ &= r dt + \sigma_i dB_{i,t}^{\mathbb{Q}^H} + \int_{\mathbb{Z}} z_i \, d\tilde{N}_C^{\mathbb{Q}^H i}(z_i,t). \end{aligned}$$

B. Independent Jump Process

We use the pricing kernel process for the independent jump process of the form:

$$\frac{d\xi_t^L}{\xi_t^L} = -rdt - (\boldsymbol{\eta}_t^L)^T d\boldsymbol{B}_t.$$

Note that there is no jump-related terms in the pricing kernel defined above because we assume that the independent jumps are purely idiosyncratic and the jump risk can be diversified away. No investor requests rewards from bearing the independent jump risk, and thus there is no market price of jump risk in ξ_t^L. On the other hand, there are still Brownian diffusion risks that are systematic and cannot be diversified away. Therefore, there are market prices of diffusion risks expressed in a vector of $\boldsymbol{\eta}_t^L$. With this specification of the pricing kernel for the independent jump process, it is easy to derive the desired results in a way that is similar to the one we stated in the common jump process.

APPENDIX 2: SIMULATION PROCEDURE FOR VALUATION OF STRUCTURED NOTES

We first discretize the whole interval into small sub-intervals of length $t_{k-1} - t_k \coloneqq \Delta > 0, \forall k \in \{1, \dots, N\}$. The price evolution of the common jump process (type H) under a martingale measure $\mathbb{Q}^H$ over each sub-interval $(t_{k-1}, t_k]$ is given by

$$S_{i,t_k}^C = S_{i,t_{k-1}}^C \exp\left(Y_{i,t_k}^C\right),$$
$$Y_{i,t_k}^C := \left(r - \lambda^{\mathbb{Q}^H} \kappa_i^{\mathbb{Q}^H} - \frac{1}{2}\sigma_i^2\right)\Delta + \sigma_i \left(B_{i,t_k}^{\mathbb{Q}^H} - B_{i,t_{k-1}}^{\mathbb{Q}^H}\right) + u_i^* \left(N_C^{\mathbb{Q}^H i}(t_k) - N_C^{\mathbb{Q}^H i}(t_{k-1})\right), \qquad \forall i \in \{1,2\}.$$

On the other hand, the price dynamics of the independent jump process (type L) under a martingale measure $\mathbb{Q}^L$ over each sub-interval $(t_{k-1}, t_k]$ is given by $(\forall i \in \{1,2\})$

$$S_{i,t_k}^I = S_{i,t_{k-1}}^I \exp\left(Y_{i,t_k}^I\right),$$
$$Y_{i,t_k}^I := \left(r - \lambda \kappa_i - \frac{1}{2}\sigma_i^2\right)\Delta + \sigma_i \left(B_{i,t_k}^{\mathbb{Q}^L} - B_{i,t_{k-1}}^{\mathbb{Q}^L}\right) + u_i^* \left(N_I^i(t_k) - N_I^i(t_{k-1})\right), \qquad \forall i \in \{1,2\}.$$

We implement the Monte Carlo simulation by drawing S sample paths of $\left\{S_{1,t_k}^{(s)}, S_{2,t_k}^{(s)}\right\}$, $k \in \{1, \dots, N\}$, $s \in \{1, \dots, S\}$, and we set S = 100,000 in this study. Our simulation procedure for each interval $(t_{k-1}, t_k]$ is illustrated in the following steps:

1. Given $\left\{S_{1,t_{k-1}}^l, S_{2,t_{k-1}}^l\right\}$ and $\left\{Y_{1,t_{k-1}}^l, Y_{2,t_{k-1}}^l\right\}$ for $\forall l \in \{C, I\}$. Generate $Z_0 \sim N(0,1)$ and $Z_1 \sim N(0,1)$ and create the following:
$$Z_2^C = \rho_{12}^H Z_1 + \sqrt{1 - (\rho_{12}^H)^2}\, Z_0\,, \quad Z_2^I = \rho_{12}^L Z_1 + \sqrt{1 - (\rho_{12}^L)^2}\, Z_0$$
2. Generate $N_C^{\mathbb{Q}^H i}(t) - N_C^{\mathbb{Q}^H i}(t_{k-1}) \sim \text{Poisson}\left(\lambda^{\mathbb{Q}^H} \Delta\right)$, $N_I^i(t_k) - N_I^i(t_{k-1}) \sim \text{Poisson}(\lambda \Delta)$, $\forall i \in \{1,2\}$.
3. If $N_C^{\mathbb{Q}^H i}(t_k) - N_C^{\mathbb{Q}^H i}(t_{k-1}) > 0$ then generate $\boldsymbol{u}^* \sim N\left(\boldsymbol{\zeta}^*, \boldsymbol{\Sigma}_J\right)$ or else set $\boldsymbol{u}^* \coloneqq (0,0)^T$.
4. If $N_I^i(t_k) - N_I^i(t_{k-1}) > 0$ then generate $u_i \sim N\left(\zeta_i, \delta_i^2\right)$ or else set $u_i \coloneqq 0$, $\forall i \in \{1,2\}$.
5. Create
$$Y_{i,t_k}^C = \left(r - \lambda^{\mathbb{Q}^H} \kappa^{\mathbb{Q}^H} - \tfrac{1}{2}\sigma_i\right)\Delta + \sigma_i \sqrt{\Delta}\, Z_i^C + u_i^*,$$

$$Y_{i,t_k}^{I} = \left(r - \lambda\kappa - \frac{1}{2}\sigma_i\right)\Delta + \sigma_i\sqrt{\Delta}\, Z_i^{I} + u_i, \quad \forall i \in \{1,2\}.$$

and set $S_{i,t_k}^{C} = S_{i,t_{k-1}}^{C} \exp\left(Y_{i,t_k}^{C}\right)$ and $S_{i,t_k}^{I} = S_{i,t_{k-1}}^{I} \exp\left(Y_{i,t_k}^{I}\right)$.

6. Repeat steps 1 to 5 for the next time interval $(t_k, t_{k+1}]$ given $\{S_{1,t_{k-1}}^{l}, S_{2,t_{k-1}}^{l}\}$ and $\{Y_{1,t_{k-1}}^{l}, Y_{2,t_{k-1}}^{l}\}$, $\forall l \in \{C, I\}$.

Each sample path generates $\left\{C_{t_k}^{(s)}\right\}_{k\in\{1,\dots,N\}}$ and $X_T^{(s)}$ based on conditions stated in case 1 to 3, and we take the sum of discounted payoffs, that is, $\sum_{k=1}^{N} \exp(-rt_i) C_{t_k}^{(s)} \Delta\, X_0 + \exp(-rT) X_T^{(s)}$, for each $s \in \{1, \dots, S\}$, $S = 100{,}000$. The final estimate of the expectation is found by ($\forall m \in \{H, L\}$)

$$\hat{V}_0^{m} = \frac{1}{S}\sum_{s=1}^{S}\left[\sum_{k=1}^{N} \exp(-rt_k) C_{t_k}^{(s)} \Delta\, X_0 + \exp(-rT) X_T^{(s)}\right].$$

APPENDIX 3: SKETCH OF PROOF: THEOREM

We prove the theorem by modifying the results in Kijima and Uchida (2005). In their study, the low-type agent (similar to type L in this study) is assumed to switch his/her type randomly and *independently* to a high type (similar to type H in this study). We modify the assumption so that all the type L investors switch randomly but *simultaneously* to type H at a constant switching rate $\lambda\,(> 0)$. Along with this modification, we revise the Markov transition structure of $\{N_{HS}(t), N_{LB}(t)\}$ where $N_{HS}(t)$ $\left(N_{LB}(t)\right)$ represents the number of type H sellers (type L buyers) at time *t*, so that the state space stays irreducible. Then, we can apply the well-known result in the theory of Markov chains and with the similar manipulation in Kijima and Uchida (2005), we derive the desired result given the assumptions stated in Section 4.

References

Bates, D. S., 2000, “Post-’87 Crash Fears in the S&P 500 Futures Option Market,” *Journal of Econometrics*, 94, pp. 181–238.

Brigo, D., M. Morini, and A. Pallavicini, 2013, *Counterparty Credit Risk, Collateral and Funding: With Pricing Cases for All Asset Classes*, John Wiley & Sons.

Cesari, G., J. Aquilina, N. Charpillon, Z. Filipovic, G. Lee, and I. Manda, 2009, *Modeling, Pricing, and Hedging Counterparty Credit Exposure*, Springer.

Cont, R., 2006, “Model Uncertainty and Its Impact on the Pricing of Derivative Instruments,” *Mathematical Finance*, 16, pp. 519–547.

Das, S. R., and R. Uppal, 2004, “Systemic Risk and International Portfolio Choice,” *Journal of*

Finance, 59, pp. 2809–2834.

Duffie, D., and N. Garleanu, 2001, "Risk and Valuation of Collateralized Debt Obligations," *Financial Analysts Journal*, 57, pp. 41–59.

——, ——, and L. H. Pedersen, 2005, "Over-the-Counter Markets," *Econometrica*, 73, pp. 1815–1847.

——, and M. Huang, 1996, "Swap Rates and Credit Quality," *Journal of Finance*, 51, pp. 921–949.

Embrechts, P., B. Wang, and R. Wang, 2014, "Aggregation-Robustness and Model Uncertainty of Regulatory Risk Measures," working paper, ETH Zurich.

Gregory, J., 2012, *Counterparty Credit Risk and Credit Value Adjustment—A Continuing Challenge for Global Financial Markets*, John Wiley & Sons (second edition).

Hull, J., and A. White, 2012, "CVA and Wrong Way Risk," *Financial Analysts Journal*, 68, pp. 58–69.

Kijima, M., and Y. Uchida, 2005, "A Markov Model for Valuing Asset Prices in a Dynamic Bargaining Market," *Quantitative Finance*, 5, pp. 277–288.

Lipton, A., and A. Sepp, 2009, "Credit Value Adjustment for Credit Default Swaps via the Structural Default Model," *Journal of Credit Risk*, 5, pp. 127–150.

Liu, J., and J. Pan, 2003, "Dynamic Derivative Strategies," *Journal of Financial Economics*, 69, pp. 401–430.

——, ——, and T. Wang, 2005, "An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks, *Review of Financial Studies*, 18, pp. 131–164.

Morini, M., 2011, *Understanding and Managing Model Risk: A Practical Guide for Quants, Traders and Validators*, John Wiley & Sons.

Naik, V., and M. Lee, 1990, "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies*, 3, pp. 493–521.

Oksendal, B., and A. Sulem, 2007, *Applied Stochastic Control of Jump Diffusions*, Springer.

Pan, J., 2002, "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journal of Financial Economics*, 63, pp. 3–50.

Redon, C., 2006, "Wrong Way Risk Modeling," *Risk*, April, pp. 90–95.

Senior Supervisors Group (SSG), 2009, "Risk Management Lessons from the Global Banking Crisis of 2008."

Pykhtin, M., and Sokol, A., 2013, "Exposure under Systemic Impact," *Risk*, September, pp. 100–105.