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Analytical Solution for the Loss Distribution of a Collateralized Loan under a Quadratic Gaussian Default Intensity Process

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Abstract

In this study, we derive an analytical solution for expected loss and the higher moment of the discounted loss distribution for a collateralized loan. To ensure nonnegative values for intensity and interest rate, we assume a quadratic Gaussian process for default intensity and discount interest rate. Correlations among default intensity, discount interest rate, and collateral value are represented by correlations among Brownian motions driving the movement of the Gaussian state variables. Given these assumptions, the expected loss or the *m*-th moment of the loss distribution is obtained by a time integral of an exponential quadratic form of the state variables. The coefficients of the form are derived by solving ordinary differential equations. In particular, with no correlation between default intensity and discount interest rate, the coefficients have explicit closed form solutions. We show numerical examples to analyze the effects of the correlation between default intensity and collateral value on expected loss and the standard deviation of the loss distribution.

Keywords: default intensity; stochastic recovery; quadratic Gaussian; expected loss; measure change

JEL classification: G21, G32, G33

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I Introduction

In credit risk valuations for loans, a lender's potential loss is given by default probability, recovery rate, and discount interest rate. Uncertainties related to loss include the correlation between default rate and recovery rate, as well as the independent volatility of these two rates. Empirical studies show a negative correlation between default rates and recovery rates (see Altman *et al.* [2005]). In the recent period of financial turmoil, regulators have paid much attention to the negative correlation associated with the countercyclicality of default rates and the procyclicality of recovery rates.

Against this backdrop, Yamashita and Yoshiba [2010] analytically evaluate the m-th moment of loss distribution with the square-root default intensity process, focusing on the negative correlation between default intensity and recovery rate. The study improves on Kijima and Miyake [2004] by keeping default intensity nonnegative. The model adopts the square-root process, a kind of affine process for default intensity (see, among others, Chen and Joslin [2011]; Duffie [2005]; Duffie, Pan, and Singleton [2000]) and solves the loss distribution with a zero fixed discount interest rate.

Another approach to representing nonnegative default intensity is to adopt a quadratic Gaussian process for default intensity. Assuming a fixed recovery rate and a discount interest rate, evaluations of the expected loss from a loan are reduced to that of survival probability. The discount bond price for a quadratic Gaussian short-term interest rate process can be applied to evaluate survival probability, since the relationship between survival probability and default intensity is the same as that between discount bond price and short-term interest rate. Most studies of quadratic Gaussian processes focus on the term structure of the interest rate (see, among others, Ahn, Dittmar, and Gallant [2002]; Chen, Filipović, and Poor [2004]; Constantinides [1992]; Jamshidian [1996]; Leippold and Wu [2003]; Piterbarg [2009]; Rogers [1997]). Here, we apply the discount bond price derived by Pelsser [1997], who assumes that the short-term interest rate follows a square function of a state variable with a Gaussian Ornstein-Uhlenbeck process. The author derives integral forms for the discount bond price. Kijima, Tanaka, and Wong [2009] give explicit closed form solutions for the discount bond price.

In a study that applies the quadratic Gaussian process to default intensity, Duffie and Liu [2001] evaluate defaultable bond prices with quadratic Gaussian default intensity and short-term interest rates, fixing the recovery rate and focusing on the negative correlation between default intensity and short-term interest rates. The defaultable discount bond price is represented as the exponential quadratic form of the state variables, similar to the non-defaultable discount bond price of Pelsser [1997]. However, in contrast to the explicit closed form solution for the discount bond price in Kijima, Tanaka, and Wong [2009], the defaultable discount bond price is given as the solution of ordinary differential equations, not as the closed form solution.

In this study, we extend the model of Duffie and Liu [2001] to incorporate stochastic recovery rates by the stochastic collateral value process. We evaluate the expected discounted loss and the m-th moment of the discounted loss as a general case. Solutions are represented by an integral of an exponential quadratic form of the state vector. We derive the ordinary differential equations satisfied by the coefficients of the form. In particular, we show closed form solutions for the coefficients with no correlation between a state variable of the discount interest rate and that of the default intensity.

The remainder of this paper is organized as follows. Section II describes our model. Section III derives solutions for the expected loss and the m-th moment of the loss for a collateralized loan. Section IV gives numerical examples of expected losses and the standard deviations of the losses. We analyze the effects of the correlation on the expected loss and the standard deviation of the loss distribution. Section V derives the condition whereby the correlation between default intensity and collateral value has the same sign as the correlation between the driving Brownian motions of the two state variables. Section VI presents our conclusions.

Appendix 1 gives a proof of the measure-changed Brownian motions used to evaluate the expected loss and the m-th moment of the discounted loss distribution. Appendix 2 demonstrates a simplified version of integration by partial integration.

II Our model

Suppose that a bank supplies a collateralized loan D with maturity T to a firm. The collateral value is denoted by A_t. Let default time τ be a nonnegative random variable defined on a probability space $(\Omega, \mathcal{F}, P)$.[1] The loss incurred by the bank at time τ is assumed[2] to be

$$L_\tau = D - \delta A_\tau, \tag{1}$$

where δ is a constant denoting the portion recovered of the collateral value.

The discount value of the loss depends on the discount interest rate, default intensity, and the collateral value, which to some extent are correlated. To represent the correlation, we assume a three-dimensional state vector $(y_t, z_t, \ln A_t)^\top$ with the correlated Gaussian Ornstein-Uhlenbeck process below.

$$d \begin{bmatrix} y_t \\ z_t \\ \ln A_t \end{bmatrix} = \begin{bmatrix} -\kappa_y y_t \\ -\kappa_z z_t \\ \mu_A - \sigma_A^2/2 \end{bmatrix} dt + S d \begin{bmatrix} W_{1,t} \\ W_{2,t} \\ W_{3,t} \end{bmatrix}, \tag{2}$$

[1]Here, we assume a risk-neutral probability P to simplify the evaluation.

[2]Similar to Yamashita and Yoshiba [2010], this assumption implies that the recovery rate may exceed 100% when the collateral value is greater than the loan amount. In fact, market recovery rates sometimes exceed 100%. If we set appropriate parameters, including δ, the recovery rate will rarely exceed 100%.

where $\Sigma = SS^\top$ is given as a constant variance-covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_y^2 & \rho_{yz}\sigma_y\sigma_z & \rho_{yA}\sigma_y\sigma_A \\ \rho_{yz}\sigma_y\sigma_z & \sigma_z^2 & \rho_{zA}\sigma_z\sigma_A \\ \rho_{yA}\sigma_y\sigma_A & \rho_{zA}\sigma_z\sigma_A & \sigma_A^2 \end{bmatrix}. \tag{3}$$

To keep discount interest rate r_t and default intensity λ_t nonnegative, we assume that these variables are represented as quadratic forms of the state variables

$$r_t = (y_t + \alpha_r + \beta_r t)^2, \tag{4}$$

and

$$\lambda_t = (z_t + \alpha_\lambda + \beta_\lambda t)^2. \tag{5}$$

To evaluate the m-th moment of the loss (1), we define filtrations. Let $(\mathcal{H}_t)_{t\geq 0}$ be a filtration generated by $\mathcal{H}_t = \sigma(1_{\{\tau\leq t\}})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be auxiliary filtration $\mathcal{F}_t = \sigma(\{W_s^A, W_s^y, W_s^z : s \leq t\})$ generated by the Brownian motions in equation (2). We also define an augmented filtration $(\mathcal{G}_t)_{t\geq 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. The default time τ is assumed to be a doubly stochastic random variable with respect to the filtration $\mathcal{F}_t$, and the default time is assumed to have a hazard rate process defined by equation (5).[3]

Now we evaluate the expected discounted loss of a collateralized loan. Let $E_t[\cdot]$ be an expectation given filtration $\mathcal{F}_t$:

$$E_t[\cdot] = E[\cdot|\mathcal{F}_t]. \tag{6}$$

If we assume $\int_t^T |L_s\lambda_s| e^{-\int_t^s (r_u+\lambda_u)du} ds$ is integrable, the expected loss for the bank is

$$\begin{aligned} &E[e^{-\int_t^\tau r_u du} L_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\ &= DE[e^{-\int_t^\tau r_u du} 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] - \delta E[e^{-\int_t^\tau r_u du} A_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\ &= 1_{\{t<\tau\}} D \int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s]ds - 1_{\{t<\tau\}}\delta \int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s A_s]ds. \end{aligned} \tag{7}$$

In this setting, the evaluation of the expected loss for the bank is decomposed by that of the discounted default probability in the first term of the right-hand side of equation (7) and that of the expected recovered collateral value in the second term of the right-hand side of the same equation.

III Solution for the expected loss and m-th moment of the loss distribution

In this section, we evaluate components in the expected loss (7) and derive an analytical solution for the expected loss. We also derive an analytical solution for higher moments of the loss.

[3]See McNeil, Frey, and Embrechts [2005] for the technical conditions for doubly stochastic random variables.

This section is organized as follows. Proposition 1 gives the first term of the expected loss (7), which indicates the discounted default probability. Proposition 2 gives the term in the special case in which no correlation exists between the discount interest rate and default intensity. In this case, the first term of the expected loss is given as a time integral of multiples of a discount rate and a time differential of a survival probability. These components of the multiple have closed form solutions. Proposition 3 gives the second term of the expected loss (7), corresponding to the expected recovered collateral value. Proposition 4 gives the term in the special case in which no correlation exists between discount interest rate and default intensity. In this case, the second term of the expected loss is given as a time integral of multiples of a measure-changed discount rate and a time differential of a measure-changed survival probability. These components of the multiple have closed form solutions. Theorem 1 gives the analytical form of the expected loss by applying Proposition 1 and Proposition 3 to equation (7) and by applying Proposition 2 and Proposition 4 to equation (7). Corollary 1 of the theorem gives the higher moments of the loss distribution.

Proposition 1. The discounted default probability is given by

$$\int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s]ds = -\lim_{w\to 0}\int_t^T \frac{d\zeta(t,s|X_t,\boldsymbol{\alpha},1,w)}{dw}ds, \tag{8}$$

where $\zeta(t,s|X_t,\boldsymbol{\alpha},m,w)$ is an evaluation of the expectation

$$\zeta(t,s|X_t,\boldsymbol{\alpha},m,w) = E_t[\exp\left(-\int_t^s (mr_u+\lambda_u)du\right)e^{w\lambda_s}], \tag{9}$$

with the two-dimensional state vector $X_t = (y_t, z_t)^\top$ and the two-dimensional parameter vector $\boldsymbol{\alpha} = (\alpha_r, \alpha_\lambda)^\top$. The expectation $\zeta(t,s|X_t,\boldsymbol{\alpha},m,w)$ is given by an exponential quadratic form of the state vector

$$\zeta(t,s|X_t,\boldsymbol{\alpha},m,w) = \exp(H_0(t,s) - H_1(t,s)\cdot X_t - X_t^\top H_2(t,s)X_t). \tag{10}$$

The coefficients $H_2(t,s)$, $H_1(t,s)$ and $H_0(t,s)$ in equation (10) are given by the solution for the ordinary differential equations below,[4]

$$\frac{dH_2(t,s)}{dt} = -\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} + (H_2(t,s)^\top + H_2(t,s))\begin{bmatrix} \kappa_y & 0 \\ 0 & \kappa_z \end{bmatrix} + 2H_2(t,s)^\top \Sigma H_2(t,s), \tag{11}$$

$$\frac{dH_1(t,s)}{dt} = -2\begin{bmatrix} m(\alpha_r+\beta_r t) \\ \alpha_\lambda + \beta_\lambda t \end{bmatrix} + \begin{bmatrix} \kappa_y & 0 \\ 0 & \kappa_z \end{bmatrix} H_1(t,s) + 2H_2(t,s)\Sigma H_1(t,s), \tag{12}$$

$$\frac{dH_0(t,s)}{dt} = m(\alpha_r+\beta_r t)^2 + (\alpha_\lambda+\beta_\lambda t)^2 + \mathrm{tr}[\Sigma H_2(t,s)] - \frac{1}{2}H_1(t,s)^\top \Sigma H_1(t,s), \tag{13}$$

[4]The coefficients $H_2(t,s)$, $H_1(t,s)$ and $H_0(t,s)$ depend on $\boldsymbol{\alpha}$, m and w but do not depend on X_t.

with boundary conditions

$$H_2(s,s) = -\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}, \; H_1(s,s) = -2w\begin{bmatrix} 0 \\ \alpha_\lambda + \beta_\lambda s \end{bmatrix}, \; H_0(s,s) = w(\alpha_\lambda + \beta_\lambda s)^2. \quad (14)$$

Proof. The integrand of the discounted default probability is transformed as follows:

$$E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s] = \lim_{w\to 0} E_t[e^{-\int_t^s (r_u+\lambda_u)du}e^{w\lambda_s}\lambda_s] \;= -\lim_{w\to 0}\frac{d\zeta(t,s|X_t,\boldsymbol{\alpha},1,w)}{dw}, \quad (15)$$

Here, $\zeta(t,s|X_t,\boldsymbol{\alpha},m,w)$ on the right-hand side of equation (15) is the expectation having the form of equation (9). This leads to equation (8). Since the expectation (9) is the expectation for the two-dimensional quadratic Gaussian process, we can evaluate this as the exponential quadratic form of equation (10). On the other hand, let

$$M_t = E_t[e^{-\int_0^s (mr_u+\lambda_u)du}e^{w\lambda_s}] = e^{-\int_0^t (mr_u+\lambda_u)du}\zeta(t,s|X_t,\boldsymbol{\alpha},m,w). \quad (16)$$

Then, M_t is a martingale, and the drift of dM_t is zero. By Ito's formula, this condition is reduced to the following partial differential equation:

$$\begin{aligned} 0 = &-(mr_t+\lambda_t)\zeta(t,s) + \frac{\partial\zeta(t,s)}{\partial t} - \kappa_y y_t\frac{\partial\zeta(t,s)}{\partial y_t} - \kappa_z z_t\frac{\partial\zeta(t,s)}{\partial z_t} \\ &+ \frac{\sigma_y^2}{2}\frac{\partial^2\zeta(t,s)}{\partial y_t^2} + \frac{\sigma_z^2}{2}\frac{\partial^2\zeta(t,s)}{\partial z_t^2} + \rho_{yz}\sigma_y\sigma_z\frac{\partial^2\zeta(t,s)}{\partial y_t\partial z_t}. \end{aligned} \quad (17)$$

Substituting equation (10) into equation (17) and collecting the terms with y_t^2, z_t^2, and $y_t z_t$ yields the ordinary differential equation (11). Similarly, collecting the first-order terms for y_t and z_t yields the ordinary differential equation (12). Collecting the constant terms yields the ordinary differential equation (13). Since

$$\zeta(s,s|X_s,\boldsymbol{\alpha},m,w) = \exp(w\lambda_s) = \exp(w(z_s+\alpha_\lambda+\beta_\lambda s)^2), \quad (18)$$

the boundary conditions are given by the equations (14). □

Proposition 2. If $\rho_{yz} = 0$, then the discounted default probability is given by

$$\int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s]ds = -\int_t^T \Gamma(t,s|y_t,\alpha_r,\beta_r,1)d_s\Gamma(t,s|z_t,\alpha_\lambda,\beta_\lambda,1), \quad (19)$$

where

$$\begin{aligned} &\Gamma(t,s|x_t,\alpha,\beta,m) \\ &= \exp\{C_0(t,s|\alpha,\beta,\kappa_x,\gamma_x,m) - C_1(t,s|\alpha,\beta,\kappa_x,\gamma_x,m)x_t - C_2(t,s|\kappa_x,\gamma_x,m)x_t^2\}, \end{aligned} \quad (20)$$

with

$$\gamma_y = \sqrt{\kappa_y^2 + 2m\sigma_y^2}, \quad \gamma_z = \sqrt{\kappa_z^2 + 2m\sigma_z^2}. \quad (21)$$

The coefficients on the right-hand side of equation (20) have closed form solutions.

$$C_2(t,s|\kappa,\gamma,m) = \frac{m(e^{2\gamma(s-t)}-1)}{(\gamma+\kappa)e^{2\gamma(s-t)}+\gamma-\kappa}, \tag{22}$$

$$\begin{aligned} C_1(t,s|\alpha,\beta,\kappa,\gamma,m) &= \frac{2m\alpha\{(\gamma+\kappa)e^{\gamma(s-t)}-(\gamma-\kappa)e^{-\gamma(s-t)}-2\kappa\}}{\gamma\{(\gamma+\kappa)e^{\gamma(s-t)}+(\gamma-\kappa)e^{-\gamma(s-t)}\}} \\ &+ \frac{2m\beta[(\gamma+\kappa)(1+\gamma t)e^{\gamma(s-t)}+(\gamma-\kappa)(1-\gamma t)e^{-\gamma(s-t)}-2\gamma\{1+\kappa s\}]}{\gamma^2\{(\gamma+\kappa)e^{\gamma(s-t)}+(\gamma-\kappa)e^{-\gamma(s-t)}\}}, \end{aligned} \tag{23}$$

$$\begin{aligned} C_0(t,s|\alpha,\beta,\kappa,\gamma,m) &= -m\alpha^2(s-t) - m\alpha\beta(s^2-t^2) - \frac{m\beta^2(s^3-t^3)}{3} + \frac{(\gamma+\kappa)(s-t)}{2} \\ &- \frac{1}{2}\ln\frac{(\gamma+\kappa)e^{2\gamma(s-t)}+\gamma-\kappa}{2\gamma} - \frac{m(\gamma^2-\kappa^2)G(t,s|\alpha,\beta,\kappa,\gamma)}{2\gamma^5\{(\gamma+\kappa)e^{\gamma(s-t)}+(\gamma-\kappa)e^{-\gamma(s-t)}\}}. \end{aligned} \tag{24}$$

The function $G(t,s|\alpha,\beta,\kappa,\gamma)$ in equation (24) is given by

$$\begin{aligned} G(t,s|\alpha,\beta,\kappa,\gamma) &= (\gamma-\kappa)\{\alpha^2\gamma^2 G_{1a}(t,s|\gamma) + 2\alpha\beta\gamma G_{2a}(t,s|\gamma) + \beta^2 G_{3a}(t,s|\gamma)\} \\ &+ (\gamma+\kappa)\{\alpha^2\gamma^2 G_{1b}(t,s|\gamma) + 2\alpha\beta\gamma G_{2b}(t,s|\gamma) + \beta^2 G_{3b}(t,s|\gamma)\}, \end{aligned} \tag{25}$$

where

$$G_{1a}(t,s|\gamma) = -e^{\gamma(s-t)} + 4 - e^{-\gamma(s-t)}(3+2\gamma(s-t)), \tag{26}$$

$$G_{1b}(t,s|\gamma) = e^{-\gamma(s-t)} - 4 + e^{\gamma(s-t)}(3-2\gamma(s-t)), \tag{27}$$

$$G_{2a}(t,s|\gamma) = e^{\gamma(s-t)}(1-\gamma s) - 2(1-\gamma(t+s)) + e^{-\gamma(s-t)}(1-\gamma(2t+s)+\gamma^2(t^2-s^2)), \tag{28}$$

$$G_{2b}(t,s|\gamma) = e^{-\gamma(s-t)}(1+\gamma s) - 2(1+\gamma(t+s)) + e^{\gamma(s-t)}(1+\gamma(2t+s)+\gamma^2(t^2-s^2)), \tag{29}$$

$$\begin{aligned} G_{3a}(t,s|\gamma) &= -4\gamma t(1-\gamma s) - e^{\gamma(s-t)}(1-\gamma s)^2 \\ &+ e^{-\gamma(s-t)}(1+2\gamma t-\gamma^2(2t^2+s^2)+\frac{2}{3}\gamma^3(t^3-s^3)), \end{aligned} \tag{30}$$

$$\begin{aligned} G_{3b}(t,s|\gamma) &= -4\gamma t(1+\gamma s) + e^{-\gamma(s-t)}(1+\gamma s)^2 \\ &+ e^{\gamma(s-t)}(-1+2\gamma t+\gamma^2(2t^2+s^2)+\frac{2}{3}\gamma^3(t^3-s^3)). \end{aligned} \tag{31}$$

Proof. If $\rho_{yz} = 0$, then

$$\begin{aligned} & E_t[\exp\left(-\int_t^s r_u du\right)\exp\left(-\int_t^s \lambda_u du\right)\lambda_s] \\ &= E_t[\exp\left(-\int_t^s r_u du\right)]E_t[\exp\left(-\int_t^s \lambda_u du\right)\lambda_s] \\ &= -E_t[\exp\left(-\int_t^s r_u du\right)]\frac{dE_t[\exp\left(-\int_t^s \lambda_u du\right)]}{ds}. \end{aligned} \tag{32}$$

Kijima, Tanaka, and Wong [2009] give the closed form solution for the discount bond price $E_t[e^{-\int_t^s r_u du}]$ by calculating the integrals shown in Pelsser [1997]. The closed form solution is given by equation (20) and equations (22)–(31). Similarly, survival probability $E_t[e^{-\int_t^s \lambda_u du}]$ is given by $\Gamma(t,s|z_t,\alpha_\lambda,\beta_\lambda,1)$, also a closed form solution. This leads to equation (19). □

Proposition 3. The expected recovered collateral value is calculated by

$$\int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s A_s]ds = -A_t \lim_{w\to 0}\int_t^T e^{\mu_A(s-t)}\frac{d\zeta(t,s|\tilde{X}_t,\tilde{\boldsymbol{\alpha}},1,w)}{dw}ds, \tag{33}$$

where $\zeta(t,s|\tilde{X}_t,\tilde{\boldsymbol{\alpha}},m,w)$ is the measure-changed expectation of $\zeta(t,s|X_t,\boldsymbol{\alpha},m,w)$ with the Radon-Nikodym density process

$$\eta(t;A) = A_t e^{-\mu_A t}/A_0. \tag{34}$$

The measure-changed expectation $\zeta(t,s|\tilde{X}_t,\tilde{\boldsymbol{\alpha}},m,w)$ can be calculated as the expectation $\zeta(t,s|X_t,\boldsymbol{\alpha},m,w)$, given that we substitute the following $\tilde{X}_t$ and $\tilde{\boldsymbol{\alpha}}$, respectively, for X_t and $\boldsymbol{\alpha}$:

$$\tilde{X}_t = \begin{pmatrix}\tilde{y}_t\\ \tilde{z}_t\end{pmatrix} = \begin{pmatrix} y_t - \rho_{yA}\sigma_A\sigma_y/\kappa_y \\ z_t - \rho_{zA}\sigma_A\sigma_z/\kappa_z\end{pmatrix},\ \tilde{\boldsymbol{\alpha}} = \begin{pmatrix}\tilde{\alpha}_r\\ \tilde{\alpha}_\lambda\end{pmatrix} = \begin{pmatrix}\alpha_r + \rho_{yA}\sigma_A\sigma_y/\kappa_y\\ \alpha_\lambda + \rho_{zA}\sigma_A\sigma_z/\kappa_z\end{pmatrix}. \tag{35}$$

Proof. Under the measure-changed probability $\tilde{P}$ with the Radon-Nikodym density process $\eta(t;A)$,

$$\tilde{W}_t^y = W_t^y - \rho_{yA}\sigma_A t,\ \tilde{W}_t^z = W_t^z - \rho_{zA}\sigma_A t,\ \tilde{W}_t^A = W_t^A - \sigma_A t, \tag{36}$$

are Brownian motions,[5] and the state variables $\tilde{y}_t$ and $\tilde{z}_t$ follow the processes

$$d\tilde{y}_t = dy_t = -\kappa_y y_t dt + \rho_{yA}\sigma_A\sigma_y dt + \sigma_y d\tilde{W}_t^y = -\kappa_y\tilde{y}_t dt + \sigma_y d\tilde{W}_t^y, \tag{37}$$

$$d\tilde{z}_t = -\kappa_z\tilde{z}_t dt + \sigma_z d\tilde{W}_t^z. \tag{38}$$

The integrand of the expected recovery value is given as

$$\begin{aligned} E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s A_s] &= E_t[\frac{\eta(s;A)}{\eta(t;A)}A_t e^{\mu_A(s-t)}e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s] \\ &= A_t e^{\mu_A(s-t)}\tilde{E}_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s], \end{aligned} \tag{39}$$

where $\tilde{E}_t[\cdot]$ is the expectation under the changed probability $\tilde{P}$. Here, $\tilde{E}_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s]$ is the integrand of the measure-changed discounted default probability with state variables $\tilde{y}_t$ and $\tilde{z}_t$. Discount interest rate r_t and default intensity λ_t are represented using $\tilde{y}_t$ and $\tilde{z}_t$, respectively:

$$r_t = (y_t+\alpha_r+\beta_r t)^2 = (\tilde{y}_t+\tilde{\alpha}_r+\beta_r t)^2,\ \lambda_t = (z_t+\alpha_\lambda+\beta_\lambda t)^2 = (\tilde{z}_t+\tilde{\alpha}_\lambda+\beta_\lambda t)^2. \tag{40}$$

Thus, the integrand of the measure-changed discounted default probability is given by

$$\tilde{E}_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s] = -\lim_{w\to 0}\frac{d\zeta(t,s|\tilde{X}_t,\tilde{\boldsymbol{\alpha}},1,w)}{dw}. \tag{41}$$

Equation (39) and equation (41) lead to this proposition. □

[5]See Appendix 1 for the proof. For measure-changed Brownian motion with Radon-Nikodym density process and Girsanov theorem, see (for example) Shreve [2004].

Proposition 4. If $\rho_{yz} = 0$, then the expected recovered collateral value is calculated by

$$\int_t^T E_t[e^{-\int_t^s (r_u+\lambda_u)du}\lambda_s A_s]ds = -A_t \int_t^T e^{\mu_A(s-t)}\Gamma(t,s|\tilde{y}_t,\tilde{\alpha}_r,\beta_r,1)d_s\Gamma(t,s|\tilde{z}_t,\tilde{\alpha}_\lambda,\beta_\lambda,1), \tag{42}$$

where $\Gamma(t,s|\tilde{y}_t,\tilde{\alpha}_r,\beta_r,1)$ and $\Gamma(t,s|\tilde{z}_t,\tilde{\alpha}_\lambda,\beta_\lambda,1)$ are given by the closed form solution as equation (20).

Proof. The proof is similar to the proof of Proposition 2. □

Proposition 3 and 4 show that the expected recovery value is given by time integration of the measure-changed discounted mean collateral return to maturity, with measure-changed survival probability as the integration measure.[6]

Theorem 1. The expected loss of the collateralized loan observed at time t is given by

$$\begin{aligned} E[e^{-\int_t^\tau r_u du}L_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] =& -1_{\{t<\tau\}}D\lim_{w\to 0}\int_t^T \frac{d\zeta(t,s|X_t,\boldsymbol{\alpha},1,w)}{dw}ds \\ &+ 1_{\{t<\tau\}}\delta A_t \lim_{w\to 0}\int_t^T e^{\mu_A(s-t)}\frac{d\zeta(t,s|\tilde{X}_t,\tilde{\boldsymbol{\alpha}},1,w)}{dw}ds. \end{aligned} \tag{43}$$

If $\rho_{yz} = 0$, the equation (43) is given by

$$\begin{aligned} E[e^{-\int_t^\tau r_u du}L_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] &= -1_{\{t<\tau\}}D\int_t^T \Gamma(t,s|y_t,\alpha_r,\beta_r,1)d_s\Gamma(t,s|z_t,\alpha_\lambda,\beta_\lambda,1) \\ + 1_{\{t<\tau\}}\delta A_t &\int_t^T e^{\mu_A(s-t)}\Gamma(t,s|\tilde{y}_t,\tilde{\alpha}_r,\beta_r,1)d_s\Gamma(t,s|\tilde{z}_t,\tilde{\alpha}_\lambda,\beta_\lambda,1), \end{aligned} \tag{44}$$

where four elements of $\Gamma(t,s|x_t,\alpha,\beta,1)$ are given by the closed form solution as equation (20).

Proof. We prove this theorem by applying Proposition 1 and Proposition 3 to equation (7) and applying Proposition 2 and Proposition 4 to equation (7). □

The higher moment of the loss distribution is also calculated by a combination of the measure-changed integral with another Radon-Nikodym density process as the following corollary for Theorem 1.

[6]If the collateral is liquid and yields no dividends, the instantaneous return of the collateral is equivalent to the discounted interest rate in risk-neutral probability. In that case, equation (42) is equivalent to $A_t\{1-\Gamma(t,T|\tilde{z}_t,\tilde{\alpha}_\lambda,\beta_\lambda,1)\}$.

Corollary 1. The m-th moment loss distribution of the collateralized loan observed at time t is expanded as follows:

$$E[(e^{-\int_t^\tau r_u du}L_\tau 1_{\{t<\tau\leq T\}})^m|\mathcal{G}_t] = \sum_{n=0}^{m} {}_mC_n D^{m-n}(-\delta)^n E[e^{-m\int_t^\tau r_u du}A_\tau^n 1_{\{t<\tau\leq T\}}|\mathcal{G}_t]. \quad (45)$$

The expectation of the right-hand side of equation (45) is given by

$$\begin{aligned} &E[e^{-m\int_t^\tau r_u du}A_\tau^n 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\ &= -1_{\{t<\tau\}}A_t^n \lim_{w\to 0}\int_t^T e^{n\{\mu_A+(n-1)\sigma_A^2/2\}(s-t)}\frac{d\zeta(t,s|X_t^{(n)},\boldsymbol{\alpha}^{(n)},m,w)}{dw}ds, \end{aligned} \quad (46)$$

where $X_t^{(n)}$ and $\boldsymbol{\alpha}^{(n)}$ are defined by

$$X_t^{(n)} = \begin{pmatrix} y_t^{(n)} \\ z_t^{(n)} \end{pmatrix} = \begin{pmatrix} y_t - n\rho_{yA}\sigma_A\sigma_y/\kappa_y \\ z_t - n\rho_{zA}\sigma_A\sigma_z/\kappa_z \end{pmatrix}, \ \boldsymbol{\alpha}^{(n)} = \begin{pmatrix} \alpha_r^{(n)} \\ \alpha_\lambda^{(n)} \end{pmatrix} = \begin{pmatrix} \alpha_r + n\rho_{yA}\sigma_A\sigma_y/\kappa_y \\ \alpha_\lambda + n\rho_{zA}\sigma_A\sigma_z/\kappa_z \end{pmatrix}. \quad (47)$$

If $\rho_{yz}=0$, then the derivative of $\zeta(\cdot)$ with respect to w in equation (46) is given by

$$\lim_{w\to 0}\frac{d\zeta(t,s|X_t^{(n)},\boldsymbol{\alpha}^{(n)},m,w)}{dw} = \Gamma(t,s|y_t^{(n)},\alpha_r^{(n)},\beta_r,m)\frac{d\Gamma(t,s|z_t^{(n)},\alpha_\lambda^{(n)},\beta_\lambda,1)}{ds}. \quad (48)$$

This gives a closed form solution for the integrand on the right-hand side of equation (46).

Proof. Equation (45) is given by the binomial expansion for the loss. The expectation on the left-hand side of equation (46) is transformed as follows:

$$E[e^{-m\int_t^\tau r_u du}A_\tau^n 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] = 1_{\{t<\tau\}}\int_t^T E_t[e^{-\int_t^s(mr_u+\lambda_u)du}\lambda_s A_s^n]ds. \quad (49)$$

We recognize the integrand on the right-hand side of equation (49). From equation (2), A_t^n has the diffusion process

$$dA_t^n = \mu_A^{(n)}A_t^n dt + \sigma_A A_t^n dW_t^A, \quad (50)$$

where

$$\mu_A^{(n)} = n\mu_A + \frac{n(n-1)\sigma_A^2}{2}. \quad (51)$$

Let $P^{(n)}$ be a changed probability measure with Radon-Nikodym density process

$$\left.\frac{dP^{(n)}}{dP}\right|_{G_t} = \eta(t;A^n) = \frac{A_t^n e^{-\mu_A^{(n)}t}}{A_0^n}. \quad (52)$$

The integrand of the right-hand side in equation (49) is calculated as follows:

$$\begin{aligned} E_t[e^{-\int_t^s(mr_u+\lambda_u)du}\lambda_s A_s^n] &= E_t[\frac{\eta(s;A^n)}{\eta(t;A^n)}A_t^n e^{\mu_A^{(n)}(s-t)}e^{-\int_t^s(mr_u+\lambda_u)du}\lambda_s] \\ &= A_t^n e^{\mu_A^{(n)}(s-t)}E_t^{(n)}[e^{-\int_t^s(mr_u+\lambda_u)du}\lambda_s], \end{aligned} \quad (53)$$

where $E_t^{(n)}[\cdot]$ is the expectation with probability measure $P^{(n)}$ given filtration F_t. Here,

$$W_t^{A(n)} = W_t^A - n\sigma_A t,\ W_t^{y(n)} = W_t^y - n\rho_{yA}\sigma_A t,\ W_t^{z(n)} = W_t^z - n\rho_{zA}\sigma_A t, \tag{54}$$

are standard Brownian motions. (See Appendix 1 for the proof.) Because discount interest rate r_t and default intensity λ_t are represented by

$$r_t = (y_t + \alpha_r + \beta_r t)^2 = (y_t^{(n)} + \alpha_r^{(n)} + \beta_r t)^2, \tag{55}$$

$$\lambda_t = (z_t + \alpha_\lambda + \beta_\lambda t)^2 = (z_t^{(n)} + \alpha_\lambda^{(n)} + \beta_\lambda t)^2, \tag{56}$$

the expectation on the right-hand side of equation (53) is given by

$$E_t^{(n)}[e^{-\int_t^s (mr_u+\lambda_u)du}\lambda_s] = -\lim_{w\to 0} \frac{d\zeta(t,s|X_t^{(n)}, \boldsymbol{\alpha}^{(n)}, m, w)}{dw}. \tag{57}$$

Substituting equation (53) with equation (57) in equation (49) leads to equation (46). If $\rho_{yz} = 0$, equation (57) is transformed as follows:

$$E_t^{(n)}[e^{-\int_t^s (mr_u+\lambda_u)du}\lambda_s] = E_t^{(n)}[e^{-m\int_t^s r_u du}]E_t^{(n)}[e^{-\int_t^s \lambda_u du}\lambda_s]. \tag{58}$$

Substituting

$$E_t^{(n)}[e^{-m\int_t^s r_u du}] = \Gamma(t,s|y_t^{(n)}, \alpha_r^{(n)}, \beta_r, m), \tag{59}$$

$$E_t^{(n)}[e^{-\int_t^s \lambda_u du}\lambda_s] = \frac{d\Gamma(t,s|z_t^{(n)}, \alpha_\lambda^{(n)}, \beta_\lambda, 1)}{ds}, \tag{60}$$

into equation (58) yields equation (48). □

IV Numerical examples

In this section, we show numerical examples of the expected loss and the standard deviation of the loss distribution observed at time 0. In particular, we focus on the correlation between default intensity and recovery rate. For simplicity, we assume in this section that $\rho_{yz} = 0$.

Referring to Kijima, Tanaka, and Wong [2009], we assign the following values to the parameters in equations (2)–(4):

$$\kappa_y = 0.09,\ \sigma_y = 0.03,\ y_0 = -0.13,\ \alpha_r = 0.22,\ \beta_r = 0. \tag{61}$$

We assign the following values for the other parameters, referring to Yamashita and Yoshiba [2010]:

$$D = A_0 = 100,\ \delta = 0.7,\ T = 1,\ \ \mu_A = 1\%,\ \sigma_A = 10\%,\ \sigma_z = 10\%. \tag{62}$$

First, let β_λ be zero. Then, α_λ denotes the square root of the mean reversion level of the default intensity, and z_0 denotes the deviation from the mean reversion level. That is, $z_0 > 0$ means that the initial state is worse than the mean reversion state; $z_0 < 0$ means that the initial state is better than the mean reversion state. We consider two patterns, $(z_0, \alpha_\lambda) = (-0.03, 0.2)$ and $(z_0, \alpha_\lambda) = (0.03, 0.17)$, for the set of z_0 and α_λ.[7]

From Theorem 1, the expected loss at time 0 is given as follows:

$$\begin{aligned} E[e^{-\int_0^\tau r_u du} L_\tau 1_{\{\tau \le T\}}] &= -D \int_0^T \Gamma(0, s|y_0, \alpha_r, \beta_r, 1) d_s \Gamma(0, s|z_0, \alpha_\lambda, \beta_\lambda, 1) \\ + \delta A_0 &\int_0^T e^{\mu_A s} \Gamma(0, s|\tilde{y}_0, \tilde{\alpha}_r, \beta_r, 1) d_s \Gamma(0, s|\tilde{z}_0, \tilde{\alpha}_\lambda, \beta_\lambda, 1). \end{aligned} \tag{63}$$

Here, the two Stieltjes integrals on the right-hand side of equation (63) are rapidly calculated by partial integration, based on the assumption that $\beta_r = 0$ in equation (61) (see Appendix 2 for details).[8]

Figure 1 illustrates the expected loss (63) with respect to the correlation ρ_{zA} in four cases of κ_z: $\kappa_z = 0.1$, 1, 5, 10. Figure 1(a) is the case of $z_0 = -0.03$ (good state); Figure 1(b) is the case of $z_0 = 0.03$ (bad state). In all cases, the expected loss increases as correlation ρ_{zA} decreases. The increments of the expected loss are greater when the mean reversion speed κ_z is less. That is, the loss tends to increase if the mean reversion speed is slow. The expected loss tends to be greater with greater κ_z in the case of $z_0 < 0$, Figure 1(a). This is because the high mean reversion speed κ_z quickly converges to an undesirable mean reversion level. Conversely, the expected loss tends to be greater with lower κ_z in the case of $z_0 > 0$, Figure 1(b). This is because the low mean reversion speed κ_z slowly restores good mean reversion levels. Almost all our results are similar to those in Yamashita and Yoshiba [2010], which assumed a square-root process for the default intensity.

Next, we introduce the trend parameter β_λ for default intensity. Figure 2 illustrates the expected loss (63) for $\beta_\lambda = 0.01$. The other parameters are the same as those in Figure 1. We see that the shape of the expected loss does not differ significantly from that in Figure 1 and that expected loss increases by about 0.05.

[7] The parameter setting for $\beta = 0$ is almost equivalent to that in Yamashita and Yoshiba [2010] as follows. First, D, A_0, δ, T, and μ_A are the same as those in Yamashita and Yoshiba [2010]. By Ito's formula, σ_z corresponds to $\sigma_h/2$ in Yamashita and Yoshiba [2010]. The diffusion term of the log value of collateral σ_A corresponds to $\sigma_A\sqrt{h_t}$ in Yamashita and Yoshiba [2010]. In Yamashita and Yoshiba [2010], the initial value of default intensity h_0 and the mean-reversion level of default intensity $\bar{h}$ are selected from the values of 4% and 3%. We set $\sigma_A = 10\%$ as the level of $h_t = 4\%$. From equation (5), $z_0 + \alpha_\lambda$ corresponds to the square root of the initial default intensity h_0 and α_λ corresponds to the square root of the mean-reversion level of default intensity $\bar{h}$. The value is selected from $\sqrt{0.04} = 0.2$ and $\sqrt{0.03} \cong 0.17$. We specify the value of α_λ, considering $z_0 = -0.03$ for the good state case and $z_0 = 0.03$ for the bad state case.

[8] The integrals in the following numerical examples are calculated by adaptive quadrature (integrare() function in R).

Figure 1 Expected loss with respect to the correlation ρ_{zA} ($\beta_\lambda = 0$)

(a) $z_0 = -0.03$, $\alpha_\lambda = 0.2$ (b) $z_0 = 0.03$, $\alpha_\lambda = 0.17$

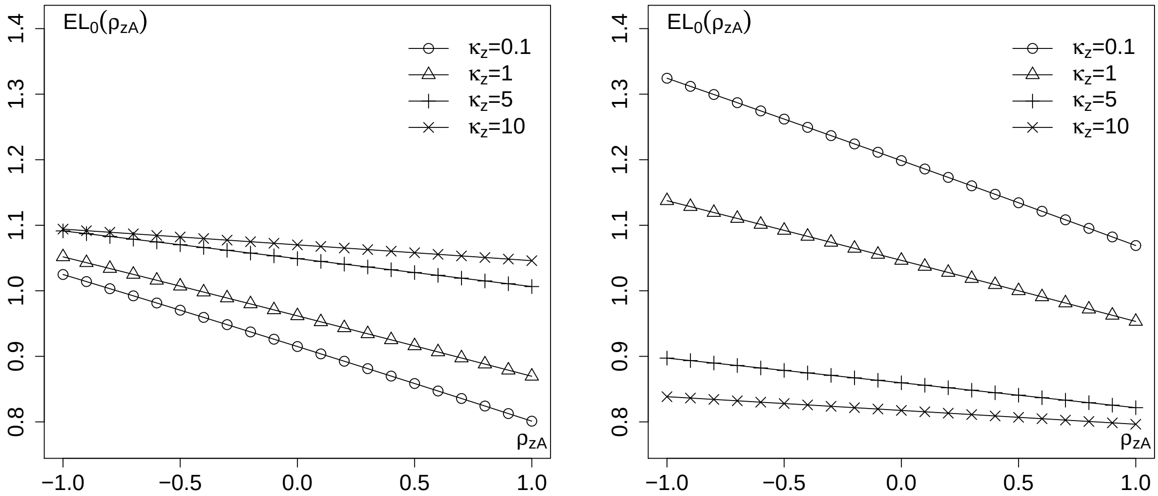
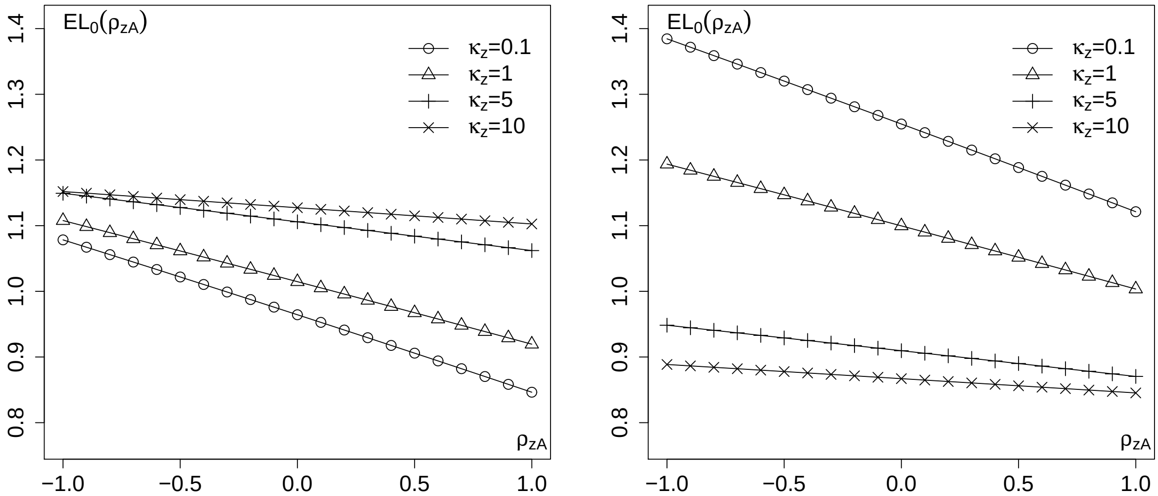


Figure 2 Expected loss with respect to the correlation ρ_{zA} ($\beta_\lambda = 0.01$)

(a) $z_0 = -0.03$, $\alpha_\lambda = 0.2$ (b) $z_0 = 0.03$, $\alpha_\lambda = 0.17$



The standard deviation of the loss distribution at time 0 is given by $\sqrt{\mathrm{var}[e^{-\int_0^\tau r_u du} L_\tau 1_{\{\tau \le T\}}]}$, where

$$\mathrm{var}[e^{-\int_0^\tau r_u du} L_\tau 1_{\{\tau \le T\}}] = E[e^{-2\int_0^\tau r_u du} L_\tau^2 1_{\{\tau \le T\}}] - (E[e^{-\int_0^\tau r_u du} L_\tau 1_{\{\tau \le T\}}])^2. \tag{64}$$

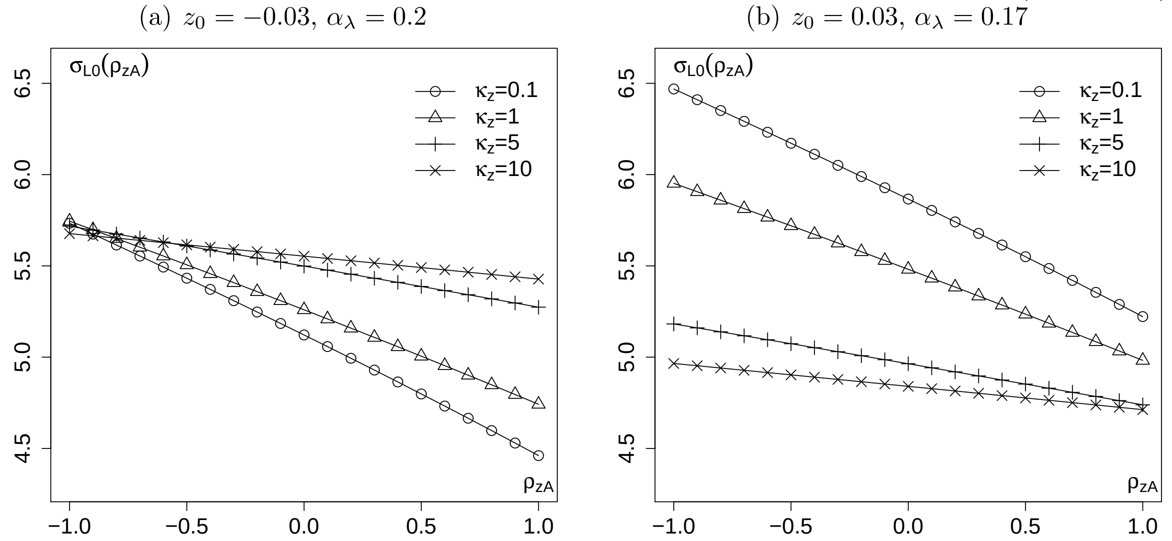
From Corollary 1, the first term on the right-hand side of equation (64) with $\beta_r = 0$ is calculated as follows:

$$\begin{aligned} E[e^{-2\int_0^\tau r_u du} L_\tau^2 1_{\{\tau \le T\}}] &= -D^2 \int_0^T \Gamma(0, s|y_0, \alpha_r, 0, 2) d_s \Gamma(0, s|z_0, \alpha_\lambda, \beta_\lambda, 1) \\ &+ 2\delta A_0 D \int_0^T e^{\mu_A s} \Gamma(0, s|\tilde{y}_0, \tilde{\alpha}_r, 0, 2) d_s \Gamma(0, s|\tilde{z}_0, \tilde{\alpha}_\lambda, \beta_\lambda, 1) \\ &- \delta^2 A_0^2 \int_0^T e^{(2\mu_A + \sigma_A^2)s} \Gamma(0, s|y_0^{(2)}, \alpha_r^{(2)}, 0, 2) d_s \Gamma(0, s|z_0^{(2)}, \alpha_\lambda^{(2)}, \beta_\lambda, 1). \end{aligned} \tag{65}$$

Here, the three Stieltjes integrals on the right-hand side of equation (65) are rapidly calculated by partial integration based on the assumption that $\beta_r = 0$. (See Appendix 2 for details.) The second term on the right-hand side of equation (64) is obtained from equation (63).

Figure 3 illustrates the standard deviation of the loss with respect to the correlation ρ_{zA}. All parameters are the same as those in Figure 2. Figure 3 shows that lower correlation increases the standard deviation of the loss. Lower κ_z has greater impact.

Figure 3 Standard deviation of the loss with respect to the correlation ρ_{zA} ($\beta_\lambda = 0.01$)



Based on the numerical results in this section, we posit that risk managers must closely examine negative correlations in terms of both expected loss and the standard deviation of the loss when mean reversion speed κ_z is slow. This result is similar to the results derived by Yamashita and Yoshiba [2010].

V Correlation between default intensity and collateral value

The correlation ρ_{zA} here that we specify is the correlation between the driving Brownian motion of the state variable z_t and that of the log value of collateral $\ln A_t$. In this section, we evaluate the correlation between default intensity and collateral value, confirming that the correlation has the same sign as ρ_{zA} if we set appropriate parameters.

Proposition 5. Based on equations (2)–(5), the correlation between default intensity λ_T and the log value of the collateral $\ln A_T$ observed at time t is calculated as follows:

$$\mathrm{corr}_t(\lambda_T, \ln A_T) = \frac{2\rho_{zA}(1 - e^{-\kappa_z(T-t)})\{\alpha_\lambda + \beta_\lambda T + z_t e^{-\kappa_z(T-t)}\}}{\sqrt{2\kappa_z(T-t)(1 - e^{-2\kappa_z(T-t)})}|\alpha_\lambda + \beta_\lambda T + z_t e^{-\kappa_z(T-t)}|}. \tag{66}$$

Proof. The covariance between the state variable z_T and log value of the collateral $\ln A_T$ at time T observed at time t is given by

$$\mathrm{cov}_t(z_T, \ln A_T) = \sigma_y \sigma_A \int_t^T e^{-\kappa(T-s)} \rho_{zA} ds = \frac{\rho_{zA}\sigma_y\sigma_A}{\kappa_z}(1 - e^{-\kappa_z(T-t)}). \tag{67}$$

The covariance between z_T^2 and $\ln A_T$ is given by

$$\begin{aligned}\mathrm{cov}_t(z_T^2, \ln A_T) &= 2\rho_{zA}\sigma_z\sigma_A z_t e^{-\kappa_z(T-t)} \int_t^T e^{-\kappa_z(T-s)} ds \\ &= \frac{2\rho_{zA}\sigma_z\sigma_A z_t e^{-\kappa_z(T-t)}(1 - e^{-\kappa_z(T-t)})}{\kappa_z}.\end{aligned} \tag{68}$$

Thus, the covariance between the default intensity λ_T and the log value of collateral $\ln A_T$ is given by

$$\begin{aligned}\mathrm{cov}_t(\lambda_T, \ln A_T) &= \mathrm{cov}_t(z_T^2 + 2(\alpha_\lambda + \beta_\lambda T)z_T, \ln A_T) \\ &= \frac{2\rho_{zA}\sigma_z\sigma_A(1 - e^{-\kappa_z(T-t)})\{\alpha_\lambda + \beta_\lambda T + z_t e^{-\kappa_z(T-t)}\}}{\kappa_z}.\end{aligned} \tag{69}$$

The variance of default intensity λ_T at time T observed at time t is given as

$$\begin{aligned}\mathrm{var}_t[\lambda_T] &= \mathrm{cov}_t(z_T^2 + 2(\alpha_\lambda + \beta_\lambda T)z_T, z_T^2 + 2(\alpha_\lambda + \beta_\lambda T)z_T) \\ &= \mathrm{var}_t[z_T^2] + 4(\alpha_\lambda + \beta_\lambda T)^2 \mathrm{var}_t[z_T] + 4(\alpha_\lambda + \beta_\lambda T)\mathrm{cov}_t(z_T, z_T^2).\end{aligned} \tag{70}$$

Here, each term on the right-hand side of equation (70) is given as follows:

$$\mathrm{var}_t[z_T] = \sigma_z^2 \int_t^T e^{-2\kappa_z(T-s)} ds = \frac{\sigma_z^2(1 - e^{-2\kappa_z(T-t)})}{2\kappa_z}, \tag{71}$$

$$\mathrm{cov}_t(y_T, y_T^2) = 2z_t\sigma_z^2 \int_t^T e^{-2\kappa_z(T-s)} ds = \frac{\sigma_z^2(1 - e^{-2\kappa_z(T-t)})z_t e^{-\kappa_z(T-t)}}{\kappa_z}, \tag{72}$$

$$\mathrm{var}_t[y_T^2] = 4\{z_t e^{-\kappa_z(T-t)}\}^2 \sigma_z^2 \int_t^T e^{-2\kappa_z(T-s)} ds = \frac{2\sigma_z^2(1 - e^{-2\kappa_z(T-t)})z_t^2 e^{-2\kappa_z(T-t)}}{\kappa_z}. \tag{73}$$

Substituting equations (71), (72), and (73) into equation (70) yields:

$$\mathrm{var}_t[\lambda_T] = \frac{2\sigma_z^2(1-e^{-2\kappa_z(T-t)})\{\alpha_\lambda + \beta_\lambda T + z_t e^{-\kappa_z(T-t)}\}^2}{\kappa_z}. \tag{74}$$

The variance of the log value of collateral $\ln A_T$ is given by

$$\mathrm{var}_t[\ln A_T] = \sigma_A^2 \int_t^T ds = \sigma_A^2(T-t). \tag{75}$$

The correlation between default intensity λ_T and the log value of the collateral $\ln A_T$ observed at time t is calculated as follows:

$$\mathrm{corr}_t(\lambda_T, \ln A_T) = \frac{\mathrm{cov}_t(\lambda_T, \ln A_T)}{\sqrt{\mathrm{var}_t[\lambda_T]}\sqrt{\mathrm{var}_t[\ln A_T]}}. \tag{76}$$

Substituting equations (69), (74) and (75) into equation (76) yields this proposition. □

Corollary 2. Based on the assumption that $\kappa_z > 0$, the conditional correlation (66) has the same sign as ρ_{zA} if

$$\alpha_\lambda + \beta_\lambda T + z_t e^{-\kappa_z(T-t)} > 0. \tag{77}$$

At time $t = 0$, the condition (77) holds if we set β_λ to be nonnegative and if $\alpha_\lambda + z_0$ is positive. The negative correlation between default intensity and the collateral value is represented by setting $\rho_{zA} < 0$, $\beta_\lambda \geq 0$, and $\alpha_\lambda + z_0 > 0$.

The latent variable y_t follows an Ornstein-Uhlenbeck process in equation (2), and the variable may assume a negative value over time. When z_t takes a value such that

$$z_t < -(\alpha_\lambda + \beta_\lambda T)e^{\kappa_z(T-t)}, \tag{78}$$

the conditional correlation between default intensity and the log value of the collateral takes the sign opposite the sign of ρ_{zA}. The condition (78) will rarely hold if we set appropriate parameters (including κ_z and σ_z), since z_t converges to the zero mean reversion level.

In cases in which $\rho_{zA} < 0$ is a realistic setting, it is not necessarily unrealistic to assume that default intensity and the collateral value have a locally positive correlation satisfying condition (78). In fact, the default intensity becomes small just before $z_t e^{-\kappa_z(T-t)} + \alpha_\lambda + \beta_\lambda T$ becomes negative. A negative correlation between default intensity and collateral value may not be clear if the firm is in a good state, with small default intensity. This implies that our model considers locally positive correlations.

VI Conclusions

Our analysis evaluated the discounted loss distribution of a collateralized loan, focusing on the correlation between default intensity and collateral value. For the default intensity process and the discount interest rate, we assumed a quadratic Gaussian process to keep their values nonnegative. The correlations among default intensity, discount interest rate, and collateral value are represented by the correlations among the three state variables ρ_{yz}, ρ_{yA}, and ρ_{zA}.

The m-th moment of the discounted loss distribution is given by a time integral of an exponential quadratic form of the state variables. The coefficients of the form are generally given by the solutions of ordinary differential equations. The solutions can be calculated rapidly by numerical methods, such as the Runge-Kutta method. If we can assume $\rho_{yz} = 0$, the ordinary differential equations will have closed form solutions, and no numerical methods need to be applied.

Numerical examples for the expected loss and the standard deviation of the loss distribution show that a decrease in correlation ρ_{zA} yields increases in the expected loss and in the standard deviation of the loss. The impact is greater when κ_z is lower. Similar to Yamashita and Yoshiba [2010], we posit that risk managers must closely examine negative correlation ρ_{zA} in terms of both expected loss and the standard deviation of the loss when the mean reversion speed κ_z is low.

Our main contribution is to obtain an analytical formulation of the m-th moment of the loss distribution of a collateralized loan under correlated stochastic default intensity, collateral value, and discount interest rate. In certain cases, the formulation becomes a closed form solution, and the value can be rapidly calculated. In other cases, solutions can be obtained by numerical methods.

Appendix 1 Correlation of Brownian motions and measure change

In this appendix, we confirm equation (36), the transformation equation of the Brownian motion in the probability measure $\tilde{P}$. More generally, we confirm equation (54), the transformation equation of the Brownian motion in the probability measure $P^{(n)}$ by the following lemma:

Lemma 1. In the probability measure $P^{(n)}$ with Radon-Nikodym density process (52), $W_t^{A(n)}$, $W_t^{y(n)}$ and $W_t^{z(n)}$ defined in equation (54) are Brownian motions.

Proof. We must demonstrate the following three points to prove W_t is a Brownian motion:

(a) (continuity) W_t is continuous, and $W_0 = 0$.

(b) (stationary normality) For any $0 \leq t_0 < t_1 < \cdots < t_N$, each $W_{t_j} - W_{t_{j-1}}$ ($j = 1, \ldots, N$) has a normal distribution $N(0, t_j - t_{j-1})$ independent of the history until t_{j-1}.

(c) (independent increment) For any $0 \le t_0 < t_1 < \cdots < t_N$, $W_{t_j} - W_{t_{j-1}}$ $(j = 1, \ldots, N)$ are mutually independent.

Here, equation (54) clearly indicates the (a) continuity of $W_t^{A(n)}$, $W_t^{y(n)}$ and $W_t^{z(n)}$. From equation (52) and equation (2),

$$\frac{\eta(t_j; A^n)}{\eta(t_{j-1}; A^n)} = \frac{A_{t_j}^n e^{-\mu_A^{(n)} t_j}}{A_{t_{j-1}}^n e^{-\mu_A^{(n)} t_{j-1}}} = e^{-n^2\sigma_A^2(t_j - t_{j-1})/2 + n\sigma_A W_{t_j}^A - n\sigma_A W_{t_{j-1}}^A}. \tag{A-1}$$

we can demonstrate (b) the stationary normality of $W_t^{A(n)}$ by the equation

$$E_{t_{j-1}}^{(n)}[\exp(\mathrm{i}z(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] = \exp(-z^2(t_j - t_{j-1})/2). \tag{A-2}$$

for any $z \in R$. In fact, from equation (54) and (A-1),

$$\begin{aligned}
&E_{t_{j-1}}^{(n)}[\exp(\mathrm{i}z(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] = E_{t_{j-1}}[\frac{\eta(t_j; A^n)}{\eta(t_{j-1}; A^n)} e^{\mathrm{i}z(W_{t_j}^A - W_{t_{j-1}}^A - n\sigma_A(t_j - t_{j-1}))}] \\
&= e^{-n^2\sigma_A^2(t_j - t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A} E_{t_{j-1}}[e^{n\sigma_A W_{t_j}^A} e^{\mathrm{i}z(W_{t_j}^A - W_{t_{j-1}}^A - n\sigma_A(t_j - t_{j-1}))}] \\
&= e^{-n^2\sigma_A^2(t_j - t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A - \mathrm{i}zn\sigma_A(t_j - t_{j-1}) - \mathrm{i}zW_{t_{j-1}}^A} E_{t_{j-1}}[e^{(n\sigma_A + \mathrm{i}z)W_{t_j}^A}] \\
&= e^{-n^2\sigma_A^2(t_j - t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A - \mathrm{i}zn\sigma_A(t_j - t_{j-1}) - \mathrm{i}zW_{t_{j-1}}^A} \\
&\quad \times e^{(n\sigma_A + \mathrm{i}z)W_{t_{j-1}}^A + (n\sigma_A + \mathrm{i}z)^2(t_j - t_{j-1})/2} \\
&= \exp(-z^2(t_j - t_{j-1})/2).
\end{aligned} \tag{A-3}$$

we can demonstrate (c) the independent increment of $W_t^{A(n)}$ by the equation

$$E^{(n)}[\exp(\mathrm{i}\sum_{j=1}^N z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] = \prod_{j=1}^N E^{(n)}[\exp(\mathrm{i}z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))]. \tag{A-4}$$

From (A-2),

$$\begin{aligned}
&E^{(n)}[\exp(\mathrm{i}\sum_{j=1}^N z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] \\
&= E^{(n)}[E_{t_{N-1}}^{(n)}[\exp(\mathrm{i}z_N(W_{t_N}^{A(n)} - W_{t_{N-1}}^{A(n)}))] \exp(\mathrm{i}\sum_{j=1}^{N-1} z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] \\
&= e^{-z_N^2(t_N - t_{N-1})^2} E^{(n)}[\exp(\mathrm{i}\sum_{j=1}^{N-1} z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))] \\
&= \cdots = \prod_{j=1}^N e^{-z_j^2(t_j - t_{j-1})^2} = \prod_{j=1}^N E^{(n)}[\exp(\mathrm{i}z_j(W_{t_j}^{A(n)} - W_{t_{j-1}}^{A(n)}))].
\end{aligned} \tag{A-5}$$

Thus, $W_t^{A(n)}$ is a standard Brownian motion.

Similarly, for $W_t^{y(n)}$ and any $z \in \mathbb{R}$, from equation (54) and (A-1),

$$\begin{aligned}
&E_{t_{j-1}}^{(n)}[\exp(\mathrm{i}z(W_{t_j}^{y(n)} - W_{t_{j-1}}^{y(n)}))] = E_{t_{j-1}}[\frac{\eta(t_j;A^n)}{\eta(t_{j-1};A^n)} e^{\mathrm{i}z(W_{t_j}^y - W_{t_{j-1}}^y - n\rho_{yA}\sigma_A(t_j-t_{j-1}))}] \\
&= e^{-n^2\sigma_A^2(t_j-t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A} E_{t_{j-1}}[e^{n\sigma_A W_{t_j}^A} e^{\mathrm{i}z(W_{t_j}^y - W_{t_{j-1}}^y - n\rho_{yA}\sigma_A(t_j-t_{j-1}))}] \\
&= e^{-n^2\sigma_A^2(t_j-t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A - \mathrm{i}zn\rho_{yA}\sigma_A(t_j-t_{j-1}) - \mathrm{i}zW_{t_{j-1}}^y} E_{t_{j-1}}[e^{n\sigma_A W_{t_j}^A + \mathrm{i}zW_{t_j}^y}] \\
&= e^{-n^2\sigma_A^2(t_j-t_{j-1})/2 - n\sigma_A W_{t_{j-1}}^A - \mathrm{i}zn\rho_{yA}\sigma_A(t_j-t_{j-1}) - \mathrm{i}zW_{t_{j-1}}^A} \\
&\quad \times e^{(n\sigma_A W_{t_{j-1}}^A + \mathrm{i}zW_{t_{j-1}}^y) + \{(n\sigma_A + \mathrm{i}\rho_{yA}z)^2 - (1-\rho_{yA}^2)z^2\}(t_j-t_{j-1})/2} \\
&= \exp(-z^2(t_j - t_{j-1})/2).
\end{aligned} \tag{A-6}$$

This demonstrates (b) the stationary normality of $W_t^{y(n)}$. The stationary normality of $W_t^{z(n)}$ is also demonstrated in the same way. Similar to equation (A-5), we can demonstrate (c) the independent increment of $W_t^{y(n)}$ and $W_t^{z(n)}$. Thus, $W_t^{y(n)}$ and $W_t^{z(n)}$ are standard Brownian motions. □

Considering Lemma 1 in the case of $n = 1$, we can confirm equation (36) in the probability measure $\tilde{P}$.

Appendix 2 Partial integration with the assumption $\beta_r = 0$

This appendix shows that the integration of the left-hand side of equation (A-7) with the assumption of $\beta_r = 0$ is easily calculated.

By partial integration,

$$\begin{aligned}
&\int_0^T e^{n(\mu_A + (n-1)\sigma_A^2/2)s} \Gamma(0,s|y_t^{(n)}, \alpha_r^{(n)}, \beta_r, m) d_s \Gamma(0,s|z_t^{(n)}, \alpha_\lambda^{(n)}, \beta_\lambda, 1) \\
&= e^{n(\mu_A + (n-1)\sigma_A^2/2)T} \Gamma(0,T|y_t^{(n)}, \alpha_r^{(n)}, \beta_r, m) \Gamma(0,T|z_t^{(n)}, \alpha_\lambda^{(n)}, \beta_\lambda, 1) - 1 \\
&\quad - \int_t^T \left[\begin{array}{c} e^{n(\mu_A + (n-1)\sigma_A^2/2)s} \Gamma(0,s|y_t^{(n)}, \alpha_r^{(n)}, \beta_r, m) \Gamma(0,s|z_t^{(n)}, \alpha_\lambda^{(n)}, \beta_\lambda, 1) \\ \times \left\{ \begin{array}{l} n(\mu_A + (n-1)\sigma_A^2/2) + \dfrac{dC_0(0,s|\alpha_r^{(n)}, \beta_r, \kappa_y, \gamma_y, m)}{ds} \\ -y_0^{(n)} \dfrac{dC_1(0,s|\alpha_r^{(n)}, \beta_r, \kappa_y, \gamma_y, m)}{ds} - y_0^{(n)2} \dfrac{dC_2(0,s|\kappa_y, \gamma_y, m)}{ds} \end{array} \right\} \end{array} \right] ds.
\end{aligned} \tag{A-7}$$

Here, if $\beta_r = 0$, then $dC_0(0,s)/ds$ and $dC_1(0,s)/ds$ are simple closed form equations as equations (A-8) and (A-9). The numerical integration of the left-hand side of equation (A-7) with $\beta_r = 0$ is easily calculated.

$$\begin{aligned}
&\frac{dC_0(t,s|\alpha,0,\kappa,\gamma,m)}{ds} = -m\alpha^2 + \frac{(\gamma+\kappa)}{2} - \frac{\gamma(\gamma+\kappa)e^{\gamma(s-t)}}{(\gamma+\kappa)e^{\gamma(s-t)} + (\gamma-\kappa)e^{-\gamma(s-t)}} \\
&\quad + \frac{m\alpha^2(\gamma^2 - \kappa^2)}{\gamma^2} \left\{ \frac{(\gamma+\kappa)e^{\gamma(s-t)} - (\gamma-\kappa)e^{-\gamma(s-t)} - 2\kappa}{(\gamma+\kappa)e^{\gamma(s-t)} + (\gamma-\kappa)e^{-\gamma(s-t)}} \right\}^2.
\end{aligned} \tag{A-8}$$

$$\frac{dC_1(t,s|\alpha,0,\kappa,\gamma,m)}{ds} = 4m\alpha\frac{\kappa\{(\gamma+\kappa)e^{\gamma(s-t)}-(\gamma-\kappa)e^{-\gamma(s-t)}\}+2(\gamma+\kappa)(\gamma-\kappa)}{\{(\gamma+\kappa)e^{\gamma(s-t)}+(\gamma-\kappa)e^{-\gamma(s-t)}\}^2}. \tag{A-9}$$

$$\frac{dC_2(t,s|\kappa,\gamma,m)}{ds} = m\left\{\frac{2\gamma}{(\gamma+\kappa)e^{\gamma(s-t)}+(\gamma-\kappa)e^{-\gamma(s-t)}}\right\}^2. \tag{A-10}$$

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