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# Analytical Solution for Expected Loss of a Collateralized Loan: A Square-root Intensity Process Negatively Correlated with Collateral Value

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**Abstract**

In this study, we derive an explicit solution for the expected loss of a collateralized loan, focusing on the negative correlation between default intensity and collateral value. Three requirements for the default intensity and the collateral value are imposed. First, the default event can happen at any time until loan maturity according to an exogenous stochastic process of default intensity. Second, default intensity and collateral value are negatively correlated. Third, the default intensity and collateral value are non-negative. To develop an explicit solution, we propose a square-root process for default intensity and an affine diffusion process for collateral value. Given these settings, we derive an explicit solution for the integrand of the expected recovery value within an extended affine model. From the derived solution, we find the expected recovery value is given by a Stieltjes integral with a measure-changed survival probability.

**Keywords:** stochastic recovery; default intensity model; affine diffusion; extended affine; survival probability; measure change

**JEL classification:** G21, G32, G33

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# I Introduction

The advanced internal ratings approach proposed under Basel II requires internationally active banks to estimate a recovery rate that adequately reflects the downturn in economic conditions (see Basel Committee on Banking Supervision [2005]). The regulations are based on empirical studies of the negative correlation between default rates and recovery rates (see Altman *et al.* [2005]). In the recent financial turmoil, regulators have paid much attention to the negative correlation associated with the counter-cyclicality of default rates and pro-cyclicality of recovery rates. In this paper, we evaluate the expected loss of a collateralized loan in a closed form, focusing on this negative correlation.

There are two major approaches to modeling credit risk in mathematical finance. One is a structural approach in which the default probability is determined endogenously based on debt-to-asset ratios. The other is a default intensity approach in which the default event is assumed to arise stochastically according to exogenous stochastic intensity.

In the structural approach, the probability that a firm will default is defined by the probability that asset values are less than debt amounts at a given time. The asset value is often assumed to follow a geometric Brownian motion, while the debt amount is assumed to be a constant. In other words, the debt-to-asset ratio determines the default probability, and the debt amount is the default boundary. In evaluating the value of a collateralized loan, Pykhtin [2003] assumed the collateral value process correlates positively with the firm's asset value process. This assumption implies a negative correlation between default probability and recovery value. Although Pykhtin [2003] derives a solution for the expected loss of the collateralized loan, Pykhtin's one-period structural model is implausible because the default event is assumed to occur, if at all, only when the loan comes to maturity. One way to sidestep this implausible assumption is to introduce a first passage time model with a default boundary (Black and Cox [1976]). However, a simple first passage time model is still unable to explain short-term credit spreads. The model implies a near-zero spread for a firm with a low debt-to-asset ratio despite the wide credit spread in a market. Incorporating incomplete information for the asset value or the default boundary is one possible solution to the problem proposed by Duffie and Lando [2001]. Chen, Collin-Dufresne, and Goldstein [2009] identify a large discrepancy between observed credit spreads and estimates based on historical default and recovery rates, dubbing this the "credit spread puzzle," and suggest the pro-cyclicality of recovery rates as one of the factors leading to the puzzle.

In the default intensity approach (see Duffie [2005] for examples), the likelihood of the default event is assumed to be linked to exogenous default intensity. This implies that the default event can happen at any point in time until maturity, a more realistic approach than that of the one-period structural model. However, in default intensity models, recovery rates are often unrealistically assumed to be constant. Some models deal with the stochastic recovery rate by the default intensity approach. For example, Bakshi, Madan, and Zhang [2006] discuss a general framework of stochastic recovery and show a class of closed-form pricing model for defaultable debt in which the recovery rate is given by a deterministic function of the default intensity. Guo, Jarrow, and Zeng [2009] introduce a double stopping time to describe post-default recovery processes. Chen and Joslin [2009] develop a generalized transform model for affine processes and introduce recovery risk as an application of their model. They derive a closed-form pricing model for defaultable debt focusing on correlations among risk-free interest rates, default intensity, and recovery rate. In their model, the negative correlation between short-term market interest rates and default intensity are expressed by a two-dimensional state vector, one of whose elements is default intensity. The recovery rate is assumed to be a deterministic function of default intensity as described by Bakshi, Madan, and Zhang [2006].

In contrast to Bakshi, Madan, and Zhang [2006] and Chen and Joslin [2009], Kijima and Miyake [2004] derive a closed-form pricing model for loans collateralized with real property focusing on the negative correlation between default intensity and collateral value. In their model, short-term market interest rates, default intensity, and the logarithmic value of the collateral follows a correlated Ornstein-Uhlenbeck process. This assumption poses the mathematical problem that the intensity which should be non-negative may become negative.

In this paper, we adopt a default intensity model for collateralized loans. To make a solution to the negative intensity problem associated with the Ornstein-Uhlenbeck process, we assume a square-root process for the default intensity, referred to Cox, Ingersoll, and Ross [1985]. We obtain an analytical solution for the expected loss and $n$-th moment of loss that simultaneously satisfies the following three requirements:

a) The default event can happen at any point in time up to loan maturity according to a stochastic process of default intensity.

b) Default intensity and collateral value are negatively correlated.

c) Default intensity and collateral value are non-negative.

The solution is obtained within the extended affine model introduced by Duffie, Pan, and Singleton [2000], who derive generalized Riccati equations characterizing extended affine models. The Riccati equations do not necessarily have explicit solutions. A more general version of this extended affine model is described by Chen and Joslin [2009]. However, with either model, whether the derived Riccati equations have explicit solutions is determined by the case in question.

In addition to the assumption of a square-root process for default intensity, we assume that the two-dimensional state vector consisting of default intensity and the logarithm of the collateral value follows an affine diffusion process. We assume a negative correlation between the driving Brownian motion of default intensity and that of the collateral value. In this setting, the expected loss consists of the survival probability and the time integral of the loan's expected recovery value. The survival probability is reduced to a basic affine model, and the solution is the same as the discount bond price in Cox, Ingersoll, and Ross [1985]. The integrand of expected recovery value is reduced to an extended affine model. The Riccati equations derived can be solved explicitly. The time integral of the expected recovery value is given as a Stieltjes integral with measure-changed survival probability.

Following this introduction, Section II describes our model of the stochastic processes for default intensity and collateral value. Section III derives solutions for expected loss and the $n$-th moment of loss. Section IV gives numerical examples of expected loss and the standard deviation of loss. Section V concludes this paper. The Appendix describes in detail the derivation of the explicit solutions for survival probability and the $n$-th moment of loss.

# II Our model

Assume that a bank supplies collateralized loan $D$ to a firm with maturity $T$. The collateral value is denoted $A_t$. For the sake of simplicity, we assume the loan to be supplied as a discount bond with zero interest rate. Despite this assumption, our following model can be directly extended for valuing a defaultable loan with a fixed interest rate.

Let default time $\tau$ be a non-negative random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that the loss incurred by the bank at time $\tau$ is given by:[1]

$$L_\tau = D - \delta A_\tau, \tag{1}$$

where $\delta$ is a constant denoting the recovered portion of the collateral value.

The default intensity or hazard rate of the firm $h_t$ is assumed to be governed by the following square-root process:

$$dh_t = \kappa(\bar{h} - h_t)dt + \sigma_h \sqrt{h_t} dW_t^h, \tag{2}$$

where $\kappa$, $\bar{h}$, and $\sigma_h$ are positive real number. $\bar{h}$ denotes mean the reversion level of default intensity. $\kappa$ denotes the speed of mean reversion. The intensity process (2) remains non-negative if the initial value $h_0$ is positive, since the instantaneous volatility

[1] This implies that the recovery rate may exceed 100%.

of the intensity is given by $\sigma_h\sqrt{h_t}$. The intensity is always positive if $2\kappa\bar{h} \geq \sigma_h^2$ (see Cox, Ingersoll, and Ross [1985]).

Let collateral value $A_t$ also be a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. The collateral value is assumed to be governed by the following diffusion process[2]:

$$dA_t = \mu_A A_t dt + \sigma_A A_t \sqrt{h_t} dW_t^A. \tag{3}$$

Furthermore, the Brownian motions in equations (2) and (3) are assumed to be correlated as follows:

$$\mathrm{cov}(dW^A, dW^h) = d[W^A, W^h] = \rho dt. \tag{4}$$

Mathematically, correlation $\rho$ can be either negative or positive. Given real-world circumstances, we focus on a negative correlation $\rho$.

We evaluate the expected value and $n$-th moments of loss (1). Let $(\mathcal{H}_t)_{t\geq 0}$ be a filtration generated by $\mathcal{H}_t = \sigma(1_{\{\tau \leq t\}})$. Let $(\mathcal{F}_t)_{t\geq 0}$ be an auxiliary filtration $\mathcal{F}_t = \sigma(\{W_s^h, W_s^A : s \leq t\})$ generated by the Brownian motions in equations (2) and (3). We also define an augmented filtration $(\mathcal{G}_t)_{t\geq 0}$ by

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t. \tag{5}$$

The default time $\tau$ is assumed to be a doubly stochastic random variable with respect to the filtration $\mathcal{F}_t = \sigma(\{W_s^h, W_s^A : s \leq t\})$,[3] and default time is assumed to have a hazard rate process defined by equation (2). Assuming the integrability of $\int_t^T |L_s h_s| \exp(-\int_t^s h_u du) ds$, the expected loss for the bank is given by:

$$\begin{aligned} E[L_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] &= DE[1_{\{t<\tau\leq T\}}|\mathcal{G}_t] - \delta E[A_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\ &= 1_{\{t<\tau\}} D(1 - \Pr[\tau > T | t < \tau]) \\ &\quad - 1_{\{t<\tau\}} \delta E[\int_t^T \exp\left(-\int_t^s h_u du\right) h_s A_s ds | \mathcal{F}_t]. \end{aligned} \tag{6}$$

In the first term of the right-hand side of equation (6), $\Pr[\tau > T | t < \tau]$ is the survival probability until time $T$ if the firm is not in default at time $t$. The second term of the right-hand side of equation (6),

$$\delta \int_t^T E[\exp\left(-\int_t^s h_u du\right) h_s A_s | \mathcal{F}_t] ds, \tag{7}$$

[2]The difference between process (3) and the geometric Brownian process $dA_t = \mu_A A_t dt + \sigma_A A_t dW_t^A$ lies in the instantaneous volatility parts: $\sigma_A\sqrt{h_t}$ and $\sigma_A$. That is, the instantaneous variance in equation (3) is proportional to default intensity and not constant.

[3]McNeil, Frey, and Embrechts [2005] discuss technical conditions for doubly stochastic random variables.

is the time integral of expected recovery. In Section III, we evaluate the survival probability $\Pr[\tau \leq T | t < \tau]$ and the expected recovery (7) and derive a solution for expected loss (6).[4]

These stochastic processes (2)–(4) are summarized as an affine diffusion process with a two-dimensional state vector, $X_t = (h_t, \ln A_t)^\top$. Using Ito's lemma, we can transform the collateral value process (3) to:

$$d \ln A_t = (\mu_A - \sigma_A^2 h_t / 2) dt + \sigma_A \sqrt{h_t} dW_t^A. \tag{8}$$

Introducing independent Brownian motions $W_{1,t}$ and $W_{2,t}$, we express the correlation of the Brownian motion (4) by:

$$W_t^A = W_{1,t}, \quad W_t^h = \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t}. \tag{9}$$

We can reduce equations (2), (8), and (9) to the following two-dimensional diffusion process:

$$dX_t = d \begin{pmatrix} h_t \\ \ln A_t \end{pmatrix} = \mu(X_t) dt + \sigma(X_t) d \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix}, \tag{10}$$

where

$$\mu(X_t) = \begin{pmatrix} \kappa \bar{h} \\ \mu_A \end{pmatrix} + \begin{pmatrix} -\kappa & 0 \\ -\sigma_A^2/2 & 0 \end{pmatrix} \begin{pmatrix} h_t \\ \ln A_t \end{pmatrix}, \tag{11}$$

$$\sigma(X_t) = \begin{pmatrix} \sigma_h \sqrt{h_t} & 0 \\ \sigma_A \rho \sqrt{h_t} & \sigma_A \sqrt{1 - \rho^2} \sqrt{h_t} \end{pmatrix}. \tag{12}$$

The diffusion process (10) is affine for two reasons. First, the drift term $\mu(X_t)$ is affine with respect to the state vector $X_t$ shown as equation (11). Second, the instantaneous variance-covariance matrix is

$$\sigma(X_t) \sigma(X_t)^\top = \begin{pmatrix} \sigma_h^2 h_t & \rho \sigma_A \sigma_h h_t \\ \rho \sigma_A \sigma_h h_t & \sigma_A^2 h_t \end{pmatrix}, \tag{13}$$

all of whose elements are linear with respect to the state vector $X_t$.

# III Solution for expected loss and $n$-th moment of loss

First, we derive a solution for the expected loss, shown as equation (6). Second, we extend the solution for the $n$-th moment of loss.

[4] Here, we evaluate the expected loss in physical probability without discounting by any interest rate. Evaluating discounted expected value with a fixed interest rate in risk-neutral probability is a direct application of our result.

## A. Expected loss

Now, we evaluate the expected loss (6) under the stochastic process for the hazard rate and collateral value expressed by equations (2)–(4).

The expected loss is composed of survival probability $\Pr[\tau > T | t < \tau]$ and the time integral of the expected recovery, shown as equation (7).

The survival probability is given by:

$$\Gamma(T-t|h_t) \equiv E[\exp\left(-\int_t^T h_s ds\right)|\mathcal{F}_t]. \tag{14}$$

The survival probability (14) is reduced to a basic affine model using one-dimensional state variable $h_t$ whose process is affine, as shown by equation (2). The survival probability is given by

$$\Gamma(T-t|h_t) = \left[\frac{2\gamma_h e^{(\gamma_h+\kappa)(T-t)/2}}{(\gamma_h+\kappa)e^{\gamma_h(T-t)}+\gamma_h-\kappa}\right]^{\frac{2\kappa\bar{h}}{\sigma_h^2}} \exp\left(\frac{2(1-e^{\gamma_h(T-t)})h_t}{(\gamma_h+\kappa)e^{\gamma_h(T-t)}+\gamma_h-\kappa}\right), \tag{15}$$

where

$$\gamma_h = \sqrt{\kappa^2 + 2\sigma_h^2}. \tag{16}$$

See Appendix 1 for details. The survival probability is the expressed in the same way as the discount bond price in the Cox-Ingersoll-Ross model. (See Cox, Ingersoll and Ross [1985], Nakagawa [1999].) Introducing the two-dimensional state vector $X_t = (h_t, \ln A_t)^\top$, we can reduce the integrand of the time integral of the expected recovery

$$\zeta(t,s) \equiv E[\exp\left(-\int_t^s h_u du\right) h_s A_s|\mathcal{F}_t], \tag{17}$$

to an extended affine form as follows:

$$\zeta(t,s) = E[\exp\left(-\int_t^s h_u du\right) e^{\ln A_s} h_s|\mathcal{F}_t], \tag{18}$$

for a fixed $s$ and varying $t$. $\zeta(t,s)$ is reduced to an extended affine model because the state vector $X_t$ has an affine diffusion process.[5] As Duffie, Pan, and Singleton [2000] show, the solution for equation (18) is given by:

$$\zeta(t,s) = (C(t) + B(t)\cdot X_t)\exp(\alpha(t) + \beta(t)\cdot X_t). \tag{19}$$

Coefficients $C(t)$, $B(t)$, $\alpha(t)$, and $\beta(t)$ satisfy Riccati equations. In this case, the Riccati

[5] Using notation proposed by Dai and Singleton [2000], we can indicate this process as $\mathbb{A}_1(2)$ affine diffusion.

equations derived from the model can be solved explicitly as follows:

$$\begin{aligned}
\alpha(t) &= \mu_A(s-t) + \frac{2\kappa\bar{h}}{\sigma_h^2}\ln\frac{2\gamma e^{\frac{\gamma+\tilde{\kappa}}{2}(s-t)}}{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})},\\
\beta(t) &\equiv (\beta_1(t),\beta_2(t))^\top, \quad \beta_1(t) = \frac{2(1-e^{\gamma(s-t)})}{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})}, \quad \beta_2(t)=1,\\
B(t) &\equiv (B_1(t),B_2(t))^\top, \quad B_1(t) = \frac{4\gamma^2 e^{\gamma(s-t)}}{\{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})\}^2}, \quad B_2(t)=0,\\
C(t) &= \frac{2\kappa\bar{h}(e^{\gamma(s-t)}-1)}{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})},
\end{aligned} \tag{20}$$

where

$$\tilde{\kappa} = \kappa - \rho\sigma_h\sigma_A, \quad \gamma = \sqrt{\tilde{\kappa}^2 + 2\sigma_h^2}. \tag{21}$$

Here, we note the term

$$\begin{aligned}
\eta(s-t|h_t) &\equiv \exp(\tilde{\alpha}(s-t) + \tilde{\beta}(s-t)h_t)\\
&= \left[\frac{2\gamma e^{(\gamma+\tilde{\kappa})(s-t)/2}}{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})}\right]^{\frac{2\kappa\bar{h}}{\sigma_h^2}} \exp\left\{\frac{2(1-e^{\gamma(s-t)})h_t}{(\gamma+\tilde{\kappa})e^{\gamma(s-t)}+(\gamma-\tilde{\kappa})}\right\},
\end{aligned} \tag{22}$$

where

$$\tilde{\alpha}(s-t) \equiv \alpha(t) - \mu_A(s-t), \quad \tilde{\beta}(s-t) \equiv \beta_1(t). \tag{23}$$

We see that $\eta(s-t|h_t)$ has the same form as survival probability $\Gamma(T-t|h_t)$. The first derivative of $\eta(s-t|h_t)$ with respect to $s$ is given by:

$$\frac{d\eta(s-t|h_t)}{ds} = (\tilde{C}(s-t) + \tilde{B}(s-t)h_t)\exp(\tilde{\alpha}(s-t) + \tilde{\beta}(s-t)h_t), \tag{24}$$

where

$$\tilde{C}(s-t) \equiv C(t), \quad \tilde{B}(s-t) \equiv B_1(t). \tag{25}$$

This leads to:

$$\zeta(t,s) = -A_t e^{\mu_A(s-t)} \left.\frac{d\eta(z|h_t)}{dz}\right|_{z=s-t}. \tag{26}$$

See Appendix 2 for a detailed derivation.

Comparing equations (15) and (22), we see that $\eta(T-t|h_t)$ can be interpreted as a measure-changed survival probability. First, many $\kappa$s are changed to $\tilde{\kappa}$s, but $\kappa\bar{h}$s are unchanged. Second, $\gamma_h$ becomes $\gamma$ if $\kappa$ is changed to $\tilde{\kappa}$. Thus, $\eta(T-t|h_t)$ is the survival probability with the following default intensity process:

$$dh_t = (\kappa\bar{h} - \tilde{\kappa}h_t)dt + \sigma_h\sqrt{h_t}d\tilde{W}_t^h, \tag{27}$$

where $\tilde{W}_t^h$ is a measure-changed Brownian motion given as:

$$d\tilde{W}_t^h = dW_t^h - \rho\sigma_A\sqrt{h_t}dt. \tag{28}$$

The measure-changed diffusion term is given by subtracting the instantaneous covariance between collateral movement and hazard rate movement from the original diffusion term:

$$\sigma_h \sqrt{h_t} d\tilde{W}_t^h = \sigma_h \sqrt{h_t} dW_t^h - \mathrm{cov}(dh_t, d\ln A_t|\mathcal{F}_t). \tag{29}$$

Furthermore, using this changed measure and equation (26), we can decompose $\zeta(t,s)$, shown as equation (17), by the expectation of the collateral value and the time differential of the survival probability, as follows:

$$\begin{aligned} E[\exp\left(-\int_t^s h_u du\right) h_s A_s|F_t] &= -Ae^{\mu_A(s-t)}\frac{d}{ds}\tilde{E}[\exp\left(-\int_t^s h_u du\right)|\mathcal{F}_t] \\ &= E[A_s|F_t]\tilde{E}[\exp\left(-\int_t^s h_u du\right) h_s|\mathcal{F}_t], \end{aligned} \tag{30}$$

where $\tilde{E}[\cdot]$ is the expectation with the measure-changed process. Using equation (26), we can evaluate the expected recovery measured at time $t$ as the following Stieltjes integral with the changed survival measure $\eta(\cdot)$.

$$\delta E[\int_t^T \exp\left(-\int_t^s h_u du\right) h_s A_s ds|\mathcal{F}_t] = -\delta A_t \int_0^{T-t} e^{\mu_A z} d\eta(z|h_t). \tag{31}$$

Substituting equations (15) and (31) into equation (6) with $t = 0$, we obtain the following expected loss for the bank at time 0:

$$E[(D - \delta A_\tau)1_{\{\tau \le T\}}] = D(1 - \Gamma(T|h_0)) + \delta A_0 \int_0^T e^{\mu_A z} d\eta(z|h_0), \tag{32}$$

where $\Gamma(T|h_0)$ and $\eta(s|h_0)$ are given by equations (15) and (22), respectively. The integral in equation (32) can be evaluated as follows:

$$\sum_{i=0}^{N-1} e^{\mu_A i\Delta}\{\eta((i+1)\Delta|h_0) - \eta(i\Delta|h_0)\}, \tag{33}$$

where $\Delta = T/N$, with large positive integer $N$. Based on this equation, we can perform high-speed computations for $N \approx 1,000$.

## B. $n$-th moment of loss

The expected loss is the basic measure for the loss distribution. The variance of loss and the higher moment of loss are also important measures for the loss distribution. The solution for the expected loss, shown as equation (32), is generalized to the $n$-th moment of loss. Using binomial expansion, we can express the $n$-th moment of loss as follows:

$$E[L_\tau^n|\mathcal{G}_t] = E[(D - \delta A_\tau)^n 1_{\{t<\tau\le T\}}|\mathcal{G}_t] = 1_{\{t<\tau\}} \sum_{i=0}^{n} {}_nC_i D^n (-\delta)^{n-i} I_{n-i}, \tag{34}$$

where

$$I_n = E[\int_t^T \exp\left(-\int_t^s h_u du\right) A_s^n h_s ds|\mathcal{F}_t]$$
$$= \int_t^T E[\exp\left(-\int_t^s h_u du\right) e^{n\ln A_s} h_s|\mathcal{F}_t]ds. \qquad (35)$$

As with our evaluation of $\zeta(t,s)$, we can evaluate the integrand of $I_n$ in an extended affine model. The derived Riccati equations have explicit solutions. Similar to the expected loss, $I_n$ can be evaluated as a Stieltjes integral with another measure-changed survival probability. (See Appendix 2 for the detailed derivation.) The default probability can be interpreted as a special case of $I_n$, $I_0$.

As an example, the standard deviation of loss is given by a combination of $I_n$s as:

$$\sqrt{\mathrm{var}[(D-\delta A_\tau)1_{\{\tau\le T\}}]} = \sqrt{D^2I_0 - 2\delta DI_1 + \delta^2 I_2 - (DI_0-\delta I_1)^2}, \qquad (36)$$

where $I_n$s are evaluated at time $t=0$.

# IV Numerical example

In this section, we show numerical examples of the expected loss, shown as equation (32), and the standard deviation of loss, shown as equation (36). We evaluate the integral of the right-hand-side of equation (32) as equation (33), where $N = 1,000$. Let $D = 100$, $T = 1$, $\delta = 0.7$, $\mu_A = 1\%$, $\sigma_A = 0.5$, $A_0 = 100$, $\sigma_h = 20\%$. Figure 1 illustrates the expected loss (left figure) and standard deviation of loss (right figure) with respect to negative correlation $\rho$ in four cases of $\kappa = 0.1, 1, 5, 10$, where $h_0 = 4\%$ and $\bar{h} = 3\%$. Figure 2 depicts the curves for $h_0 = 3\%$ and $\bar{h} = 4\%$. We see that the increase in the absolute value of correlation yields an increase in expected loss, with larger impact with lower $\kappa$. The increase in the absolute value of correlation also yields an increase in the standard deviation of loss. Again, the impact is larger when $\kappa$ is lower. Based on these numerical results, we would posit that risk managers must closely examine negative correlations in terms of both expected loss and the standard deviation of loss when the mean-reversion speed of default intensity is slow.

# V Conclusions

We obtained analytical solutions for the expected loss and $n$-th moment of loss distribution of a collateralized loan, simultaneously satisfying the following three requirements:

a) The default event can happen at any point in time up to loan maturity according to a stochastic process of default intensity.

Figure 1: Expected loss and standard deviation of loss against correlation $\rho$ ($h_0 > \bar{h}$)

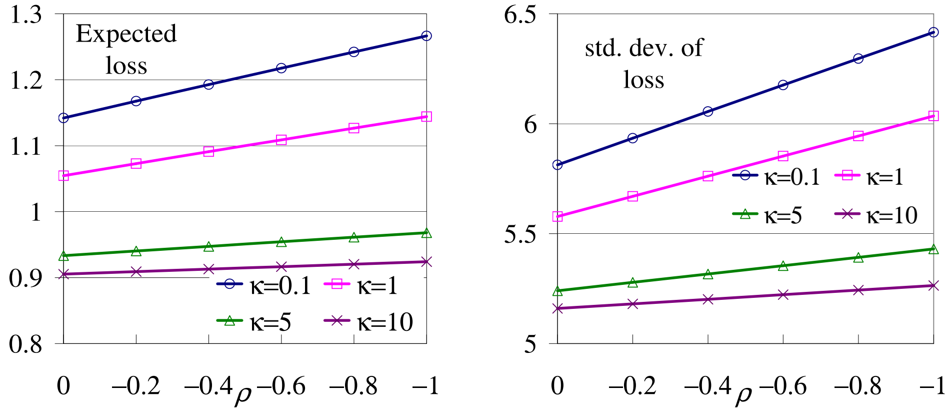
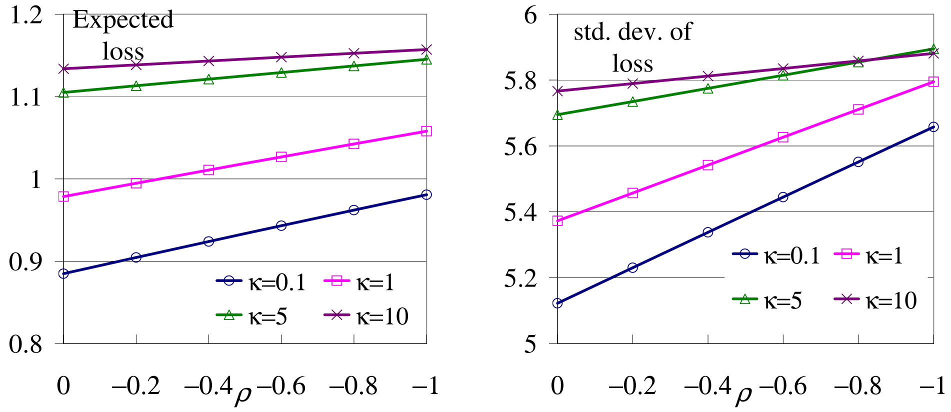



Figure 2: Expected loss and standard deviation of loss against correlation $\rho$ ($h_0 < \bar{h}$)



b) Default intensity and collateral value are negatively correlated.

c) Default intensity and collateral value are non-negative.

The expected loss consists of two parts: (i) the product of the loan amount and the default probability (one minus the survival probability) and (ii) expected recovery value at default. The survival probability in part (i) can be evaluated explicitly within a basic affine model, as with the Cox-Ingersoll-Ross discount bond price. We show the part (ii) is reduced to a Stieltjes integral with a measure-changed survival probability measure. We extend explicit formulations to $n$-th moment of loss using other measure-changed survival probabilities.

Since we have obtained analytical formulations for the expected loss and $n$-th moment of loss, we can evaluate various sensitivities for the expected loss, standard deviation, skewness, or kurtosis of loss. Although we note expected loss and $n$-th moment of loss, we can also approximate the value-at-risk by these $n$-th moments of loss.

Numerical examples show that the increase in the absolute value of the negative correlation between default intensity and collateral value yields an increase in the expected loss or the standard deviation of the loss. The impact is large when the mean-reversion speed of the default intensity is slow. Based on these numerical results, we posit that risk managers should pay close attention to the negative correlation both in terms of the expected loss and of the standard deviation of loss when the speed of mean-reversion of default intensity is slow.

# Appendix 1 Derivation of survival probability

The survival probability can be evaluated in a basic affine model by introducing the state variable $X_t = h_t$ as:

$$E[\exp\left(-\int_t^T h_s ds\right)|F_t] = \exp(\alpha_h(t) + \beta_h(t)h_t). \tag{A.1}$$

Here, $\alpha_h(t)$ and $\beta_h(t)$ satisfy the following ordinarily differential equations.

$$\frac{d\beta_h(t)}{dt} = 1 + \kappa\beta_h(t) - \frac{1}{2}\sigma_h^2\beta_h(t)^2, \tag{A.2}$$

$$\frac{d\alpha_h(t)}{dt} = -\kappa\bar{h}\beta_h(t). \tag{A.3}$$

The boundary conditions are given by:

$$\beta_h(T) = 0, \quad \alpha_h(T) = 0. \tag{A.4}$$

From equation (A.2) with the boundary condition (A.4), $\beta_h(t)$ is given by:

$$\beta_h(t) = \frac{2(1 - e^{\gamma_h(T-t)})}{(\gamma_h + \kappa)e^{\gamma_h(T-t)} + \gamma_h - \kappa}, \tag{A.5}$$

where

$$\gamma_h = \sqrt{\kappa^2 + 2\sigma_h^2}. \tag{A.6}$$

See Appendix 3 for a derivation of the solution.

Substituting equations (A.4) and (A.5) into equation (A.3) yields:

$$\alpha_h(t) = \alpha_h(t) - \alpha_h(T) = \kappa\bar{h}\int_t^T \beta_h(s)ds = \frac{2\kappa\bar{h}}{\sigma_h^2}\ln\frac{2\gamma_h e^{\frac{\gamma_h+\kappa}{2}(T-t)}}{(\gamma_h + \kappa)e^{\gamma_h(T-t)} + \gamma_h - \kappa}. \tag{A.7}$$

Substituting equations (A.5) and (A.7) into equation (A.1) yields survival probability as follows:

$$\begin{aligned} \Gamma(T-t|h_t) &= E[\exp\left(-\int_t^T h_s ds\right)|\mathcal{F}_t] \\ &= \left[\frac{2\gamma_h e^{(\gamma_h+\kappa)(T-t)/2}}{(\gamma_h + \kappa)e^{\gamma_h(T-t)} + \gamma_h - \kappa}\right]^{\frac{2\kappa\bar{h}}{\sigma_h^2}} \exp\left(\frac{2(1 - e^{\gamma_h(T-t)})h_t}{(\gamma_h + \kappa)e^{\gamma_h(T-t)} + \gamma_h - \kappa}\right). \end{aligned} \tag{A.8}$$

This expression is the same as the discount bond price in Cox, Ingersoll, and Ross [1985]. See also Nakagawa [1999].

# Appendix 2 Evaluation of $n$-th moment of loss within an extended affine model

Expected loss, shown as equation (6), or the $n$-th moment of loss, shown as equation (34), is given by a combination of $I_n$, shown as equation (35). This appendix evaluates $I_n$ within an extended affine model.

As shown in Section II, a two-dimensional state vector $X_t = (h_t, \ln A_t)^\top$ has an affine diffusion:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \tag{A.9}$$

The drift vector $\mu(X_t)$ is an affine function of $X_t$. Each element of the instantaneous variance-covariance matrix $\sigma(X_t)\sigma(X_t)^\top$ is a linear function of $X_t$.

Duffie, Pan, and Singleton [2000] show that for the well-behaved affine diffusion (A.9), the expectation

$$\phi(v, w, X_t, t, T) = E[\exp\left(-\int_t^T R(X_u)du\right)(v \cdot X_T)e^{w \cdot X_T}|\mathcal{F}_t], \tag{A.10}$$

where

$$R(X_u) = r_0 + r_1 \cdot X_u, \tag{A.11}$$

can be evaluated as follows:

$$\phi(v, w, X_t, t, T) = (C(t) + B(t) \cdot X_t)\exp(\alpha(t) + \beta(t) \cdot X_t). \tag{A.12}$$

They derive Riccati ordinary differential equations satisfied by the coefficients in equation (A.12), $C(t)$, $B(t)$, $\alpha(t)$, and $\beta(t)$.

Here, let

$$r_0 = 0 \text{ and } r_1 = (1, 0);\ \text{that is, } R(X_u) = h_u, \tag{A.13}$$

$$w = (0, n);\ \text{that is, } e^{w \cdot X_T} = e^{n \ln A_T} = A_T^n, \tag{A.14}$$

$$v = (1, 0);\ \text{that is, } v \cdot X_T = h_T. \tag{A.15}$$

Then, an integrand of $I_n$ is given by an extended affine form, as follows:

$$\phi(v, w, X_t, t, T) = E[\exp\left(-\int_t^T h_u du\right) A_T^n h_T|\mathcal{F}_t]. \tag{A.16}$$

Following Duffie, Pan, and Singleton [2000], we obtain the following ordinary differential equations satisfied by the coefficients in equation (A.12).

$$\begin{aligned}\frac{d\beta_1(t)}{dt} &= 1 + \kappa\beta_1(t) + \frac{\sigma_A^2}{2}\beta_2(t) - \frac{1}{2}\beta(t)^\top \begin{pmatrix} \sigma_h^2 & \rho\sigma_h\sigma_A \\ \rho\sigma_h\sigma_A & \sigma_A^2 \end{pmatrix} \beta(t) \\ &= 1 + \kappa\beta_1(t) + \frac{\sigma_A^2}{2}\beta_2(t) - \frac{\sigma_h^2}{2}\beta_1(t)^2 - \rho\sigma_h\sigma_A\beta_1(t)\beta_2(t) - \frac{\sigma_A^2}{2}\beta_2(t)^2,\end{aligned} \tag{A.17}$$

$$\frac{d\beta_2(t)}{dt} = 0, \tag{A.18}$$

$$\frac{d\alpha(t)}{dt} = -\begin{pmatrix} \kappa\bar{h} \\ \mu_A \end{pmatrix} \cdot \beta(t) = -\kappa\bar{h}\beta_1(t) - \mu_A\beta_2(t), \tag{A.19}$$

$$\begin{aligned} -\frac{dB_1(t)}{dt} &= -\kappa B_1(t) - \frac{\sigma_A^2}{2}B_2(t) + \beta(t)^\top \begin{pmatrix} \sigma_h^2 & \rho\sigma_h\sigma_A \\ \rho\sigma_h\sigma_A & \sigma_A^2 \end{pmatrix} B(t) \\ &= -\kappa B_1(t) - \frac{\sigma_A^2}{2}B_2(t) + \sigma_h^2\beta_1(t)B_1(t) \\ &\quad + \rho\sigma_h\sigma_A(\beta_1(t)B_2(t) + \beta_2(t)B_1(t)) + \sigma_A^2\beta_2(t)B_2(t), \end{aligned} \tag{A.20}$$

$$-\frac{dB_2(t)}{dt} = 0, \tag{A.21}$$

$$-\frac{dC(t)}{dt} = \begin{pmatrix} \kappa\bar{h} \\ \mu_A \end{pmatrix} \cdot B(t) = \kappa\bar{h}B_1(t) + \mu_A B_2(t), \tag{A.22}$$

where $\beta(t) = (\beta_1(t), \beta_2(t))^\top$ and $B(t) = (B_1(t), B_2(t))^\top$. The boundary conditions are given by:

$$\beta_1(T) = 0, \quad \beta_2(T) = n, \quad \alpha(T) = 0, \tag{A.23}$$

$$B_1(T) = 1, \quad B_2(T) = 0, \quad C(T) = 0. \tag{A.24}$$

First, we solve the ordinary equations (A.17), (A.18) and (A.19) with boundary conditions (A.23). Equation (A.18) with boundary condition (A.23) specifies $\beta_2(t)$ as follows:

$$\beta_2(t) = n. \tag{A.25}$$

Substituting equation (A.25) into equation (A.17) yields

$$\frac{d\beta_1(t)}{dt} = 1 + \frac{n(1-n)\sigma_A^2}{2} + (\kappa - n\rho\sigma_h\sigma_A)\beta_1(t) - \frac{\sigma_h^2}{2}\beta_1(t)^2. \tag{A.26}$$

Equation (A.26) is a Riccati equation with constant coefficients. The ordinary differential equation (A.26) with a boundary condition can be solved explicitly. See Appendix 3 for the derivation. The solution with boundary condition (A.23) is given by:

$$\begin{aligned} \beta_1(t) &= \frac{(\tilde{\kappa}_n + \gamma_n) + (\tilde{\kappa}_n - \gamma_n)\tilde{\delta}_n e^{\gamma_n(T-t)}}{\sigma_h^2(\tilde{\delta}_n e^{\gamma_n(T-t)} + 1)} = \frac{(\gamma_n - \tilde{\kappa}_n)(\gamma_n + \tilde{\kappa}_n)(1 - e^{\gamma_n(T-t)})}{\sigma_h^2\{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)\}} \\ &= \frac{\{2 + n(1-n)\sigma_A^2\}(1 - e^{\gamma_n(T-t)})}{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)}, \end{aligned} \tag{A.27}$$

where

$$\tilde{\kappa}_n = \kappa - n\rho\sigma_h\sigma_A, \tag{A.28}$$

$$\gamma_n = \sqrt{\tilde{\kappa}_n^2 + \sigma_h^2\{2 + n(1-n)\sigma_A^2\}}, \tag{A.29}$$

$$\tilde{\lambda}_n = \frac{\tilde{\kappa}_n + \gamma_n}{-\tilde{\kappa}_n + \gamma_n}. \tag{A.30}$$

Substituting equations (A.25) and (A.27) into equation (A.19) and integrating with boundary condition (A.23) yields:

$$\begin{aligned}\alpha(t) &= \alpha(t) - \alpha(T) = \int_t^T \{\kappa\bar{h}\beta_1(s) + n\mu_A\}ds \\ &= n\mu_A(T-t) + \frac{\kappa\bar{h}(\gamma_n - \tilde{\kappa}_n)(\gamma_n + \tilde{\kappa}_n)}{\sigma_h^2}\int_t^T \frac{(1-e^{\gamma_n(T-s)})}{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-s)} + (\gamma_n - \tilde{\kappa}_n)}ds \\ &= \left\{n\mu_A + \frac{\kappa\bar{h}(\gamma_n + \tilde{\kappa}_n)}{\sigma_h^2}\right\}(T-t) + \frac{2\kappa\bar{h}}{\sigma_h^2}\ln\frac{2\gamma_n}{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)}.\end{aligned} \tag{A.31}$$

Next, we solve the ordinary equations (A.20), (A.21), and (A.22) with boundary conditions (A.24). Equation (A.21) with boundary condition (A.24) specifies $B_2(t)$ as follows:

$$B_2(t) = 0. \tag{A.32}$$

Substituting equations (A.32) and (A.25) into equation (A.20) yields:

$$-\frac{dB_1(t)}{dt} = -\kappa B_1(t) + \sigma_h^2\beta_1(t)B_1(t) + n\rho\sigma_h\sigma_A B_1(t). \tag{A.33}$$

Integrating equation (A.33) with equation (A.27) and boundary condition (A.24) yields:

$$\begin{aligned}\ln B_1(t) &= -\int_t^T \{\tilde{\kappa}_n - \sigma_h^2\beta_1(s)\}ds \\ &= -\tilde{\kappa}_n(T-t) + \sigma_h^2\int_t^T \beta_1(s)ds \\ &= \gamma_n(T-t) + 2\ln\frac{2\gamma_n}{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)}.\end{aligned} \tag{A.34}$$

Equation (A.34) is equivalent to:

$$B_1(t) = \frac{4\gamma_n^2 e^{\gamma_n(T-t)}}{\{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)\}^2}. \tag{A.35}$$

Substituting equation (A.32) into equation (A.22) yields:

$$-\frac{dC(t)}{dt} = \kappa\bar{h}B_1(t). \tag{A.36}$$

Integrating equation (A.36) with equation (A.35) and boundary condition (A.24) yields:

$$\begin{aligned}C(t) &= \kappa\bar{h}\int_t^T B_1(s)ds = 4\gamma_n^2\kappa\bar{h}\int_t^T \frac{e^{\gamma_n(T-s)}}{\{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-s)} + (\gamma_n - \tilde{\kappa}_n)\}^2}ds \\ &= -\frac{4\gamma_n\kappa\bar{h}}{(\gamma_n + \tilde{\kappa}_n)}\int_{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n(T-t)}+(\gamma_n-\tilde{\kappa}_n)}^{2\gamma_n} \frac{1}{z^2}dz \\ &= \frac{2\kappa\bar{h}(e^{\gamma_n(T-t)} - 1)}{(\gamma_n + \tilde{\kappa}_n)e^{\gamma_n(T-t)} + (\gamma_n - \tilde{\kappa}_n)}.\end{aligned} \tag{A.37}$$

Thus, we evaluate equation (A.16) as follows:

$$\begin{aligned} &E[\exp\left(-\int_t^T h_u du\right) A_T^n h_T|\mathcal{F}_t] \\ &\quad = A_t^n e^{n\mu_A(T-t)}(\tilde{C}(T-t) + \tilde{B}(T-t)h_t)\exp(\tilde{\alpha}(T-t) + \tilde{\beta}(T-t)h_t), \end{aligned} \tag{A.38}$$

where

$$\tilde{C}(z) = \frac{2\kappa\bar{h}(e^{\gamma_n z}-1)}{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)}, \tag{A.39}$$

$$\tilde{B}(z) = \frac{4\gamma_n^2 e^{\gamma_n z}}{\{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)\}^2}, \tag{A.40}$$

$$\tilde{\alpha}(z) = \frac{\kappa\bar{h}(\gamma_n+\tilde{\kappa}_n)}{\sigma_h^2}z - \frac{2\kappa\bar{h}}{\sigma_h^2}\ln\frac{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)}{2\gamma_n}, \tag{A.41}$$

$$\tilde{\beta}(z) = \frac{(\gamma_n-\tilde{\kappa}_n)(\gamma_n+\tilde{\kappa}_n)}{\sigma_h^2}\frac{(1-e^{\gamma_n z})}{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)}. \tag{A.42}$$

Here, let

$$\begin{aligned} &\eta_n(z|h_t) \equiv \exp(\tilde{\alpha}(z) + \tilde{\beta}(z)h_t) \\ &= \left[\frac{2\gamma_n e^{(\gamma_n+\tilde{\kappa}_n)z/2}}{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)}\right]^{\frac{2\kappa\bar{h}}{\sigma_h^2}} \exp\left\{\frac{\{2+n(1-n)\sigma_A^2\}(1-e^{\gamma_n z})h_t}{(\gamma_n+\tilde{\kappa}_n)e^{\gamma_n z}+(\gamma_n-\tilde{\kappa}_n)}\right\}. \end{aligned} \tag{A.43}$$

Comparing equation (A.8) with equation (A.43), we find that $\eta_n(z|h_t)$ corresponds to a kind of survival probability wherein parameters $\kappa$, $\gamma_h$, $\bar{h}$, $h_t$ become, respectively, $\tilde{\kappa}_n$, $\gamma_n$, $\kappa\bar{h}/\tilde{\kappa}_n$, $\{1+n(1-n)\sigma_A^2/2\}h$. We also find that

$$\frac{\partial\tilde{\alpha}(z)}{\partial z} = \frac{\tilde{\kappa}_n^2-\gamma_n^2}{2\sigma_h^2}\tilde{C}(z) \tag{A.44}$$

From equation (A.44), we see that the first derivative of $\eta_n(z|h_t)$ with respect to time to maturity $z$ is given by:

$$\frac{d\eta_n(z|h_t)}{dz} = \frac{\tilde{\kappa}_n^2-\gamma_n^2}{2\sigma_h^2}\exp(\tilde{\alpha}(z)+\tilde{\beta}(z)h_t)\{\tilde{C}(z)+\tilde{B}(z)h_t\}. \tag{A.45}$$

Substituting equation (A.45) into equation (A.38) with $T = s$ yields:

$$E[\exp\left(-\int_t^s h_u du\right) A_s^n h_s|F_t] \ = A_t^n e^{n\mu_A(s-t)}\frac{2\sigma_h^2}{\tilde{\kappa}_n^2-\gamma_n^2}\left.\frac{d\eta_n(z|h_t)}{dz}\right|_{z=s-t}. \tag{A.46}$$

In conclusion, from equations (35) and (A.29), $I_n$ is given by:

$$\begin{aligned} I_n &= \frac{-A_t^n}{1+n(1-n)\sigma_A^2/2}\int_t^T e^{n\mu_A(s-t)}d\eta_n(s-t|h_t) \\ &= \frac{-A_t^n}{1+n(1-n)\sigma_A^2/2}\int_0^{T-t} e^{n\mu_A z}d\eta_n(z|h_t). \end{aligned} \tag{A.47}$$

# Appendix 3 Solution for a Riccati equation

Assume an ordinary differential equation

$$\frac{dy(t)}{dt} = -\frac{1}{2}a^2 y(t)^2 + by(t) + c, \tag{A.48}$$

with boundary condition

$$y(T) = g, \tag{A.49}$$

where $a$, $b$, $c$, $g$ are constants and $c \geq 0$. The ordinary differential equation has a solution

$$y(t) = \frac{b + \gamma + (b - \gamma)\lambda e^{\gamma(T-t)}}{a^2(\lambda e^{\gamma(T-t)} + 1)}, \tag{A.50}$$

where

$$\gamma = \sqrt{b^2 + 2a^2 c}, \tag{A.51}$$

$$\lambda = \frac{-a^2 g + b + \gamma}{a^2 g - b + \gamma}. \tag{A.52}$$

*Proof.* Equation (A.48) is equivalent to:

$$\frac{dy(t)}{dt} = -\frac{1}{2}a^2(y(t) - y_1)(y(t) - y_2), \tag{A.53}$$

where

$$y_1 = \frac{b + \gamma}{a^2}, \quad y_2 = \frac{b - \gamma}{a^2}, \quad \gamma = \sqrt{b^2 + 2a^2 c}. \tag{A.54}$$

Integrating equation (A.53) yields:

$$\begin{aligned} -\frac{1}{2}a^2(T - t) &= \int_{y(t)}^{y(T)} \frac{dy(s)}{(y(s) - y_1)(y(s) - y_2)} \\ &= \frac{1}{y_1 - y_2} \int_{y(t)}^{y(T)} \left\{ \frac{1}{y(s) - y_1} - \frac{1}{y(s) - y_2} \right\} dy(s) \\ &= \frac{1}{y_1 - y_2} \left\{ \ln \frac{y(T) - y_1}{y(t) - y_1} - \ln \frac{y(T) - y_2}{y(t) - y_2} \right\} = \frac{a^2}{2\gamma} \left\{ \ln \frac{g - y_1}{g - y_2} \frac{y(t) - y_2}{y(t) - y_1} \right\}. \end{aligned} \tag{A.55}$$

Rearranging equation (A.55) gives equation (A.50). $\square$

# References


Altman, Edward I., Brooks Brady, Andrea Resti, and Andrea Sironi, “The PD/LGD Link: Empirical Evidence from the Bond Market,” in *Recovery Risk*, Altman, E. I., A. Resti, and A. Sironi (eds.), Risk Books, 2005, pp.217–233.

Bakshi, Gurdip, Dilip Madan, and Frank Zhang, “Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates,” working paper, 2006.

Basel Committee on Banking Supervision, “International Convergence of Capital Measurement and Capital Standards,” Basel Committee Publications No. 107, November 2005.

Black, Fischer and John C. Cox, “Valuing Corporate Securities: Some Effects of Bond Indenture Provisions,” *Journal of Finance*, **31**(2), 1976, pp.351–367.

Chen, Hui and Scott Joslin, “Generalized Transform Analysis of Affine Processes and Asset Pricing Application,” working paper, 2009.

Chen, Long, Pierre Collin-Dufresne, and Robert S. Goldstein, “On the Relation Between the Credit Spread Puzzle and the Equity Premium Puzzle,” *Review of Financial Studies*, **22**(9), 2009, pp.3367–3409.

Cox, John C., Jonathan E. Ingersoll Jr., and Stephen A. Ross, “A Theory of the Term Structure of Interest Rates,” *Econometrica*, **53**(2), 1985, pp.385–407.

Dai, Qiang and Kenneth J. Singleton, “Specification Analysis of Affine Term Structure Models,” *Journal of Finance*, **55**(5), 2000, pp.1943–1978.

Duffie, Darrell, Jun Pan, and Kenneth J. Singleton, “Transform Analysis and Asset Pricing for Affine Jump-diffusions,” *Econometrica*, **68**(6), 2000, pp.1343–1376.

——— and David Lando,“Term Structure of Credit Spreads with Incomplete Accounting Information,” *Econometrica*, **69**(3), 2001, pp.633–664.

——— and Kenneth J. Singleton, *Credit Risk: Pricing, Measurement, and Management*, Princeton University Press, 2003.

——— , “Credit Risk Modeling with Affine Processes,” *Journal of Banking and Finance*, **29**(11), 2005, pp.2751–2802.

Guo, Xin, Robert A. Jarrow, and Yan Zeng, “Modeling the Recovery Rate in a Reduced Form Model,” *Mathematical Finance*, **19**(1), 2009, pp.73–97.

Kijima, Masaaki and Yusuke Miyake, "On the Term Structure of Lending Interest Rates When a Fraction of Collateral is Recovered upon Default," *Japan Journal of Industrial and Applied Mathematics*, **21**(1), 2004, pp.35–56.

McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts, *Quantitative Risk Management: Concepts, Techniques, and Tools*, Princeton University Press, 2005.

Nakagawa, Hidetoshi, "Valuation of Default Swap with Affine-type Hazard Rate," *Proceeding of the Japan Academy Ser. A*, **75**(3), 1999, pp.43–46.

Pykhtin, Michael, "Unexpected Recovery Risk," *Risk*, **16**(8), 2003, pp.74–78.