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A Synthesis of "Market Microstructure"
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A Pricing Theory under a Finite Number of Securities Issued: A Synthesis of "Market Microstructure" and "Mathematical Finance"

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Abstract

Traditional finance theory generally assumes a frictionless market, in which a risk premium is described only by price volatility. In reality, however, the risk premium is influenced by a range of factors including the market microstructure. This paper constructs a novel no-arbitrage and complete model that explicitly incorporates among the market microstructure factors a constraint on a finite number of securities issued. From the theoretical perspective, the model is a synthesis of market microstructure and mathematical finance in that it makes it possible to derive a risk-neutral price applicable to a market with a detailed market microstructure. We also calibrate the model to show that the price in the Japanese government bond futures market is significantly affected by the factor of number of securities issued.

Keywords: Security price; Number of securities issued; Risk neutral pricing rule; Market microstructure; No-arbitrage; Quasi risk aversion; Quasi risk neutral measure

JEL classification: D49, G01, G12

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1. Introduction

It is widely recognized that a number of factors besides fundamentals affect security prices.[1] Market participants frequently observe that the announcement of a stock split and stock buyback by a firm triggers an immediate rise in the firm's equity price, even though such actions do not substantially affect its profitability. In addition, the sharp drops in trading volume accompanied a plunge in equity prices following the collapse of Lehman Brothers in 2008 and during the European debt crisis of 2009 are a recent memory.

The recognition is supported by numerous empirical studies. For example, Amihud and Mendelson (1991) demonstrate the existence of a significant price difference between U.S. Treasury bills and notes with the same remaining maturities and cash flows. Amihud, Mendelson and Lauterbach (1997) use price data from the Tel Aviv stock exchange to show that stock prices increase with greater frequency of trading. Uchida and Yoshikawa (2014) show that a stock split and a change in the floating stock ratio cause substantial price changes in Japanese stock market. Concerning several major currency pairs, Mancini, Ranaldo, and Wrampelmeyer (2013) provide evidence on the relationship between the foreign exchange rate and the short-term funding costs that reflect the creditworthiness of individual financial institutions. Earlier studies suggest that the actual transacted prices depend on a range of factors other than fundamentals, regardless of the type of the assets.

Because traditional finance theory, which assumes a frictionless market where the risk premium is described only by price volatility, cannot analyze the effect of a number of factors on prices, many theoretical studies have sought to overcome such a limitation. These studies can be divided broadly into two schools.[2] The first, which we call "Market Microstructure," constructs a specific model that analyzes the market microstructure and

[1] These factors consists of a range of elements: for example, the number of securities issued, the frequency of the trading related to the ease of finding counterparties, the size of orders, a wide difference in risk appetite among market participants, and information asymmetry.

[2] Several streams of research do not belong to the two schools described; for example, Acharya and Pedersen (2005) propose a liquidity-adjusted capital asset pricing model (CAPM). This is an extension of the CAPM and considers the effect of liquidity explicitly.

incorporates the detailed market structure to derive price, trade volume, and trade strategy. For example, Duffie, Gârleanu, and Pederson (2005) and Kijima and Uchida (2005), utilize dynamic matching and bargaining games to consider a securities market with a finite number of participants who cannot immediately find each other as counterparties. They incorporate the explicit structure of trading frequency to derive security prices in the market. Almgren and Chriss (2000) and Alfonsi and Schied (2010) derive optimal execution strategies for a large size of orders by considering the market impact. The second school, which we call "Mathematical Finance," constructs an elegant theoretical model that deals with incompleteness in the market to show a pricing theory. Kijima and Tamura (2012) derive mathematical conditions for security prices to be in equilibrium with the existence of transaction costs. Çetin, Jarrow, and Protter (2004) consider security prices dependent on trading size. They derive the no-arbitrage and complete conditions considering the size of trades, although they do not impose any constraint on the finiteness of securities issued. Once we construct the no-arbitrage complete market model incorporating factors in the market, security prices can be expressed by the discounted expectation of the underlying asset at the time of maturity with the equivalent martingale measure. This representation is convenient because of the ease of calculation under normal conditions. Gerhold *et al.* (2014) consider the liquidity premium as one factor that lowers security prices to derive the pricing formulas, trading policy, and trading volume explicitly.[3]

Our paper presents the no-arbitrage and complete model with the explicit consideration of the market microstructure regarding the finiteness of securities issued.[4] From the theoretical perspective, our model is a synthesis of the two schools of studies described above in that it makes it possible to derive a pricing theory under a constraint on a finite

[3] They do not discuss no-arbitrage and completeness directly. However, their research deals with liquidity risk utilizing the shadow price, which formulates transaction costs under no-arbitrage and completeness. In this sense, we categorize it as mathematical finance.

[4] This paper also implies that the security price depends on the difference between market participants' preferences. Utilizing our model, Uchida and Yoshikawa (2014) consider whether the difference in the presence in the market between two types of market participants affects security prices.

number of securities issued. We also calibrate the model to show that prices in the Japanese government bond (JGB) futures market are significantly affected by the number of securities issued.

This paper is structured as follows. Section 2 gives the assumptions and preliminary calculations to derive the model. Section 3, the main part of the paper, derives the pricing formula explicitly. In Section 4, we demonstrate the numerical application of our model to the long-term JGB futures market. Finally, in Section 5, we summarize the conclusion and make some brief comments concerning future research. The proofs of the theorem and propositions are provided in the Appendix.

2. Model

We introduce a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. For simplicity, we consider a one-period binomial model that consists of time 0 and $T > 0$. More precisely, $\Omega := \{\omega_1, \omega_2\}$, $\mathcal{F} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \Omega\}$ and $\mathbb{F} = (\mathcal{F}_t)_{t=0,T}$ with the structure $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. The probability measure is defined as $\mathbb{P}(\omega_1) = p$ and $\mathbb{P}(\omega_2) = 1 - p$.

In this paper, we only consider one risky asset and one risk-free asset. Let $V = \{V_t, t = 0, T\}$ be a random variable defined on the filtered probability space given above. We describe V as the total value of the underlying asset. $X = \{X_t, t = 0, T\}$ stands for the price of the risky security. We assume the number of the securities issued, θ^M, to be constant. Therefore, the equality $V_t = \theta^M X_t$ is satisfied at $t = 0, T$.

Let $V_T(\omega_1) = V_1$ and $V_T(\omega_2) = V_2$. We can assume $V_2 < V_1$ without loss of generality. At the same time, X_T is given by $X_T(\omega_1) = X_1 = V_1/\theta^M$ and $X_T(\omega_2) = X_2 = V_2/\theta^M$. We also postulate that the risk-free rate is zero.

We assume that market participants consist of two types, where the type is given by $i \in \{H, L\}$. Let $U_i(\cdot)$ be utility functions of these types. We also assume that the support of U_i is a positive line and that they are continuous and second-order differential functions that satisfy $U_i^{'} > 0$ and $U_i^{''} < 0$. Now, we consider a portfolio that is owned by each participant. It consists of θ_i of the risky security and η_i of the risk-free asset. We describe it as (θ_i, η_i). With given initial endowments c_i, the participants allocate them to risky securities and the risk-free asset at $t = 0$ as (θ_i, η_i); i.e., $c_i = \theta_i X_0 + \eta_i$. We call (θ_i, η_i) the strategy of the participant $i \in \{H, L\}$. In this setting, we also assume that the following market clearing condition holds:

$$\theta_H + \theta_L = \theta^M, \qquad \theta_H \geq 0, \theta_L \geq 0.$$ [5]

Each participant maximizes his/her expected utility given by the following expected utility maximization problem for $i \in \{H, L\}$:

$$u_i(c_i) \coloneqq \max_{\theta \in \Theta} \{pU_i\big(c_i + \theta_i(X_1 - X_0)\big) + (1 - p)U_i\big(c_i + \theta_i(X_2 - X_0)\big)\},$$

where Θ is the set of all feasible strategies. This entails that a strategy in Θ satisfies the market clearing condition.

Now we consider the first-order condition. It is difficult for both participants to attain the optimal strategies simultaneously because of the finiteness of the number of securities issued. Indeed, a strategy of one participant is often blocked by that of the other participant, because the sum of orders by each participant does not always satisfy the market clearing condition. At the same time, the traded security price must be based on the agreement of both market participants. Therefore, we introduce the dependency of X_0 on θ_i, one of the key features of our model,[6] such that

$$\partial X_0 / \partial \theta_i \neq 0 \text{ for } i \in \{H, L\}.$$

At the same time, we do not impose any concrete form of this derivative, because each participant cannot control the security price in the way he/she wants.

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[5] We impose the short-selling constraint here.

[6] Lucas (1978) also consider the case where the price of goods is dependent on the economic states and derives an equilibrium under the market clearing condition and market completeness. The difference between Lucas (1978) and this paper can be summarized as follows: (1) Lucas (1978) admits the existence of a representative agent, and (2) Lucas (1978) imposes independency between θ_i and X_0.

Since neither participant can control the future value of the underlying asset, we need the following condition:

$$\partial X_1/\partial\theta_i = 0, \partial X_2/\partial\theta_i = 0 \text{ for } i \in \{H, L\}.$$

Therefore, we obtain the first-order condition as follows:

$$\begin{aligned} &pU_i'\big(c_i + \theta_i(X_1 - X_0)\big)\left((X_1 - X_0) - \theta_i \frac{\partial X_0}{\partial\theta_i}\right) \\ &+(1-p)U_i'\big(c_i + \theta_i(X_2 - X_0)\big)\left((X_2 - X_0) - \theta_i \frac{\partial X_0}{\partial\theta_i}\right) = 0. \end{aligned} \tag{1}$$

For each type $i \in \{H, L\}$, we call X_0 satisfying (1) an optimal security price and denote it as X_0^i. We rewrite (1) as $G_i\big(\theta_i, X_0^i\big) = 0$. Note that X_0^i does not mean the traded security price, up to this point. To attain the trading feasibility, we need $X_0^H = X_0^L$. Then, we reach the security price X_0 as the solution of the following equations

$$\begin{cases} G_H(\theta_H, X_0^H) = 0 \\ G_L(\theta_L, X_0^L) = 0 \\ X_0 = X_0^H = X_0^L\ . \\ \theta^M = \theta_H + \theta_L \\ \theta_H \geq 0, \theta_L \geq 0 \end{cases} \tag{2}$$

Remark 1

The price $X_0^i, i \in \{H, L\}$ can be considered as an extension of Davis's price (Davis [2007]). Davis (2007) proposes a pricing principle of contingent claims in the incomplete market. Let ξ be $\mathcal{F}_{\mathrm{T}}$-measurable random variable, which generates a contingent payoff of a security at $t = T$. Davis (2007) concludes that the security price can be represented by $E^{\mathbb{Q}_i}[\xi]$, utilizing a martingale measure $\mathbb{Q}_i$ that is derived by the utility maximization of the participant who is concerned only with the uncertainty of the underlying assets. Note that Davis's price only reflects the premium against this kind of uncertainty,[7] while it does not take account of the effect of the finiteness of the number of securities issued. If we apply Davis's pricing principle to a one-period binomial model, like our model, the martingale measure $\mathbb{Q}_i$ is given by $\mathbb{Q}_i = \{q_i, 1-q_i\}$ for $i \in \{H, L\}$:

$$q_i = \frac{pU_i^{'}(W_1)}{E[U_i^{'}(W_T)\,]},$$

where $W_T = \{W_1, W_2\}$ is the terminal wealth. With the notation of our model, we have $W_T := c_i + \theta_i(X_T - X_0^i)$. Utilizing $\mathbb{Q}_i$, the first-order condition $G_i\big(\theta_i, X_0^i\big) = 0$ in (2) can be rewritten as,

$$E^{\mathbb{Q}_i}\left[X_T - X_0^i - \theta_i \frac{\partial X_0^i}{\partial \theta_i}\right] = 0$$
$$\Leftrightarrow X_0^i = E^{\mathbb{Q}_i}[X_T] - \theta_i \frac{\partial X_0^i}{\partial \theta_i}.$$

This explicitly shows that X_0^i is the extension of Davis's price. Indeed, the second term of the right-hand side of the above equation, $\theta_i\, \partial X_0^i/\partial\theta_i$, will vanish if participants attain the optimal solution independently. This independence seems to result in the similar meanings that the assumption of introducing the representative agents embeds.

Hereafter, we apply the exponential utility function with risk aversion γ_i for $i \in \{H, L\}$, where we postulate $\gamma_H \geq \gamma_L > 0$ without loss of generality. With this specification, U_i and q_i for $i \in \{H, L\}$ can be expressed as:

$$\begin{cases} U_i(x) := -e^{-\gamma_i x} \\ q_i = \dfrac{pe^{-\gamma_i\theta_i X_1}}{pe^{-\gamma_i\theta_i X_1} + (1-p)e^{-\gamma_i\theta_i X_2}} \end{cases}. \tag{3}$$

In the next section, we obtain the pricing formula by solving (2) under (3).

[7] This premium can be calculated when the degree of the convexity of the utility function is given.

3. Main result

We explain the main theorem and the related propositions. All the proofs are provided in the Appendix.

THEOREM 1 *(Pricing formula and uniqueness)*

For the exponential utility function, security price X_0 *satisfies the following equation (4), which is a unique solution of (2):*

$$\theta^M X_0 = -\frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H \theta_H X_T}\right] - \frac{1}{\gamma_L}\ln E\left[e^{-\gamma_L \theta_L X_T}\right] + C_H + C_L, \qquad (4)$$

where θ_H, θ_L *are optimal strategies of market participants, and* C_H, C_L *are constants of integration.*

Therefore, X_0 *uniquely exists with given* C_H, C_L.

PROOF. See Appendix 1.

Incorporating the no-arbitrage condition in the assumption, we can modify (4) into another form that does not contain C_H, C_L.

Remark 2

In this paper, we consider the no-arbitrage condition as $X_2 < X_0 < X_1$, which is equivalent to $V_2 < \theta^M X_0 < V_1$. Because we impose the short-selling constraint, we cannot find an arbitrage strategy even under the case of $X_2 < X_1 < X_0$. The expected price of the security is, however, strictly below the present one when $X_2 < X_1 < X_0$ holds. This fact leads to a contradiction of the zero risk-free interest rate assumption, so that the value of cash denominated by the risky security diverges under the multi-period setting. Therefore, we exclude $X_2 < X_1 < X_0$ from our no-arbitrage condition.

PROPOSITION 2 *(Quasi-risk-neutral representation)*

The no-arbitrage condition holds if and only if there exists $\hat{\gamma}^M$ *satisfying the following equation:*

$$\theta^M X_0 = -\frac{1}{\hat{\gamma}^M} \ln E[e^{-\hat{\gamma}^M \theta^M X_T}].$$

In this case, the security price is also given by

$$\theta^M X_0 = E^{\widehat{\mathbb{Q}}}[\theta^M X_T],$$

where $\widehat{\mathbb{Q}} \coloneqq \{\hat{q}, 1-\hat{q}\}$ *is defined as*

$$\hat{q} \coloneqq -\frac{1}{\hat{\gamma}^M (V_1 - V_2)} \ln(p e^{-\hat{\gamma}^M (V_1 - V_2)} + (1-p)).$$

This implies $\hat{q} \in [0,1]$.

PROOF. See Appendix 2.

We call $\hat{\gamma}^M$ quasi-risk aversion and $\widehat{\mathbb{Q}}$ quasi-risk-neutral measure. In our model, quasi-risk-neutral measure reflects all kinds of market frictions. Comparing the pricing formulas utilizing the quasi-risk-neutral measure $\widehat{\mathbb{Q}}$ with the formula utilizing the risk-neutral measure $\mathbb{Q}_i$ shown in Davis-type representation $\mathrm{E}^{\mathbb{Q}_i}[X_T] - \theta_i\, \partial X_0^i / \partial \theta_i$, we find that the difference lies in the adjustment term $\theta_i\, \partial X_0^i / \partial \theta_i$.

The next proposition shows the range that the no-arbitrage condition holds for $C_H + C_L$.

PROPOSITION 3 *(No-arbitrage condition)*

The no-arbitrage condition holds, if

$$K_2 < C_H + C_L < K_1,$$

where $K_2 \coloneqq 1/\gamma_H \ln\ Ee^{-\gamma_H (V_T - V_2)}, K_1 \coloneqq (1-p)(V_1 - V_2)$.

PROOF. See Appendix 3.

Note that the security price reflects the distribution of the underlying asset and is always between X_1 and X_2 under the no-arbitrage condition. To satisfy this condition, even if the probability of going to the upper node is close to one, the security price needs to be less than X_1. To meet this upper restriction, $C_H + C_L$ cannot exceed the value of K_1. Concerning the lower restriction, we impose the additional condition, $V_2 \geq 0$, to reach a similar result and get $C_H + C_L > K_2$. Notice that the asymmetry between the value of K_1 and K_2 comes from $V_2 \geq 0$.

Now we proceed to detailed interpretations of the previous theorem and propositions. First, to prepare for the discussion, we define $f(\gamma)(\theta) := -1/\gamma\theta \ln E\left[e^{-\gamma\theta X_T}\right]$, which can be thought to be certainty equivalent[8] for the agent whose risk aversion and risky security inventory are $\gamma > 0$ and $\theta > 0$, respectively. Second, we consider the hypothetical situation that the inventory of the security does not affect the price. We can describe it as $\partial X_0/\partial\theta_i = 0$ for each i so that the optimal solution for this case is

$$\begin{cases} \theta_H^{hypo} = \dfrac{\gamma_L}{\gamma_H + \gamma_L}\theta^M \\ \theta_L^{hypo} = \dfrac{\gamma_H}{\gamma_H + \gamma_L}\theta^M \end{cases}.$$

The sum of certainty-equivalent values of all the securities issued is given by $\theta_H^{hypo} f(\gamma_H)\left(\theta_H^{hypo}\right) + \theta_L^{hypo} f(\gamma_L)\left(\theta_L^{hypo}\right)$. Then, we can find the following relationship through easy calculation:

$$\theta_H^{hypo} f(\gamma_H)\left(\theta_H^{hypo}\right) + \theta_L^{hypo} f(\gamma_L)\left(\theta_L^{hypo}\right) = \theta^M f(\gamma_{h-ave}^M)(\theta^M),$$

$$\text{where } \frac{1}{\gamma_{h-ave}^M} = \frac{1}{\gamma_H} + \frac{1}{\gamma_L}.$$

[8] As is well known, the value of a security under certainty-equivalence reflects the premium against the uncertainty of the underlying assets. The size of this premium is determined by the degree of convexity of the utility function.

$\theta^M f(\gamma_{h-ave}^M)(\theta^M)$ denotes the certainty-equivalent value of all the issued securities owned by the single hypothetical agent whose risk aversion is γ_{h-ave}^M.[9]

Now we can rewrite our pricing formula (4) in the following form:

$$\begin{aligned}\theta^M X_0 = \theta^M E[X_T] + \theta^M(f(\gamma_{h-ave}^M)(\theta^M) - E[X_T]) \\ + \left(\theta_H f(\gamma_H)(\theta_H) + \theta_L f(\gamma_L)(\theta_L) - \theta^M f(\gamma_{h-ave}^M)(\theta^M)\right) \\ + (C_H + C_L)\end{aligned}$$

$$\begin{aligned}\Leftrightarrow\ X_0 = E[X_T] + (f(\gamma_{h-ave}^M)(\theta^M) - E[X_T]) \\ + \frac{1}{\theta^M}\left(\theta_H f(\gamma_H)(\theta_H) + \theta_L f(\gamma_L)(\theta_L) - \theta^M f(\gamma_{h-ave}^M)(\theta^M)\right) \\ + \frac{1}{\theta^M}(C_H + C_L).\end{aligned} \tag{5}$$

The above equation shows that the security price is the sum of the fundamental value and three types of premiums related to uncertainty and finiteness of the securities issued. The first term of the right-hand side of (5), $E[X_T]$, is the fundamental value per unit size of the securities issued.[10] The second term, $f(\gamma_{h-ave}^M)(\theta^M) - E[X_T]$, expresses the uncertainty of the underlying asset, and we call this term the *uncertainty premium* (UP). Note that if the agent is risk neutral, then his/her certainty-equivalent value of the security is the expectation $E[X_T]$ under the physical measure. Therefore, if γ_{h-ave}^M goes to zero, this term also approaches zero. On the other hand, the second term is less than zero for $\gamma_{h-ave}^M > 0$. Furthermore, this term does not reflect the constraint of the finiteness of the securities issued. This is because we consider the hypothetical case, $\partial X_0/\partial\theta_i = 0$. The

[9] Back (2010) argued the case that risk aversion of the representative agent can be substituted for that of two types of agents, and that it is given by the harmonic mean of the two. The discussion here has a formal similarity with Back (2010).

[10] In this discussion, we assume that the expectation of the future value of the underlying asset is equal to the fundamental value. A similar argument is used by Brunnermeier and Pedersen (2009), for example.

third term, $1/\theta^M \left(\theta_H f(\gamma_H)(\theta_H) + \theta_L f(\gamma_L)(\theta_L) - \theta^M f(\gamma_{h-ave}^M)(\theta^M)\right)$, appears as the reflection of the constraint of the finiteness of the number of securities issued, and we call this term the *first type of finite number premium* (FNP1). As shown above, when the constraint does not bind, we need the condition that $\partial X_0/\partial\theta_i = 0$ for each i, so that $\theta_H^{hypo} f(\gamma_H)(\theta_H^{hypo}) + \theta_L^{hypo} f(\gamma_L)(\theta_L^{hypo})$ is equal to $\theta^M f(\gamma_{h-ave}^M)(\theta^M)$. In this case, this term is zero. However, because the main purpose of our model is to consider this type of constraint, it does not usually hold. The fourth term, $1/\theta^M\,(C_H + C_L)$, reflects the finiteness of the number of securities issued and we call it the *second type of finite number premium* (FNP2). This comes from the fact that we do not specify the boundary condition of the first and second equations in (2). In other words, this contains the constants of integration shown in (4). To define the boundary condition, we need to consider the security price when the number of securities issued is zero or infinite. However, we do not have any criteria to determine such hypothetical cases. Therefore, we discuss X_0 with given C_H, C_L in this paper.[11]

According to Proposition 2, under the no-arbitrage condition the pricing formula is simplified as $X_0 = E^{\hat{\mathbb{Q}}}[X_T]$ with the quasi-risk-neutral measure. Because one of the key features of mathematical finance is characterized by the risk-neutral pricing rule, this proposition shows the formal similarity between methods of mathematical finance and

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[11] Aside the specification of the boundary condition, there might be several methods for specifying FNP2. One would be to utilize the game-theoretic method. Because our model consists of a zero-sum economy with two agents, this might help to derive an equilibrium price in our model. However, a solution in game theory is often an ordinal number, not a cardinal number, and this is not appropriate for the application of real market data. A second method would be to introduce another optimization principle. However, it is difficult to specify which method is better than others with distinct criteria. A third method would be to incorporate a dynamics into C_H, C_L and discuss the convergence. Although it might be effective, our model is not a multi-period model but a one-period model, so its application might be excessive. Furthermore, as Proposition 2 shows, FNP2 is absorbed into the quasi-risk aversion $\hat{\gamma}^M$. Therefore, we do not apply these methods, but we do not consider it to be a flaw in our model. Because our model is simple, there are many possibilities for its extension. We will discuss several directions of such an extension in the conclusion.

those of our model, although our model contains elaborate assumptions concerning the market microstructure. Furthermore, utilizing this proposition, we can incorporate all of UP, FNP1, and FNP2 into the quasi-risk-aversion $\hat{\gamma}^M$ at once. This facilitates the direct understanding of our model. For example, the higher the quasi-risk aversion $\hat{\gamma}^M$, the lower the security price.

4. Numerical experiments on the JGB futures market

In this section, we apply our model to JGBs and the JGB futures market. In the application, first, we determine model parameters. Second, we calculate the range of the no-arbitrage condition. Finally, we show the existence of FNP2.

Recall that our model parameters consist of $X_0, V_T, \theta^M, \mathbb{P}, \gamma_H, \gamma_L, C_H$ and C_L. At first, we specify X_0, V_T, θ^M and $\mathbb{P}$ through market data and refer to the earlier empirical research, Pardo (2012), for γ_H and γ_L. With these parameters in hand, we get C_H and C_L by calibrating the model and θ_H and θ_L by using $X_0^H = X_0^L$ and $\theta_H + \theta_L = \theta^M$.

We consider the price of JGB futures as X and use the data of 10 JGB futures with delivery months from March 2011 to September 2013. JGB futures are a product for trading the hypothetical underlying JGB called the "standard" JGB[12] on the future fixing date. Because JGB futures are settled by the cheapest deliverable JGB, we describe the issued number of this cheapest JGB as θ^M.[13] The price is defined as that of the end of the day. The sampling period of the data is from 40 to 10 days before the last trading date.[14]

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[12] The hypothetical underlying JGB is defined as a bond with a 10-year maturity and a 6 % coupon. This standardization provides more liquid trading circumstances. There are two types of JGB futures, a medium-term one and a long-term one. The details mentioned above are for the long-term one that we deal with.

[13] More precisely, we consider the quotient of the total amount of issued bonds and the unit price at the issue date.

[14] The last trading date is defined as seven days before the settlement dates, which are on the 20th of March, June, September, and December. If the 20th is a holiday, then the settlement date is the following business day. For example, the data of JGB futures with a delivery month of September 2013 consists of closing prices from July 19, 2013 to August 30, 2013. We use Bloomberg for the

We describe the growth rate of price as u (upward) and d (downward), assuming $ud = 1$. We refer to the volatility index Japan distributed by Osaka University for volatility, σ.[15] Without loss of generality, we approximate that the interest rate is zero. Then, we obtain $\mathbb{P} = \{p, 1-p\}$, utilizing $u, d,$ and σ.

Let $Y := \{Y_t, t = 0, T\}$ be the price of JGB.[16] Note that u, d are the growth rate of Y, because they are obtained with the volatility of JGBs.[17] However, X is settled by Y at $t = T$, then we define X_T, applying $X_1 = Y_1 = uY_0$ and $X_2 = Y_2 = uY_0$ and specify $V_T = \theta^M X_T$.[18] Many empirical studies on risk aversion have been conducted, so we can utilize much information from them.[19] In this paper, we apply the risk aversion estimated by Pardo (2012).[20,21]

We show an application of Theorem 1 and Proposition 2. Utilizing calibrated parameters, we calculate model prices and the range of the no-arbitrage condition. Figures 1 and 2 show them on the delivery month of September 2013 and March 2013, respectively. We observe the model and market price are in the range of no-arbitrage. The difficulty in finding the arbitrage opportunity is due to the large and liquid characteristics of the JGB futures market. JGB futures with other delivery months show similar traits.

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price data.

15 Data collection period is the same that that of JGB futures.

16 The price of JGBs is also available on Bloomberg, and the data collection period is the same as that that for JGB futures.

17 The probability space is common for JGBs and JGB futures, because JGB futures are derivatives of JGBs.

18 We use the price of JGB futures as X_0.

19 Friend and Blume (1975) is one of the best-known studies.

20 Our model can also calibrate the risk aversion from market data. However, this is outside our focus.

21 Risk aversion by Pardo (2012) is implicitly given as pertaining to the representative agent. Since the representative agent's risk aversion is given by the harmonic mean of each participant under the assumption of constant absolute risk aversion (CARA) utility, we define γ_H, γ_L such that $1/\gamma_{\mathrm{H}} + 1/\gamma_L = 1/0.9690$, where 0.9690 is that of Pardo (2012). In Figure 1 and Figure 2, below, we apply $\gamma_H = 10.659$, $\gamma_L = 1.066$, which are consistent with Pardo (2012) in the sense of the harmonic mean.

Figure 1: Market price of JGB futures with a delivery month of September 2013, corresponding model price, and the range of no-arbitrage

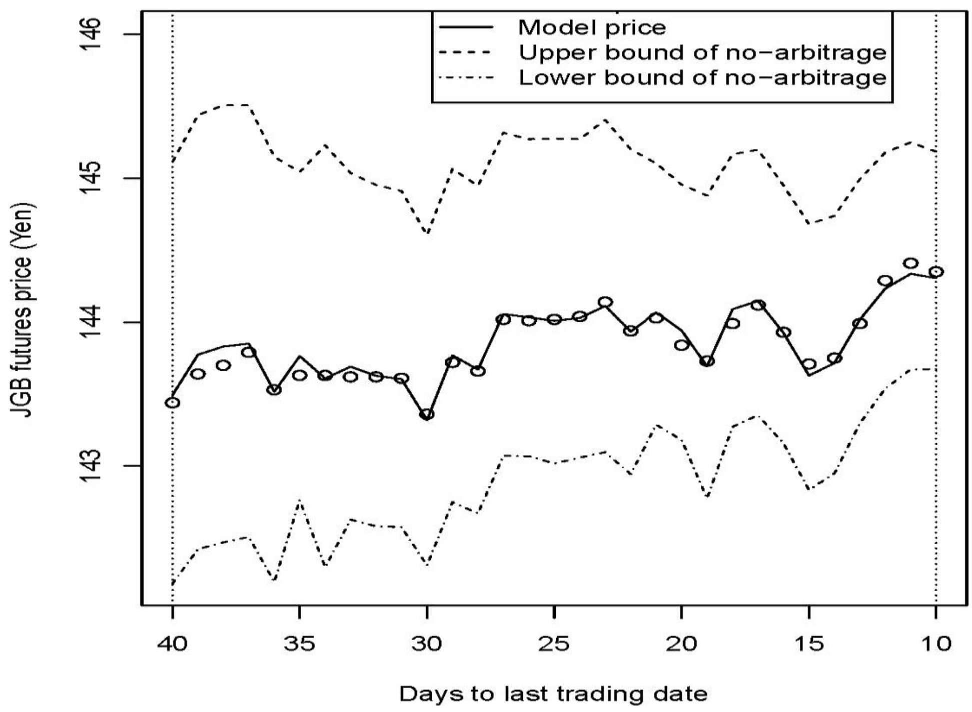
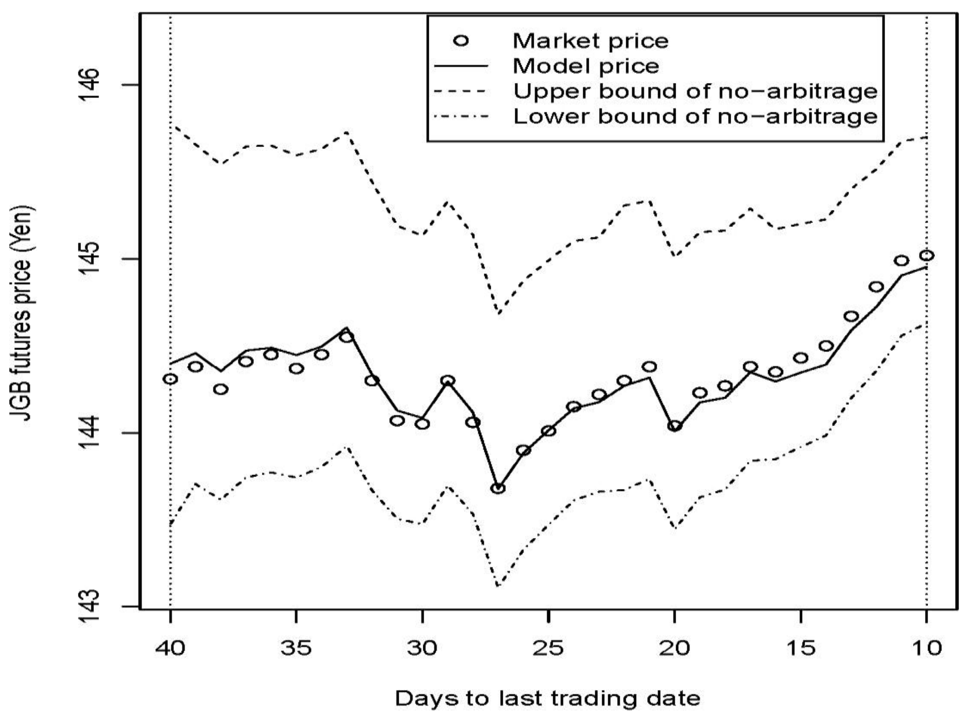


Figure 2: Market price of JGB futures with a delivery month of March 2013, corresponding model price, and the range of no-arbitrage



Because FNP2 is the constant of integration in (4), it might be zero. If $C_H = C_L = 0$, the security price given by our formula is identical to that explained by Davis's price. We conduct a statistical test, comparing Akaike information criterions (AICs), to confirm that the FNP2 term is not zero. For almost all JGB futures, FNP2 is not equal to zero.

Figures 3 and 4 show the estimated FNP2. The horizontal axis of each figure is the number of securities issued, and the vertical axis is the ratio of FNP2, $(C_H + C_L)X_0/\theta^M$. Figures next to the triangles indicate the delivery months of JGB futures. Figure 3 shows the case of $\gamma_H = \gamma_L = 1.938$. Figure 4 shows the case of $\gamma_H = 10.659,\ \gamma_L = 1.066$. Risk aversion refers to the estimate shown in Pardo (2012). In both Figures 3 and 4, FNP2 increases as the number of securities issued decreases. Comparing these two figures, we observe that the absolute values of FNP2 in Figure 4 are smaller than those in Figure 3. This implies that the difference in risk aversion increases the impact of FNP2. We can conjecture, for example, that if a participant with extremely small risk aversion entered the JGB futures market, the risk premium due to FNP2 would decrease.

Figure 3: FNP2 per total value. $\gamma_H = \gamma_L = 1.938$. Numbers next to the triangles are delivery months of JGB futures.

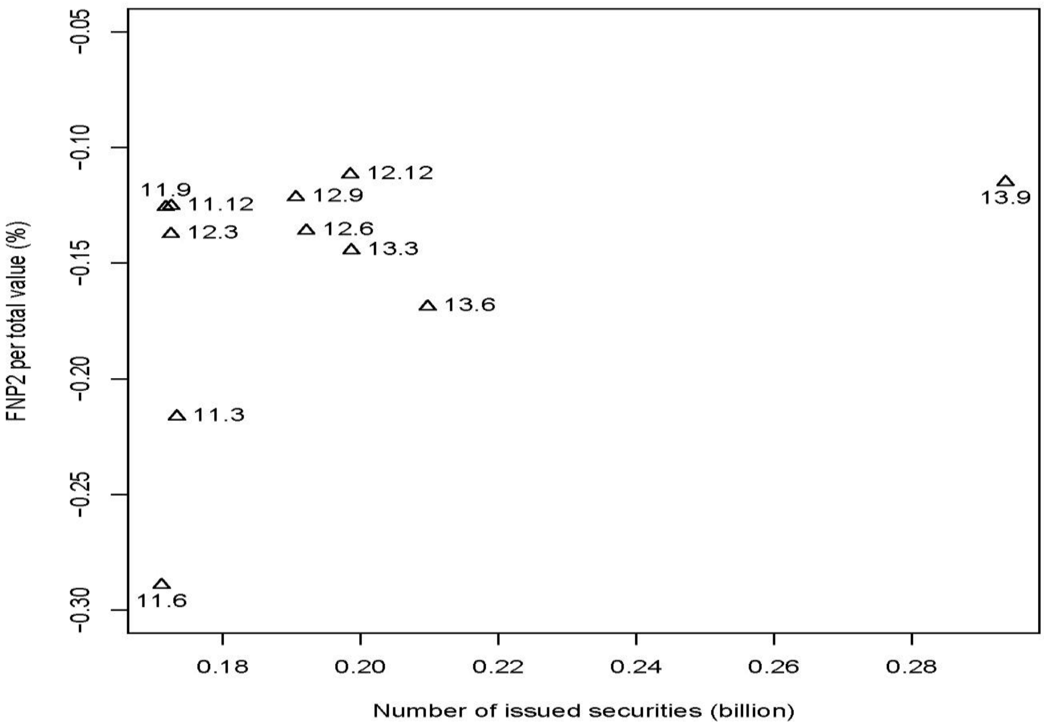
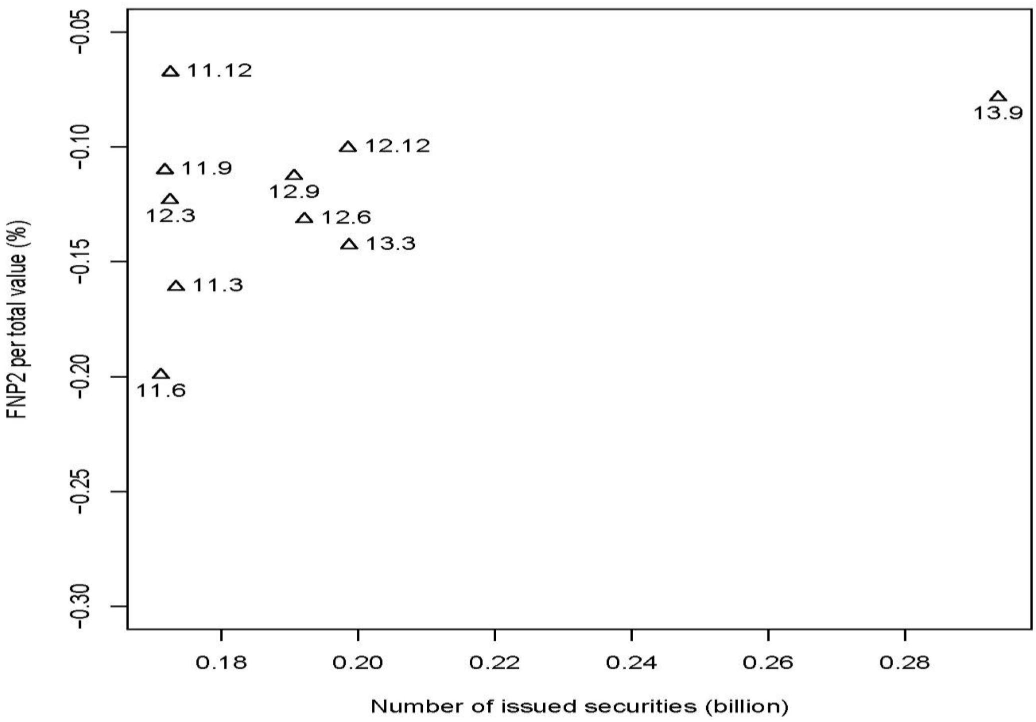


Figure 4: FNP2 per total value. $\gamma_H = 10.659, \gamma_L = 1.066$. Numbers next to the triangles are delivery months of JGB futures.



5. Conclusion

This paper constructs a no-arbitrage and complete model incorporating the constraint on the number of securities issued as one of the market microstructure factors. The model clarifies the mechanism in which the constraint influences security prices. This mechanism consists of two effects, called the first and second types of the finite number premium (FNP1 and FNP2). In particular, FNP2 reflects the boundary condition at zero and the infinite number of securities issued. Because the condition is hypothetical, we do not specify FNP2 endogenously. Instead, we introduce the quasi-risk-neutral measure and give the risk-neutral pricing representation that contains all of the risk premiums related to the security pricing. We also show the numerical examples applying our model to the data of the JGB futures market and demonstrate the existence of FNP2 in the market.

Here we briefly discuss potential topics for future research. First, because our model is simple in that the underlying asset is described by a one-period binomial model, it is difficult to analyze the relationship between the security price and the higher-order moment of the price distribution. In this regard, we need to extend our pricing formula with the introduction of a multi-period or multinomial model. Second, with the quasi-risk-neutral measure, we can determine the security price without any endogenous specification of FNP2. At the same time, our model is insufficient to clarify the mathematical structure in terms of how the security price depends on the number of securities issued. To address this issue, it is desirable to deal with FNP2 endogenously with the optimization principle including the Max-Min principle.

Appendix

Appendix 1 (Proof of Theorem 1)

As shown in Section 2, the first-order condition $G_i\left(\theta_i, X_0^i\right) = 0, i \in \{H, L\}$ is rewritten as follows:

$$
\begin{aligned}
& X_0^i = E^{\mathbb{Q}_i}[X_T] - \theta_i \frac{\partial X_0^i}{\partial \theta_i} \\
& \Leftrightarrow \frac{\partial \theta_i X_0^i}{\partial \theta_i} = E^{\mathbb{Q}_i}[X_T] \\
& \Leftrightarrow \theta_i X_0^i = \int E^{\mathbb{Q}_i}[X_T] d\theta_i + C_i,
\end{aligned}
$$

where C_i is a constant of integration. Plugging the definition of $\mathbb{Q}_i$ shown in (3) into this formula, we obtain

$$
\begin{aligned}
\theta_i X_0^i &= \int (q_i X_1 + (1 - q_i) X_2) d\theta_i + C_i \\
&= (X_1 - X_2) \int q_i d\theta_i + \int X_2 d\theta_i + C_i \\
&= (X_1 - X_2) \int \frac{p e^{-\gamma_i \theta_i X_1}}{p e^{-\gamma_i \theta_i X_1} + (1 - p) e^{-\gamma_i \theta_i X_2}} d\theta_i + X_2 \theta_i + C_i \\
&= \frac{(X_1 - X_2)}{-\gamma_i (X_1 - X_2)} \ln\left(p e^{-\gamma_i \theta_i (X_1 - X_2)} + (1 - p)\right) + X_2 \theta_i + C_i \\
&= -\frac{1}{\gamma_i} \ln\left(p e^{-\gamma_i \theta_i X_1} + (1 - p) e^{-\gamma_i \theta_i X_2}\right) + C_i \\
&= -\frac{1}{\gamma_i} \ln E\left[e^{-\gamma_i \theta_i X_T}\right] + C_i \\
&= \theta_i f(\gamma_i)(\theta_i) + C_i.
\end{aligned}
$$

Since $E\left[e^{-\gamma_i \theta_i X_T}\right]$ is monotone and bounded on the interval $\theta_i \in [0, \theta^M]$, so is $\theta_i f(\gamma_i)(\theta_i) = -1/\gamma_i \ln E\left[e^{-\gamma_i \theta_i X_T}\right]$. Because of this boundedness, there exist C_H and C_L satisfying $X_0^H = X_0^L$ and $\theta_H + \theta_L = \theta^M$.

Through the summation of $\theta_H X_0^H$ and $\theta_L X_0^L$ with the condition $\mathrm{X}_0 = X_0^H = X_0^L$, the security price (4) is derived and X_0 is unique due to the monotonicity of $\theta_i f(\gamma_i)(\theta_i)$.

Q.E.D.

Appendix 2 (Proof of Proposition 2)

First, we show "if." Note that for any $\gamma > 0$ and $\theta \in [0, \theta^M]$, we always have $X_2 \le f(\gamma)(\theta) \le X_1$. If there exists $\hat{\gamma}^M$ satisfying $\theta^M X_0 = -1/\hat{\gamma}^M \ln E[e^{-\hat{\gamma}^M \theta^M X_T}]$, we obtain $X_0 = f(\hat{\gamma}^M)(\theta^M)$. Then we obtain $X_2 \le X_0 \le X_1$, where both of the equalities hold if and only if $X_1 = X_2$. Therefore, we reach

$$X_2 < X_0 < X_1.$$

Next, we show "only if." To prepare, we show that it satisfies $\hat{q} \in [0,1]$ for any positive $\hat{\gamma}^M$. Since $pe^{-\hat{\gamma}^M (V_1 - V_2)} + (1-p) \le 1$,

$$\hat{q} = -\frac{1}{\hat{\gamma}^M (V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M (V_1 - V_2)} + (1-p)) \ge 0.$$

On the other hand,

$$\begin{aligned}
\hat{q} &= -\frac{1}{\hat{\gamma}^M (V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M (V_1 - V_2)} + (1-p)) \\
&= -\frac{\ln[(pe^{-\hat{\gamma}^M V_1} + (1-p)e^{-\hat{\gamma}^M V_2})e^{\hat{\gamma}^M V_2}]}{\hat{\gamma}^M (V_1 - V_2)} \\
&= -\frac{\ln[pe^{-\hat{\gamma}^M V_1} + (1-p)e^{-\hat{\gamma}^M V_2}] + \ln[e^{\hat{\gamma}^M V_2}]}{\hat{\gamma}^M (V_1 - V_2)} \\
&= -\frac{\ln E[e^{-\hat{\gamma}^M V_T}] + \hat{\gamma}^M V_2}{\hat{\gamma}^M (V_1 - V_2)} \\
&\le -\frac{E[\ln e^{-\hat{\gamma}^M V_T}] + \hat{\gamma}^M V_2}{\hat{\gamma}^M (V_1 - V_2)} \\
&= p \le 1.
\end{aligned}$$

On line 5, we apply Jensen's inequality. Since the no-arbitrage condition holds, a martingale measure exists and we can choose $\hat{\gamma}^M$ so that $\theta^M X_0 = E^{\hat{\mathbb{Q}}}[\theta^M X_T]$.[22] By the utilization of this $\hat{\mathbb{Q}}$, it also holds that

$$E^{\hat{\mathbb{Q}}}[\theta^M X_T] = \hat{q}\theta^M X_1 + (1-\hat{q})\theta^M X_2$$

[22] We implicitly assume that $X_0 \le E[X_T]$, because the security price cannot exceed the fundamental value due to the risk premium.

$$
\begin{aligned}
&= -\frac{1}{\hat{\gamma}^M(V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M(V_1 - V_2)} + (1-p))\theta^M X_1 \\
&\quad +(1 + \frac{1}{\hat{\gamma}^M(V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M(V_1 - V_2)} + (1-p)))\theta^M X_2 \\
&= -\frac{V_1}{\hat{\gamma}^M(V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M(V_1 - V_2)} + (1-p)) \\
&\quad +(V_2 + \frac{V_2}{\hat{\gamma}^M(V_1 - V_2)} \ln(pe^{-\hat{\gamma}^M(V_1 - V_2)} + (1-p))) \\
&= V_2 - \frac{1}{\hat{\gamma}^M} \ln(pe^{-\hat{\gamma}^M(V_1 - V_2)} + (1-p)) \\
&= -\frac{1}{\hat{\gamma}^M} \ln(pe^{-\hat{\gamma}^M V_1} + (1-p)e^{-\hat{\gamma}^M V_2}) \\
&= -\frac{1}{\hat{\gamma}^M} \ln E\left[e^{-\hat{\gamma}^M \theta^M X_T}\right].
\end{aligned}
$$

This argument shows that there exists $\hat{\gamma}^M$ satisfying $\theta^M X_0 = -\frac{1}{\hat{\gamma}^M} \ln E\left[e^{-\hat{\gamma}^M \theta^M X_T}\right]$, when the no-arbitrage condition holds.

Q.E.D.

Appendix 3 (Proof of Proposition 3)

The no-arbitrage condition $X_2 < X_0 < X_1$ is equivalent to $V_2 < \theta^M X_0 < V_1$. Applying Jensen's inequality to (3), we obtain

$$
\begin{aligned}
\theta^M X_0 &\leq -\frac{1}{\gamma_H} E\left[\ln e^{-\gamma_H \theta_H X_T}\right] - \frac{1}{\gamma_L} E\left[\ln e^{-\gamma_L \theta_L X_T}\right] + C_H + C_L \qquad (6) \\
&= E[\theta^M X_T] + C_H + C_L \\
&= E[V_T] + C_H + C_L.
\end{aligned}
$$

On the other side, we already have $C_H + C_L < K_1 = (1-p)(V_1 - V_2)$ as the assumption. Therefore, utilizing (6), we obtain

$$
\begin{aligned}
&\theta^M X_0 \leq E[V_T] + C_H + C_L \\
&\Leftrightarrow \theta^M X_0 < E[V_T] + (1-p)(V_1 - V_2) \\
&\Leftrightarrow \theta^M X_0 < V_1 .
\end{aligned}
$$

Next, we need to specify a lower bound of $C_H + C_L$. For any $\theta \geq 0$, $-1/\gamma_H \ln E\left[e^{-\gamma_H \theta X_T}\right] \leq -1/\gamma_L \ln E\left[e^{-\gamma_L \theta X_T}\right]$, because $\gamma_H \geq \gamma_L$. Therefore,

$$
\begin{aligned}
\theta^M X_0 &= -\frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H\theta_H X_T}\right] - \frac{1}{\gamma_L}\ln E\left[e^{-\gamma_L\theta_L X_T}\right] + C_H + C_L \\
&\geq -\frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H\theta_H X_T}\right] - \frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H\theta_L X_T}\right] + C_H + C_L \\
&\geq -\frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H\theta_H X_T}\right] E\left[e^{-\gamma_H\theta_L X_T}\right] + C_H + C_L.
\end{aligned}
$$

Here, we consider the minimal value of $-E\left[e^{-\gamma_H\theta_H X_T}\right]E\left[e^{-\gamma_H\theta_L X_T}\right]$. For simplicity, we define $\alpha \coloneqq \theta_H/\theta^M$ and we introduce

$$
h(\alpha) \coloneqq -E[e^{-\gamma_H\alpha V_T}]E\left[e^{-\gamma_H(1-\alpha)V_T}\right] = -E\left[e^{-\gamma_H\theta_H X_T}\right]E\left[e^{-\gamma_H\theta_L X_T}\right].
$$

By elementary calculation, we obtain

$$
\begin{aligned}
\frac{\partial h}{\partial \alpha} &= -\frac{\partial E[e^{-\gamma_H\alpha V_T}]}{\partial \alpha} E\left[e^{-\gamma_H(1-\alpha)V_T}\right] - E[e^{-\gamma_H\alpha V_T}]\frac{\partial E\left[e^{-\gamma_H(1-\alpha)V_T}\right]}{\partial \alpha} \\
&= -E[-\gamma_H V_T e^{-\gamma_H\alpha V_T}]E\left[e^{-\gamma_H(1-\alpha)V_T}\right] - E[e^{-\gamma_H\alpha V_T}]E\left[\gamma_H V_T e^{-\gamma_H(1-\alpha)V_T}\right] \\
&= \gamma_H E[e^{-\gamma_H\alpha V_T}]E\left[e^{-\gamma_H(1-\alpha)V_T}\right]\left(\frac{E[V_T e^{-\gamma_H\alpha V_T}]}{E[e^{-\gamma_H\alpha V_T}]} - \frac{E\left[V_T e^{-\gamma_H(1-\alpha)V_T}\right]}{E\left[e^{-\gamma_H(1-\alpha)V_T}\right]}\right) \\
&= \gamma_H E[e^{-\gamma_H\alpha V_T}]E\left[e^{-\gamma_H(1-\alpha)V_T}\right]\left(\frac{pe^{-\gamma_H\alpha V_1}}{E[e^{-\gamma_H\alpha V_T}]} - \frac{pe^{-\gamma_H(1-\alpha)V_1}}{E\left[e^{-\gamma_H(1-\alpha)V_T}\right]}\right)(V_1 - V_2) \\
&= \gamma_H\left(pe^{-\gamma_H\alpha V_1}E\left[e^{-\gamma_H(1-\alpha)V_T}\right] - pe^{-\gamma_H(1-\alpha)V_1}E[e^{-\gamma_H\alpha V_T}]\right)(V_1 - V_2) \\
&= \gamma_H(pe^{-\gamma_H\alpha V_1}\left(pe^{-\gamma_H(1-\alpha)V_1} + (1-p)e^{-\gamma_H(1-\alpha)V_2}\right) \\
&\quad -pe^{-\gamma_H(1-\alpha)V_1}(pe^{-\gamma_H\alpha V_1} + (1-p)e^{-\gamma_H\alpha V_2}))(V_1 - V_2) \\
&= \gamma_H p(1-p)\left(e^{-\gamma_H(\alpha V_1 + (1-\alpha)V_2)} - e^{-\gamma_H((1-\alpha)V_1 + \alpha V_2)}\right)(V_1 - V_2).
\end{aligned}
$$

On line 4, we use $\frac{(1-p)e^{-\gamma_H\alpha V_2}}{E\left[e^{-\gamma_H\alpha V_T}\right]} = 1 - \frac{pe^{-\gamma_H\alpha V_1}}{E\left[e^{-\gamma_H\alpha V_T}\right]}$ and $\frac{(1-p)e^{-\gamma_H(1-\alpha)V_2}}{E\left[e^{-\gamma_H(1-\alpha)V_T}\right]} = 1 - \frac{pe^{-\gamma_H(1-\alpha)V_1}}{E\left[e^{-\gamma_H(1-\alpha)V_T}\right]}$, and we obtain $\frac{\partial h}{\partial \alpha}(0) = \gamma_H p(1-p)(e^{-\gamma_H V_2} - e^{-\gamma_H V_1})(V_1 - V_2) > 0$. Furthermore,

$$
\begin{aligned}
\frac{\partial^2 h}{\partial \alpha^2} &= \gamma_H p(1-p)\big(-\gamma_H(V_1 - V_2)e^{-\gamma_H(\alpha V_1 + (1-\alpha)V_2)} \\
&\qquad - \gamma_H(V_1 - V_2)e^{-\gamma_H((1-\alpha)V_1 + \alpha V_2)}\big)(V_1 - V_2) \\
&= -\gamma_H^2 p(1-p)\left(e^{-\gamma_H(\alpha V_1 + (1-\alpha)V_2)} + e^{-\gamma_H((1-\alpha)V_1 + \alpha V_2)}\right)(V_1 - V_2)^2 \leq 0.
\end{aligned}
$$

Therefore, $\min_{\alpha\in[0,1]} h(\alpha) = h(0) = h(1) = -E[e^{-\gamma_H V_T}]$, and it holds that

$$\theta^M X_0 \geq -\frac{1}{\gamma_H}\ln E[e^{-\gamma_H V_T}] + C_H + C_L. \tag{7}$$

By the no-arbitrage condition, it needs to satisfy $\theta^M X_0 \geq V_2$. Plugging this into (7), we obtain

$$-\frac{1}{\gamma_H}\ln E[e^{-\gamma_H V_T}] + C_H + C_L > V_2$$
$$\Leftrightarrow C_H + C_L > V_2 + \frac{1}{\gamma_H}\ln E[e^{-\gamma_H V_T}] = \frac{1}{\gamma_H}\ln E\left[e^{-\gamma_H (V_T - V_2)}\right].$$

Q.E.D.

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