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Spurious Regressions in Technical Trading: Momentum or Contrarian?

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Abstract

This paper investigates the spurious effect in forecasting asset returns when signals from technical trading rules are used as predictors. Against economic intuition, the simulation result shows that, even if past information has non predictive power, buy or sell signals based on the difference between the short-period and long-period moving averages of past asset prices can be statistically significant when the forecast horizon is relatively long. The theory implies that both 'momentum' and 'contrarian' strategies can be falsely supported, while the probability of obtaining each result depends on the type of the test statistics employed. Several modifications to these test statistics are considered for the purpose of avoiding spurious regressions. They are applied to the stock market index and the foreign exchange rate in order to reconsider the predictive power of technical trading rules.

Keywords: Efficient market hypothesis; Nonstationary time series; Random walk; Technical analysis
JEL classification: C12, C22, C25, G11, G15

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1 Introduction

In an important Monte Carlo simulation study, Granger and Newbold (1974) showed the strong tendency of finding a significant slope coefficient and a relatively high degree of fit in terms of the R^2 when independent random walks are regressed on one another. Such a form of spurious regressions, or nonsense regressions, was theoretically examined by Phillips (1986) who provided a full account for the limiting behavior of the statistics using the Functional Central Limit Theorem (FCLT). The pair of simulation and theoretical studies made scholars widely aware of the practical importance of nonstationary time-series econometric analysis and resulted in a considerable number of works that followed.

Technical trading rules usually refer to a set of trading strategies that involve the prediction of future asset price movements using the history of its own past movements. In this paper, we show the possibility of finding a spurious correlation between signals from technical trading rules and future asset returns when in fact the asset price follows a random walk process. In such a case, past information should not have any predictive power but the difference between the short-period and long-period moving averages of past asset prices can be falsely detected as a statistically 'significant' predictor for a relatively long horizon forecast. Following the Granger-Newbold-Phillips tradition, we examine this technical analysis version of spurious regressions first by presenting simulation evidence and then by accounting for the phenomenon theoretically using the FCLT. We focus on random walks as in the classic spurious regression example á la Granger and Newbold (1974) because an asset price movement implied by the efficient market hypothesis is typically characterized by a random walk process. However, our analysis differs from prototypical spurious regressions in the following two aspects. First, the source of nonstandard limiting distributions of statistics is not the fact that variables used in the regression are integrated of order one, denoted by I(1). Instead, it is the fact that the long-horizon return and the long moving average asymptotically behave as I(1) variables when both the forecast horizon and the window length of the moving average are approximated by a nontrivial fraction of the sample size. A similar asymptotic approximation has been employed in the literature on the long-run predictability of equity returns (e.g., Richardson and Stock, 1989, and Valkanov, 2003). Second, since buy or sell signal in technical analysis is often constructed from the sign of the deviation of the short moving average from the long moving average, the theory of nonlinear transformation of I(1) processes plays an important role in our analysis. We show that using a discrete signal as a predictor, in place of the continuous regressor, does not only make the t-statistic divergent but also results in the divergence of the slope estimator.

As Brock, Lakonishok and LeBaron (1992) pointed out, wide-spread use of technical analysis among dealers in the stock market is evident from technical reports by major brokerage firms and newsletters by various experts. Regarding the foreign exchange market, a more direct evidence is available in Taylor and Allen (1992) who reported that about two thirds of the respondents of a survey among major UK dealers were using some form of moving average rules in their decision making (see also Cheung and Chinn (2001) for the similar results among dealers located in the US). According to Brock, Lakonishok and LeBaron (1992), one of the simplest moving average rules is to initiate buy (sell) signals when the short-period moving average of the past asset prices crosses the long-period moving average from below (above). The same buy signal is generated as long as the short moving average is above the long moving average.[1] In this paper, we interpret a trading strategy adopting this rule as a 'momentum' strategy because the traders are expecting buy (sell) signals to capture the upward (downward) shift in the trend after smoothing out the noise component from the original series. Similarly, for a 'contrarian' trader who exploits profit opportunities from the return reversals, the buy signal can be generated when the short moving average is below the long moving average. For both 'momentum' and 'contrarian' traders, using this simple moving average rule divides all the trading days into the ones with either buy or sell signals. In practice, the choice between 'momentum' and 'contrarian' trading strategies may depend on the investment horizon, type of assets and market conditions.[2]

[1]Brock, Lakonishok and LeBaron (1992), in their analysis, refered to this rule as the variable-length moving average rule without a band. They also considered the fixed-length moving average rule which generates signals only at the time of the crossing of the moving averages.

[2]In the literature on predictability of excess returns, it is not uncommon to find dependence of

Regardless of the direction of the signals, practitioners' strong belief of the profitability, or predictability, based solely on past information is clearly against the notion of market efficiency. This fact has lead an increasing number of academic researchers to turn to statistical analysis on technical trading rules. For example, Brock, Lakonishok and LeBaron (1992) utilized the bootstrap method and provided statistical support on the validity of using moving average rules. Other studies on the predictive performance of technical trading rules include: Gençay (1998) who provided evidence on linear and nonlinear predictability in equity returns; Sullivan, Timmermann and White (1999) who employed a reality check procedure to reduce the data snooping bias; and LeBaron (1999) who found evidence of predictability in foreign exchange rates during the period of central bank intervention. It should be noted that all of these empirical studies considered the case of a relatively short investment horizon, typically one to ten days.

For a typical trader, the relative importance of the fundamental analysis in comparison to the technical analysis is known to increase as the investment horizon becomes longer.[3] However, according to Taylor and Allen (1992, Table 3(A)), even at relatively long horizons of three and six months, 37 and 25 percents of traders, respectively, still consider technical analysis more important than, or at least as important as, fundamental analysis. In this paper, we focus more on a relatively longer horizon and point out the pitfalls of using technical trading rules in prediction.

At this point, the following simple example is helpful in understanding the problem. Consider a hypothetical daily asset price series (in logs) generated from a pure random walk process and construct a (buy signal) dummy variable which is given a value of one if the spot price is larger than the long moving average of past 50 days, and zero otherwise.[4] Let us here present a simple simulation result using the

the sign of serial correlation on the choice of horizons or lag lengths. For example, see Jegadeesh (1990) and Cutler, Poterba and Summers (1991).

[3]Menknoff and Taylor (2007) summarize the world-wide evidence on the dependence of relative importance between technical and fundamental analyses on the horizon. Commonly used predictors in the fundamental analysis include price-earning ratios for equity returns, and monetary aggregates for the foreign exchange rate returns.

[4]The spot rate corresponds to the short moving average with a window length of one day. Also, note that the same combination of short and long moving averages appears in Brock, Lakonishok and LeBaron (1992).

forecasting regression of future returns at a short horizon of 1 day, as well as at a long horizon of 50 days, on the dummy variable using 250 observations which roughly corresponds to the number of trading days in a year. For the forecasting regression with a short horizon, the t-value turns out to be greater than 1.64 in absolute value for 11 percent of 10,000 replications, which is very close to the nominal size of 10 percent. In contrast, in the case of a long horizon, on 77 percent of all occasions, the dummy variable is found to be a significant predictor using the same critical value. Furthermore, with the one-sided test using a nominal size of 5 percent, the frequency of finding a significantly positive coefficient, namely a 'momentum' result, is 15 percent while that of finding a significantly negative coefficient, namely a 'contrarian' result, is as large as 62 percent. Similar but somewhat different results can be obtained with other choices of the forecast horizon and the window length of moving average, which will be presented in the simulation section.

The remainder of the paper is organized as follows. In the simulation part of this paper, section 2, we systematically investigate the spurious effects in three selected types of simple technical trading rules using various combinations of the forecast horizon and the window length of moving average. In Section 3, we present theoretical results to account for the simulation evidence provided in Section 2. The results include the limiting distributions of the slope estimator, its t-statistic and the R^2 for the dummy variable regression. Several possibilities of avoiding spurious regressions are also discussed. Section 4 provides the empirical applications to stock returns and foreign exchange rate returns. Some concluding remarks are made in Section 5.

2 Model and Simulation

2.1 Three Variants of Technical Trading Rules

Let z_t be the (log of) representative asset price series. In typical technical trading rules, the buy or sell signal often depends on

$$TTR_t^{(S,L)} = MA_t(S) - MA_t(L)$$

where $MA_t(S) = (1/S)\sum_{j=0}^{S-1} z_{t-j}$ and $MA_t(L) = (1/L)\sum_{j=0}^{L-1} z_{t-j}$ are the short-period and long-period moving averages, respectively, with a restriction $1 \leq S << L$. In what follows, we examine three variants of simple moving average rules based on $TTR_t^{(S,L)}$ designed to predict H period ahead (log) returns, $y_{t,H} = z_{t+H} - z_t$.

The first procedure is to regress $y_{t,H}$ on a constant and $TTR_t^{(S,L)}$ using a sample of size T, which gives the least squares regression,

$$y_{t,H} = \widehat{\alpha}_C + \widehat{\beta}_C TTR_t^{(S,L)} + \widehat{u}_t, \quad t = L, ..., T + L - 1. \tag{1}$$

Here the regression sample starts at $t = L$ since the first observation of the regressor, $TTR_t^{(S,L)}$, consists of prices over the $t = 1, ..., L$ period. Note that the predictor used in this technical trading regression is a continuous random variable instead of a discrete indicator. A non-zero slope coefficient estimate $\widehat{\beta}_C$ implies that the demeaned future returns are proportional to the demeaned current $TTR_t^{(S,L)}$ and we interpret a positive (negative) coefficient as suggesting the profitability from 'momentum' ('contrarian') strategy. Similar continuous technical trading rules have been considered in Gençay's (1998) study on the comparison of linear to nonlinear forecasts. Here, we evaluate the forecasting performance of the linear regression model by testing the significance of the predictor using a t-statistic, $t(\widehat{\beta}_C)$, and also by reporting the coefficient of determination, denoted by R_C^2.

The second approach relies on a more commonly used indicator. The regressor $TTR_t^{(S,L)}$ in (1) is replaced by its discrete transformation, $\mathbf{1}\{TTR_t^{(S,L)} > 0\}$, which is given a value of one if $TTR_t^{(S,L)}$ is positive, and zero otherwise. Similar to its continuous counterpart, we examine the t-statistic, $t(\widehat{\beta}_D)$, and the coefficient of determination, R_D^2, from the discrete version of the technical trading regression,

$$y_{t,H} = \widehat{\alpha}_D + \widehat{\beta}_D \mathbf{1}\{TTR_t^{(S,L)} > 0\} + \widehat{v}_t, \quad t = L, ..., T + L - 1. \tag{2}$$

A typical interpretation for the discrete indicator is a buy signal (based on the 'momentum' strategy) if $\widehat{\beta}_D > 0$ and a sell signal (based on the 'contrarian' strategy) if $\widehat{\beta}_D < 0$, because $\widehat{\beta}_D$ represents the average difference in future returns between the cases of positive and negative $TTR_t^{(S,L)}$.

The third procedure we consider is to test the sign predictability based on the proportion of correctly predicted signs of future returns using the sign of $TTR_t^{(S,L)}$ as a predictor. Let $n = \sum_{t=L}^{T+L-1} \mathbf{1}\{TTR_t^{(S,L)} > 0\}$ be the total number of observations with a positive sign of $TTR_t^{(S,L)}$. Also let $n_+ = \sum_{t=L}^{T+L-1} \mathbf{1}\{y_{t,H} > 0, TTR_t^{(S,L)} > 0\}$ be the number of observations with both signs of $y_{t,H}$ and $TTR_t^{(S,L)}$ being positive. Then, the classic proportion t-statistic designed to detect a significant deviation of the binomial success probability from 0.5 is given by,

$$t_+ = \frac{\widehat{p}_+ - 0.5}{\sqrt{0.5^2/n}} \tag{3}$$

where $\widehat{p}_+ = n_+/n$ is the proportion of successes in predicting correct sign of returns among the observations with positive $TTR_t^{(S,L)}$. When $\widehat{p}_+$ is significantly greater (less) than 50 percent, the 'momentum' ('contrarian') strategy may be justified. Note that the proportion $\widehat{p}_+$ has a useful interpretation as the slope coefficient estimator in the regression of $\mathbf{1}\{y_{t,H} > 0\}$ on $\mathbf{1}\{TTR_t^{(S,L)} > 0\}$ with no intercept term. Thus, some similarity between the characteristics of $\widehat{p}_+$ and those of $\widehat{\beta}_C$ and $\widehat{\beta}_D$ may be expected. We can also consider an alternative proportion test based only on the observations with negative $TTR_t^{(S,L)}$,

$$t_- = \frac{\widehat{p}_- - 0.5}{\sqrt{0.5^2/m}} \tag{4}$$

where $\widehat{p}_- = m_-/m$, $m = \sum_{t=L}^{T+L-1} \mathbf{1}\{TTR_t^{(S,L)} < 0\} = T-n$ and $m_- = \sum_{t=L}^{T+L-1} \mathbf{1}\{y_{t,H} < 0, TTR_t^{(S,L)} < 0\}$. Again, if $\widehat{p}_-$ is significantly greater (less) than 50 percent, it can be interpreted as a 'momentum' ('contrarian') result.

In what follows, we conduct a Monte Carlo simulation to address potential problems of applying the above three procedures in finite sample.[5]

2.2 A Monte Carlo Simulation

We adapt the prototype spurious regression setup á la Granger and Newbold (1974) in the sense that z_t is generated by a random walk process without a drift,

$$z_t = z_{t-1} + \varepsilon_t, \quad t = 1, 2, ... \tag{5}$$

[5] While more complicated variants of moving average rules are likely to be used by professionals in practice, we believe three procedures represent the core idea behind moving average rules and thus serve for our purpose of pointing out the possibilities of spurious regressions.

Here, ε_t is drawn from an independent N(0,1) population and the initial condition is given by $z_0 = 0$. The finite sample properties of the statistics from three variants of technical trading rules are examined using the artificial price series repeatedly generated from (5) in $10,000$ replications. We evaluate the effect of increasing the sample size (T) by reporting the results for $T = 100, 500$ and 1000. The length of the short period (S) is simply set to one, thus $MA_t(S) = z_t$. We also evaluate the effect of changing the ratio among the forecast horizon (H), the length of the long period (L) and the sample size (T). For this purpose, it is convenient to introduce the fixed ratio among $H : L : T$ expressed by an additional set of notations $h : \ell : 1 - h - \ell$ where $h > 0$, $\ell > 0$ and $h+\ell < 1$. In simulation, we consider all possible combinations of $h \in \{0.1, 0.2, 0.3\}$ and $\ell \in \{0.1, 0.2, 0.3\}$. Using this ratio, both H and L are given by $H = [\frac{h}{1-h-\ell}T]$ and $L = [\frac{\ell}{1-h-\ell}T]$, respectively, where $[x]$ is the integer part of x.[6]

Table 1 reports the result from the continuous technical trading regression (1). The first three blocks of columns show the means of $\widehat{\beta}_C$ and $t(\widehat{\beta}_C)$, in absolute values, and R_C^2, respectively. The fourth and fifth blocks show the frequencies of rejecting the null hypotheses of non-positive and non-negative slope coefficient, respectively, using one-sided tests with a 5 percent level of significance. There are a number of notable features observed in this table.

First, for any combination of h and ℓ, $\widehat{\beta}_C$ in absolute value shows no tendency of convergence to zero when the sample size increases. Instead, it seems to have a stable non-zero mean value which is increasing in both h and ℓ.

Second, in contrast to $\widehat{\beta}_C$, for any choice of h and ℓ, $t(\widehat{\beta}_C)$ in absolute value is evidently increasing with the sample size. Furthermore, even for the case of $T = 100$, $h = 0.1$ and $\ell = 0.1$, a combination which provides the smallest average t-value of 2.46, it still implies a very high chance of finding a significant slope coefficient. Indeed the results from the same combination reported in the fourth and fifth blocks indicate that, at the 10 percent significance level, the two-sided test rejects the null hypothesis of a zero slope coefficient for 59 ($= 10 + 49$) percent of all occasions. The largest average t-value is obtained in the case of $T = 1000$, $h = 0.3$ and $\ell = 0.3$, as a

[6] Alternatively, both H and L can be considered as fractions of the total number of observation, $T^* = T + H + L - 1$, in the limit. Namely, $H/T^* \to h$ and $L/T^* \to \ell$ as $T \to \infty$.

result of t-values not only increasing in T but also in h and ℓ. Its average t-value of 30.4 is twelve times larger than in the case of $T = 100$, $h = 0.1$ and $\ell = 0.1$. From the last two blocks, the frequency of finding a significant slope coefficient for this combination becomes as large as 97 $(= 10 + 87)$ percent!

Third, for any given values of h and ℓ, the average of R_C^2 seems quite stable irrespective of the sample size. It also shows that R_C^2 is monotonically increasing in h and ℓ, from the smallest value of 8 percent with $h = 0.1$ and $\ell = 0.1$ to the largest value of around 40 percent with $h = 0.3$ and $\ell = 0.3$. The average value of 40 percent implies that a relatively long horizon and a relatively long period in moving average can produce a very reasonable fit in the technical trading regression.

Fourth, from the comparison of the fourth and fifth blocks of columns, frequencies of (wrongly) finding a significantly negative slope coefficient are much higher than (wrongly) finding a significantly positive slope coefficient. This implies that, if regression-based technical trading rules are used, one would find empirical support for the 'contrarian' strategy more often than the 'momentum' strategy. In most cases, the rejection of the non-positive hypothesis is at least five times more frequent than the rejection of the non-negative one. Indeed, when $T = 100$, the probability of finding a significantly positive coefficient in the longest forecasting horizon case of $h = 0.3$ is almost identical to the nominal size of 5 percent, and thus the spurious effect is almost negligible. Another interesting fact is that, while the frequency of finding a positive coefficient decreases as h increases, the frequency of finding a negative coefficient is increasing in h. In addition, for smaller values of h, the frequency of finding a positive coefficient is increasing in ℓ but the frequency of finding a negative coefficient is decreasing in ℓ. However, for all cases, when both h and ℓ are fixed, the probability of finding the spurious effect increases monotonically as the sample size increases.

Table 2 reports the result from the discrete technical trading regression (2) using the same format as the continuous version. Overall, the results are more or less similar to the continuous case in terms of the dependence on T, h and ℓ, except for the behavior of the slope estimator $\widehat{\beta}_D$. When T is fixed, $\widehat{\beta}_D$ in absolute value is increasing in both h and ℓ as $\widehat{\beta}_C$ in Table 1. However, unlike $\widehat{\beta}_C$ from the continuous technical trading regression, $\widehat{\beta}_D$ in absolute value clearly increases as the sample size T increases. In this sense, there is a stronger spurious effect on the slope coefficient in the discrete case than in the continuous case. While the average absolute value of $t(\widehat{\beta}_D)$ is slightly lower than that of $t(\widehat{\beta}_C)$ for the same combination of T, h and ℓ, the ratio of the two does not depend on T, which suggests a common divergence rate of t-statistics. The coefficient of determination R_D^2 is also slightly smaller than R_C^2 but shows no sign of convergence to zero. The frequency of finding a significantly positive slope is slightly higher in the discrete case than in the continuous case. Therefore, the smaller absolute value of $t(\widehat{\beta}_D)$ is mainly due to the lower frequency of finding a significantly negative coefficient.

Lastly, Table 3 reports the result of the proportion tests (3) and (4). The upper panel of the table shows the average absolute value of t_+ and frequencies of finding the 'momentum' and 'contrarian' results based on t_+. Here, a 'momentum' ('contrarian') result corresponds to a rejection of the hypothesis that the probability of success in forecasting positive returns is less (greater) than or equal to 50 percent based on the one-sided t test at the 5 percent significance level. The lower panel of the table shows the corresponding numbers based on t_-. Note that the results using t_+ and t_- turn out to be almost indistinguishable. Similar to the regression t-statistics, the proportion t-statistics in absolute value increase as the sample size T increases. In addition, they are again monotonically increasing in both h and ℓ. The degree of dependence of rejection frequency on the change in T, h and ℓ, is also similar to the regression case. However, compared to the regression results in Tables 1 and 2 (corresponding numbers for the same combination of T, h and ℓ), the frequency of the 'momentum' results is much higher and the frequency of the 'contrarian' results is much lower. For the proportion tests, the probability of 'momentum' results is indeed more than two-thirds of the probability of 'contrarian' results for all cases. This contrasts with previous two regression cases in which 'contrarian' results are dominant.

In summary, our simulation results suggest the presence of spurious effects in forecasting future returns with all of the three technical trading rules for many different combinations of sample size, horizon length and the window length of moving average.

3 Theory

3.1 Main Result

This section provides the theoretical foundation for the results observed in the simulation. All the proofs of theorems in this section are given in the Appendix.

We first introduce the assumption of the innovation in the random walk process (5). Let $\{\mathcal{F}_t\}_1^\infty$ be a filtration adapted to the sequence $\{\varepsilon_t\}_1^\infty$.

Assumption 1

(a) *The adapted stochastic sequence* $\{\varepsilon_t, \mathcal{F}_t\}$ *is a martingale difference sequence such that* $\sigma_t^2 \equiv E(\varepsilon_t^2) < \Delta < \infty$;

(b) $E|\varepsilon_t|^{2+\delta} < \Delta' < \infty$ *for some* $\delta > 0$ *and for all* t;

(c) $\sigma^2 \equiv \lim_{T\to\infty} T^{-1} \sum_{t=1}^T \sigma_t^2$ *exists and finite and* $\sigma^2 > \delta' > 0$.

This assumption not only includes the independent and identically distributed errors considered in our simulation as a special case, but also allows for more general white noise processes.[7] Under this assumption, the FCLT holds, and as $T \to \infty$, $T^{-1/2} \sum_{t=1}^{[rT]} \varepsilon_t \Rightarrow \sigma W(r)$ where $\Rightarrow$ denotes weak convergence, and $W(r)$ is standard Browninan motion on $C[0,1]$. By generalizing this result, the asymptotic approximation for the distribution of the statistics associated with the continuous technical trading regression (1) can be obtained, which is provided in the following theorem.

Theorem 1. *Suppose the innovation sequence* $\{\varepsilon_t\}_1^\infty$ *in* (5) *satisfies Assumption 1. Let* $H = [\frac{h}{1-h-\ell}T]$ *and* $L = [\frac{\ell}{1-h-\ell}T]$, *where* $h, \ell > 0$ *and* $h + \ell < 1$, *and* $S/L \to 0$ *as* $T \to \infty$. *Then, as* $T \to \infty$,

(a) $\widehat{\beta}_C \Rightarrow \frac{\int_\ell^{1-h} W_{\ell,h}(r)V_{\ell,h}(r)\mathrm{d}r}{\int_\ell^{1-h} W_{\ell,h}^2(r)\mathrm{d}r} = \zeta_C$;

(b) $T^{-1/2}\widehat{\alpha}_C \Rightarrow a^{-3/2}\sigma \left(\int_\ell^{1-h} V_h(r)\mathrm{d}r - \zeta_C \int_\ell^{1-h} W_\ell(r)\mathrm{d}r\right)$;

[7] Note that our assumption of the martingale property of the asset price is one of the main implications of the efficient market hypothesis. In practice, however, the hypothesis has been often examined under stronger assumptions on the successive price changes. See Campbell, Lo and MacKinlay (1997) for the classification of random walk models depending on the strength of restrictions on the error term.

(c) $T^{-1/2}t(\widehat{\beta}_C) \Rightarrow \zeta_C \left(\frac{\int_\ell^{1-h} W_{\ell,h}^2(r)\mathrm{d}r}{\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r - \zeta_C^2 \int_\ell^{1-h} W_{\ell,h}^2(r)\mathrm{d}r} \right)^{1/2}$;

(d) $T^{-1/2}t(\widehat{\alpha}_C) \Rightarrow \frac{a^{-1/2}\left(\int_\ell^{1-h} V_h(r)\mathrm{d}r - \zeta_C \int_\ell^{1-h} W_\ell(r)\mathrm{d}r\right)\left(\int_\ell^{1-h} W_{\ell,h}^2(r)\mathrm{d}r\right)^{1/2}}{\left(\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r - \zeta_C^2 \int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r\right)^{1/2}\left(\int_\ell^{1-h} W_\ell^2(r)\mathrm{d}r\right)^{1/2}}$;

(e) $R_C^2 \Rightarrow \frac{\zeta_C^2 \int_\ell^{1-h} W_{\ell,h}^2(r)\mathrm{d}r}{\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r}$;

where

$$W_\ell(r) \equiv W(r) - \tfrac{1}{\ell}\int_{r-\ell}^r W(s)\mathrm{d}s, \quad W_{\ell,h}(r) \equiv W_\ell(r) - \tfrac{1}{a}\int_\ell^{1-h} W_\ell(s)\mathrm{d}s,$$
$$V_h(r) \equiv W(r+h) - W(r), \quad V_{\ell,h}(r) \equiv V_h(r) - \tfrac{1}{a}\int_\ell^{1-h} V_h(s)\mathrm{d}s,$$

and $a \equiv 1 - \ell - h.$

To derive an asymptotic approximation for the result in the simulation, we maintain the assumption of fixing the ratio of H and L to T represented by h and ℓ.[8] According to Brock, Lakonishok and LeBaron (1992), combinations such as $(S, L) = (1, 50), (1, 150), (5, 150), (1, 200)$ and $(2, 200)$ have been popularly employed in practice. In order to approximate the relationship, $S << L$, we impose an additional assumption $S/L \to 0$ which implies that the length of short-period, S, either is a fixed constant or grows at a slower rate than T (and thus at a slower rate than L). For the purpose of comparing our results with those from the original spurious regression, Theorem 1 includes the analysis of the regression intercept $\widehat{\alpha}_C$, and its t-statistic, $t(\widehat{\alpha}_C)$.

In many respects, our results of the continuous technical trading regression are analogues to Theorem 1 of Phillips (1986) which states the limiting distribution of estimators in the regression of two I(1) variables. Parts (a) and (b) of Theorem 1 show that neither $\widehat{\beta}_C$ nor $\widehat{\alpha}_C$ converges to zero in probability. Instead, $\widehat{\beta}_C$ has a non-degenerate limiting distribution, accounting for the observation of its stable mean value in the simulation irrespective of the sample size. Furthermore, the dependence of the limiting distribution on both h and ℓ also explains the fact that the mean

[8]If we fix the values of H and L instead of fixing the values of h and ℓ, it leads to different limiting distributions. However, since any asymptotic analysis is only a theoretical device to provide a robust approximation to the finite sample distribution, usefulness of each theory depends on the quality of approximation. In the current case, we find that the theory using fixed h and ℓ provides a much better approximation for our simulation findings than the alternative theory. See Richardson and Stock (1989) for a similar argument.

value differs among the choices of the forecasting horizon and the window length of moving average. As in the case of the original spurious regression, the distribution of $\widehat{\alpha}_C$ diverges at the rate $\sqrt{T}$. Parts (c) and (d) of Theorem 1 show that the distributions of conventional t-statistics, $t(\widehat{\beta}_C)$ and $t(\widehat{\alpha}_C)$, also diverge at the rate $\sqrt{T}$. The divergence of $t(\widehat{\beta}_C)$ explains the fact that the rejection frequency of the test in simulation increases with the sample size, using a fixed critical value of 1.64 from N(0,1). Finally, part (e) of Theorem 1 shows the non-degenerate distribution of the coefficient of determination R_C^2, which is consistent with our simulation evidence.

We now turn to the results of the discrete technical trading regression (2).

Theorem 2. *Suppose all of the assumptions in Theorem 1 are satisfied. Then, as* $T \to \infty$,

(a) $T^{-1/2}\widehat{\beta}_D \Rightarrow a^{-1/2}\sigma \left(\frac{\int_\ell^{1-h} V_{\ell,h}(r)U_{\ell,h}(r)\mathrm{d}r}{\int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r} \right) = a^{-1/2}\sigma\zeta_D$;

(b) $T^{-1/2}\widehat{\alpha}_D \Rightarrow a^{-3/2}\sigma \left(\int_\ell^{1-h} V_h(r)\mathrm{d}r - \zeta_D \int_\ell^{1-h} \mathbf{1}\left(W_\ell(r)\right)\mathrm{d}r \right)$;

(c) $T^{-1/2}t(\widehat{\beta}_D) \Rightarrow \zeta_D \left(\frac{\int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r}{\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r - \zeta_D^2 \int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r} \right)^{1/2}$;

(d) $T^{-1/2}t(\widehat{\alpha}_D) \Rightarrow \frac{a^{-1/2}\left(\int_\ell^{1-h} V_{\ell,h}(r)\mathrm{d}r - \zeta_D \int_\ell^{1-h} \mathbf{1}(W_\ell(r))\mathrm{d}r\right)\left(\int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r\right)^{1/2}}{\left(\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r - \zeta_D^2 \int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r\right)^{1/2}\left(\int_\ell^{1-h} \mathbf{1}(W_\ell(r))\mathrm{d}r\right)^{1/2}}$;

(e) $R_D^2 \Rightarrow \frac{\zeta_D^2 \int_\ell^{1-h} U_{\ell,h}^2(r)\mathrm{d}r}{\int_\ell^{1-h} V_{\ell,h}^2(r)\mathrm{d}r}$;

where $\mathbf{1}(x) = \mathbf{1}\{x > 0\}$ *and* $U_{\ell,h}(r) \equiv \mathbf{1}\left(W_\ell(r)\right) - \frac{1}{a}\int_\ell^{1-h} \mathbf{1}\left(W_\ell(s)\right)\mathrm{d}s$.

Derivation of the results in Theorem 2 relies on the theory of nonlinear transformation of integrated time series available in Park and Phillips (1999) because of the presence of a discrete regressor. Recall that the most distinct feature in the simulation results of the discrete technical trading regression compared to the continuous version was the growing slope coefficients. Part (a) of Theorem 2 implies the divergence of the distribution of $\widehat{\beta}_D$ and thus explains our observation in the simulation. In fact, the comparison of parts (a) and (b) shows that both $\widehat{\beta}_D$ and $\widehat{\alpha}_D$ diverge at the rate of $\sqrt{T}$. The source of this similarity between the asymptotic properties of the regression intercept and the slope is that, for both cases, the partial sums of 1's (or squares of 1's) in the denominator of the least squares estimator do not diverge as fast as the partial sums of I(1) variables (see Appendix for the detail). In spite of the divergence of $\widehat{\beta}_D$, the asymptotic behavior of associated t-statistics, $t(\widehat{\beta}_D)$, is very similar to that of $t(\widehat{\beta}_C)$. Parts (c) and (d) of Theorem 2 show that an increase in sample size leads higher t-statistics, and thus more frequent significant coefficients based on conventional critical values. Part (e) shows that a moderate value of R_D^2 on average is expected. In summary, the major conclusion on spurious effects holds irrespective of the nonlinear transformation of the regressor $TTR_t^{(S,L)}$ in technical trading regressions.

Following theorem states the asymptotic approximation of the statistical behavior of proportion tests (3) and (4).

Theorem 3. *Suppose all of the assumptions in Theorem 1 are satisfied. Then, as* $T \to \infty$,

(a) $(\widehat{p}_+, \widehat{p}_-) \Rightarrow \left(\frac{\int_\ell^{1-h} \mathbf{1}(V_h(r))\mathbf{1}(W_\ell(r))\mathrm{d}r}{\int_\ell^{1-h} \mathbf{1}(W_\ell(r))\mathrm{d}r}, \frac{\int_\ell^{1-h} [1-\mathbf{1}(V_h(r))][1-\mathbf{1}(W_\ell(r))]\mathrm{d}r}{\int_\ell^{1-h} [1-\mathbf{1}(W_\ell(r))]\mathrm{d}r} \right) = (\kappa_+, \kappa_-)$;

(b) $\left(T^{-1/2}t_+, T^{-1/2}t_-\right)$

$\Rightarrow \left(\frac{\kappa_+ - 0.5}{0.5a^{1/2}} \left[\int_\ell^{1-h} \mathbf{1}\left(W_\ell(r)\right) \mathrm{d}r \right]^{1/2}, \frac{\kappa_- - 0.5}{0.5a^{1/2}} \left[\int_\ell^{1-h} [1 - \mathbf{1}\left(W_\ell(r)\right)] \mathrm{d}r \right]^{1/2} \right) = (\nu_+, \nu_-)$;

(c) $\kappa_+ \overset{d}{=} \kappa_-$ *and* $\nu_+ \overset{d}{=} \nu_-$ *where the symbol* $\overset{d}{=}$ *denotes that the two random variables have the same marginal distribution.*

Part (a) of Theorem 3 shows the non-degenerate distributions of the observed proportions of successes, $\widehat{p}_+$ and $\widehat{p}_-$, instead of their convergence to a fixed value of 0.5. Part (b) shows that, once again, divergence of the distribution of the conventional proportion t-statistics, t_+ and t_-, which accounts for the observed increasing frequency of finding a significant deviation from 0.5 for the tests with a larger sample size. It should be noted that the asymptotic results for the pair of the statistics (part (b)), as well as for the pair of proportions (part (a)), imply weak convergence jointly to a pair of random variables. However, part (c) shows that two limit random variables have the same marginal distribution. This explains why almost identical results between t_+ and t_- were obtained in our simulation.

3.2 Discussion

In Theorems 1 to 3, we have theoretically shown that all three technical trading rules we consider can potentially produce statistically significant evidence of forecastability even if the past information actually has no predictive power. Let us now discuss whether we can reduce the risk of obtaining such a false conclusion in practice, by taking our theoretical results into consideration. In what follows, we point out two feasible procedures which may be useful in avoiding the spurious regressions.

The first procedure utilizes the fact that all three t-statistics, namely, $t(\widehat{\beta}_C)$, $t(\widehat{\beta}_D)$, and t_+ (or t_-), converge to well-defined distributions once they are normalized by $\sqrt{T}$. Each graph in the first column of Figure 1 displays the densities of $t(\widehat{\beta}_C)$, $t(\widehat{\beta}_D)$, and t_+, respectively, from the simulation result in the previous section assuming $h = 0.1$ and $\ell = 0.3$. In all three cases, the observed densities are much more dispersed for $T = 500$ than for $T = 100$. The second column of Figure 1 displays densities of the same statistics rescaled by $\sqrt{T}$ as suggested by Theorems 1 to 3. The shapes of the densities of rescaled t-statistics for $T = 100$ are very similar to those for $T = 500$. This fact suggests that our asymptotic approximation works very well in finite sample even when the sample size is as small as $T = 100$. Therefore, we may use the limiting distribution of the rescaled t-statistics to conduct a test on the slope coefficient and the proportion of successes in sign prediction.

A similar rescaling argument can be found in Phillips (1986) in his original study of spurious regressions. The rescaled t-statistic has also been used by Valkanov (2003) in the context of long-horizon regressions. In our case, while the limiting distributions differ depending on h and ℓ, they are known in each application. The limiting distributions of our rescaled t-statistics are, therefore, free of nuisance parameters. For each of three statistics, the critical values for all possible combinations of $h \in \{0.1, 0.2, 0.3\}$ and $\ell \in \{0.1, 0.2, 0.3\}$ are tabulated in Tables 4 to 6.[9] Among the three limiting distributions, the one for the proportion test t_+ has its median closer to zero than the other two, which implies that probabilities of obtaining 'momentum' and 'contrarian' results in the limit are closest with this test.

[9] They are obtained from 10,000 iterations of generating Brownian motion approximated by partial sums of standard normal random variables with 10,000 steps.

The second procedure we consider is to normalize t-statistics using the heteroskedasticity and autocorrelation consistent (HAC) standard errors, which are often used to construct a test robust to the presence of serially correlated errors in a regression model. For example, the HAC t-statistic for $\widehat{\beta}_C$ is defined as $t_{HAC}(\widehat{\beta}_C) \equiv \widehat{\beta}_C / s_{HAC}(\widehat{\beta}_C)$, where

$$s_{HAC}(\widehat{\beta}_C) \equiv T^{1/2}\widehat{\omega}\left(\sum_{t=L}^{L+T-1}\left\{\widetilde{TTR}_t^{(S,L)}\right\}^2\right)^{-1}, \quad \widehat{\omega}^2 \equiv \sum_{j=-T+1}^{T-1} k\left(\tfrac{j}{M}\right)\widehat{\gamma}(j),$$

$$\widehat{\gamma}(j) \equiv \begin{cases} \frac{1}{T}\sum\limits_{t=L}^{T+L-1-j} \widetilde{TTR}_{t+j}^{(S,L)}\widehat{u}_{t+j}\widehat{u}_t\widetilde{TTR}_t^{(S,L)} & \text{for} \quad j \geq 0 \\ \frac{1}{T}\sum\limits_{t=-j+L}^{T+L-1} \widetilde{TTR}_{t+j}^{(S,L)}\widehat{u}_{t+j}\widehat{u}_t\widetilde{TTR}_t^{(S,L)} & \text{for} \quad j < 0, \end{cases}$$

$\widetilde{TTR}_t^{(S,L)} \equiv TTR_t^{(S,L)} - T^{-1}\sum_{i=L}^{L+1-T} TTR_i^{(S,L)}$, $k(x)$ is a kernel function and M is the bandwidth parameter. In a classical regression model, the HAC t-statistic converges to a standard normal distribution when M grows at a rate slower than T, but as shown by Kiefer and Vogelsang (2002), it converges to a nonstandard distribution when M grows at the same rate as T. Later, Sun (2004) pointed out that when Kiefer and Vogelsang's (2002) asymptotic approximation is used in the original spurious regression with I(1) variables, the non-rescaled t-statistic becomes convergent instead of diverging at the rate $\sqrt{T}$. We close this section by showing that $t_{HAC}(\widehat{\beta}_C)$ in the technical trading regression also becomes convergent if the bandwidth grows at the rate proportional to sample size.

Theorem 4. *Suppose all of the assumptions in Theorem 1 hold,* $M = bT$, $b \in (0,1]$ *and the kernel function belongs to the following class*: $\mathcal{K} = \{k(\cdot) : \mathbb{R} \to [-1,1] | k(0) = 1,\ k(x) = k(-x),\ \forall x \in \mathbb{R},\ \int_{-\infty}^{\infty} k^2(x)\mathrm{d}x < \infty,$ *and* $k(\cdot)$ *is continuous at* 0 *as well as all but a finite number of other points*$\}$. *Then, as* $T \to \infty$,

$$t_{HAC}(\widehat{\beta}_C) \Rightarrow \frac{\int_\ell^{1-h} W_{\ell,h}(r)V_{\ell.h}(r)\mathrm{d}r}{\left(\int_\ell^{1-h}\int_\ell^{1-h} W_{\ell,h}(r)Q_{\ell,h}(r)k\left(\frac{r-s}{b}\right)W_{\ell,h}(s)Q_{\ell,h}(s)\mathrm{d}r\mathrm{d}s\right)^{1/2}},$$

$$\textit{where} \quad Q_{\ell,h}(r) \equiv V_h(r) - \zeta_C W_\ell(r) - \frac{1}{a}\left(\int_\ell^{1-h} V_h(r)\mathrm{d}r - \zeta_C\int_\ell^{1-h} W_\ell(r)\mathrm{d}r\right).$$

The class of the kernel allowed in Theorem 4 is fairly general and includes most of the kernel functions commonly used in practice. As in the result of Theorem 1, the limiting distribution of HAC t-statistic depends both on h and ℓ. Here, in addition, it depends on b and the choice of the kernel function $k(x)$. Since HAC standard errors are primarily used in the regression framework, a similar HAC t-statistic can be constructed for $\widehat{\beta}_D$, but not for the proportion test.[10]

4 Empirical Applications

In this section, we apply our proposed procedures to the stock price index and foreign exchange rate data to reconsider the predictive power of technical trading rules. In particular, we examine the daily closing spot rates (at 5:00PM) of the Tokyo stock price index (TOPIX) and the yen/dollar exchange rate in the Tokyo market for all trading days over the 2004-2006 period.[11] To construct $TTR_t^{(S,L)}$, we employ five combinations of lengths of the short and long period considered by Brock, Lakonishok and LeBaron (1992), namely, $(S, L) = (1, 50), (1, 150), (5, 150), (1, 200)$ and $(2, 200)$. The upper graphs in Figures 2 and 3 present the TOPIX and yen/dollar rate exchange rate series from 2005-2006, respectively, along with the long moving average with its length $L = 50$. The lower graph in each figure shows constructed $TTR_t^{(1,50)}$ series. Note that all of the price series are expressed in logs in computing the moving averages.[12] In this empirical exercise, since the critical values of the rescaled t-statistics for the three procedures in Tables 4 to 6 are only available for selected pairs of h and ℓ, we choose the sample size T and the forecasting horizon H as follows. For all five pairs of (S, L), we first set $h = 0.1$. We then set $\ell = 0.1$ for $(S, L) = (1, 50)$, $\ell = 0.2$ for the pairs with $L = 150$, and $\ell = 0.3$ for the pairs with $L = 200$. This leads to combinations of $(H, T) = (50, 400), (75, 525)$ and $(66, 400)$, respectively in each of the three cases of long periods. Initial observations in the forecasting regression and in

[10] While not reported in the paper, we tabulated the critical values of $t_{HAC}(\widehat{\beta}_C)$ and $t_{HAC}(\widehat{\beta}_D)$ using the Bartlett kernel with $b = 1$. The critical values for all possible combinations of h and ℓ are available upon request from the authors.

[11] Using only two to three years of the sample is considered less subject to the problem of structural changes in the estimation than in the case of using a longer sample period. This also justfies the validity of our asymptotic approximation under the assumption of large H and L relative to the sample size T.

[12] Therefore, our moving average can be considered as the geometric average of the price level rather than the arithmetic avereage of the price level.

the computation of the proportion of successes are selected so that the final prediction period becomes 2006/12/29. For the HAC t-statistics, the Bartlett kernel is employed along with a large bandwidth assumption $b = 1$. Since the HAC t-statistics are shown to be convergent in our theoretical analysis, the critical values are simply taken from Kiefer and Vogelsang (2002), instead of from our new computations.

Table 7 presents the results from the TOPIX series. For the continuous technical trading regression, the slope coefficient for $(S, L) = (1, 50)$ is positive and is significantly different from zero at the 5 percent level based on a conventional t-statistic. For $(S, L) = (1, 200)$ and $(2, 200)$, in contrast, the slope becomes negative and is significant at the same level. This interesting observation of switching from a 'momentum' result to a 'contrarian' result seems to be consistent with our simulation evidence where increasing the window length of moving average reduces the probability of the 'momentum' result and raises the probability of the 'contrarian' result. When the rescaled t-statistics and the large bandwidth HAC t-statistics are used, two-sided tests cannot reject the hypothesis of a zero slope coefficient. For the discrete technical trading regression, the slope coefficients are positive in all five cases, but none of the two-sided tests using the t-statistics, the rescaled t-statistics and the HAC t-statistics provides significant 'momentum' results. For both continuous and discrete regressions, R^2's are very small. With the conventional proportion t-statistics based on the positive sign of $TTR_t^{(S,L)}$, significant 'momentum' results are obtained for all five cases. All of the significant results, however, disappear if the conventional test is replaced by the new test based on the rescaled proportion t-statistic.

Table 8 presents the corresponding results from the yen/dollar exchange rate. For both continuous and discrete technical trading regressions, all of the regression slope estimates are negative and are significantly different from zero based on conventional t-statistics. The coefficient of determination is often greater than 10 percent in our sample. Thus, one may easily interpret this evidence as empirical support of the 'contrarian' strategy. When the rescaled t-statistics are used, all of the 'contrarian' results become insignificant. When the HAC t-statistics are used, instead, only four out of ten cases remain significant.[13] The proportions of successes in detecting signs

[13] Our new critical values of the HAC t-statistics are larger in absolute value than those of Kiefer

are also greater than 0.5, and the conventional proportion tests suggest that these deviations are significant in the cases of $(S, L) = (1, 50)$, $(1, 200)$ and $(2, 200)$. When the rescaled proportion t-statistics are used, again all of the 'momentum' results become insignificant.

Note that the forecasting horizon considered in this empirical example is approximately two to three months. The survey results show that a significant number of practitioners still rely on technical trading rules in their decision making even when the trading horizons are within this range. Our analysis suggests that there is a need for careful investigation before reaching to a conclusion on the predictability of the technical trading rules at a relatively long horizon.

5 Conclusion

The popularity of technical trading rules among dealers in both equity and foreign exchange markets has long been considered a puzzle because of its violation of the efficient market hypothesis. In this paper, we pointed out that the technical trading regression used for asset return forecasts at a relatively long horizon and the classical spurious regression problem considered by Granger and Newbold (1974) and Phillips (1986) have many features in common.

Our simulation showed that, even if the price follows a random walk, and thus the past information has no predictive power as the market efficiency suggests, buy or sell signals constructed from the difference between the short-period and long-period moving averages of past values are wrongly statistically significant in most of the occasions. Furthermore, R^2 could not be trusted as a measure of regression fit in the usual sense.

In the theoretical analysis, our asymptotic approximation turned out to be useful in explaining the simulation findings. It revealed that both 'momentum' and 'contrarian' results could be falsely found in practice, while the probability of obtaining each result depends on (i) the forecast horizon, (ii) the window length of the long-period moving average, and (iii) the type of the test statistics employed. We also

and Vogelsang (2002). If we use the new critical values instead, all ten cases become insignificant.

introduced two procedures which potentially reduce the risk of spurious effects in technical trading regressions, namely, the rescaled t-statistic and the HAC t-statistic. When these methods were applied to the stock market index and foreign exchange rate, the predictive power of technical trading rules was insignificant in many cases.

A Appendix

Denote $TTR_t^{(S,L)}$ and $y_{t,H}$ by x_t and y_t, respectively, for notational simplicity. In the proofs, the summation and the integral are taken from $t = L$ to $T + L - 1$ and from ℓ to $1 - h$, respectively, unless otherwise stated, and we suppress the argument of continuous time, r, where there is no ambiguity for notational convenience. For example, $W_\ell(r)$ and $\int W_\ell(r)\mathrm{d}r$ are written, respectively, as W_ℓ and $\int W_\ell$.

Lemma 1

Suppose that ε_t satisfies Assumption 1 and $S/L \to 0$ as $T \to \infty$, then

$$\text{(a)}\quad T^{-\frac{3}{2}}\sum_{t=L}^{T+L-1} x_t \Rightarrow a^{-\frac{3}{2}}\sigma\int_{\ell}^{1-h} W_\ell(r)\mathrm{d}r,\quad T^{-\frac{3}{2}}\sum_{t=L}^{T+L-1} y_t \Rightarrow a^{-\frac{3}{2}}\sigma\int_{\ell}^{1-h} V_h(r)\mathrm{d}r,$$

$$\text{(b)}\quad T^{-2}\sum_{t=L}^{T+L-1} x_t^2 \Rightarrow a^{-2}\sigma^2\int_{\ell}^{1-h} W_\ell^2(r)\mathrm{d}r,\quad T^{-2}\sum_{t=L}^{T+L-1} y_t^2 \Rightarrow a^{-2}\sigma^2\int_{\ell}^{1-h} V_h^2(r)\mathrm{d}r,$$

$$\text{(c)}\quad T^{-2}\sum_{t=L}^{T+L-1} (x_t-\bar{x})^2 \Rightarrow a^{-2}\sigma^2\int_{\ell}^{1-h} W_{\ell,h}^2(s)\mathrm{d}s,\quad T^{-2}\sum_{t=L}^{T+L-1} (y_t-\bar{y})^2 \Rightarrow a^{-2}\sigma^2\int_{\ell}^{1-h} V_{\ell,h}^2(s)\mathrm{d}s,$$

$$\text{(d)}\quad T^{-2}\sum_{t=L}^{T+L-1} (x_t-\bar{x})(y_t-\bar{y}) \Rightarrow a^{-2}\sigma^2\int_{\ell}^{1-h} W_{\ell,h}(r)V_{\ell,h}(r)\mathrm{d}r,$$

where

$$W_\ell(r) \equiv W(r) - \frac{1}{\ell}\int_{r-\ell}^{r} W(s)\mathrm{d}s,\quad W_{\ell,h}(r) \equiv W_\ell(r) - \frac{1}{a}\int_{\ell}^{1-h} W_\ell(s)\mathrm{d}s,$$

$$V_h(r) \equiv W(r+h) - W(r),\quad V_{\ell,h}(r) \equiv V_h(r) - \frac{1}{a}\int_{\ell}^{1-h} V_h(s)\mathrm{d}s,$$

$W(r)$ is a standard Brownian motion on $[0,1]$, $\bar{x} \equiv T^{-1}\sum_{t=L}^{T+L-1} x_t$,

$\bar{y} \equiv T^{-1}\sum_{t=L}^{T+L-1} y_t$ and $a \equiv 1 - h - \ell$.

Proof of Lemma 1

Let $T^* \equiv L + T + H - 1$, i.e., the entire number of observations. In the proof, we use T^* instead of T for the normalization. It is easy to obtain the results with T as in the Lemma 1 by multiplying $(T/T^*)^k$, which converges to a^k, where $k = -3/2$ or -2. For example, $T^{-3/2}\sum x_t$ is written as $(T/T^*)^{-3/2}T^{*-3/2}\sum x_t$.

First, we provide the two results given in equations (A.1) and (A.2), which are repeatedly used in the rest of the proofs. Notice that x_t can be written as $x_t = w_{t,L} - w_{t,S}$, where

$$w_{t,L} \equiv z_t - \frac{1}{L}(z_t + z_{t-1} + \cdots + z_{t-L+1}) \quad\text{and}\quad w_{t,S} \equiv z_t - \frac{1}{S}(z_t + z_{t-1} + \cdots + z_{t-S+1}).$$

Write

$$\begin{aligned} w_{t,L} &= \tfrac{1}{L}(z_t - z_{t-1}) + \tfrac{1}{L}(z_t - z_{t-2}) + \cdots + \tfrac{1}{L}(z_t - z_{t-L+1}) \\ &= \tfrac{1}{L}\varepsilon_t + \tfrac{1}{L}(\varepsilon_t + \varepsilon_{t-1}) + \cdots + \tfrac{1}{L}(\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_{t-L+2}) \\ &= \tfrac{1}{L}\sum_{j=t-L+1}^{t-1}\left(\sum_{i=1}^{t}\varepsilon_i - \sum_{i=1}^{j}\varepsilon_i\right). \end{aligned}$$

By the FCLT for a Martingale difference sequence (MDS) and the continuous mapping theorem

(CMT), we have

$$\begin{aligned} T^{*-\frac{1}{2}} w_{t,L} &= \tfrac{T^*}{L} T^{*-1} \sum_{j=t-L+1}^{t-1} T^{*-\frac{1}{2}} \left(\sum_{i=1}^{t} \varepsilon_i - \sum_{i=1}^{j} \varepsilon_i \right) \\ &\Rightarrow \tfrac{\sigma}{\ell} \int_{r-\ell}^{r} [W(r) - W(s)] \mathrm{d}s \\ &= \sigma W_\ell(r), \end{aligned}$$

where r is determined so that $t = [rT^*]$. Next, we show that $w_{t,S} \xrightarrow{p} 0$. Write

$$\begin{aligned} w_{t,S} &= \tfrac{1}{S}(z_t - z_{t-1}) + \tfrac{1}{S}(z_t - z_{t-2}) + \cdots + \tfrac{1}{S}(z_t - z_{t-S+1}) \\ &= \tfrac{1}{S} \sum_{j=1}^{S-1} \sum_{i=1}^{j} \varepsilon_{t-i+1} \\ &= \tfrac{1}{S} \sum_{i=1}^{S-1} i \varepsilon_{t-S+1+i}. \end{aligned}$$

Since $E(\varepsilon_t) = 0 \ \forall t$, we have $E(T^{*-1/2} w_{t,S}) = 0$. Since $\{\varepsilon_t\}$ is a MDS with variances bounded by Δ, we have, if $S/L \to 0$ as $T \to \infty$,

$$\mathrm{var}(T^{*-\frac{1}{2}} w_{t,S}) \leq \frac{1}{T^*} \frac{\Delta}{S^2} \sum_{i=1}^{S-1} i^2 = \frac{1}{T^*} \frac{\Delta}{S^2} \frac{S(S-1)(2S-1)}{6} \approx \frac{S}{T^*} = \frac{S}{L} \frac{L}{T^*} \to 0,$$

where $\approx$ denotes that the both sides have asymptotically the same order as $T^* \to \infty$. Hence, we have

$$\begin{aligned} T^{*-\frac{1}{2}} x_t &= T^{*-\frac{1}{2}} w_{t,L} - T^{*-\frac{1}{2}} w_{t,S} \\ &\Rightarrow \sigma W_\ell(r). \end{aligned} \tag{A.1}$$

By the FCLT, we have

$$\begin{aligned} T^{*-\frac{1}{2}} y_t &= T^{*-\frac{1}{2}} (z_{t+h} - z_t) \\ &= T^{*-\frac{1}{2}} \sum_{i=1}^{t+h} \varepsilon_i - T^{*-\frac{1}{2}} \sum_{i=1}^{t} \varepsilon_i \\ &\Rightarrow \sigma W(r+h) - \sigma W(r) \\ &= \sigma V_h(r). \end{aligned} \tag{A.2}$$

For part (a), using the result in (A.1) and applying the CMT, we obtain

$$\begin{aligned} T^{*-\frac{3}{2}} \sum x_t &= T^{*-1} \sum T^{*-\frac{1}{2}} x_t \\ &\Rightarrow \sigma \int W_\ell. \end{aligned}$$

Multiplying both sides by $(T/T^*)^{-3/2}$ gives the first result of part (a). Arguments entirely analogous to those of the proof of the first result of part (a) yield the second result of part (a) and part (b).

For part(c), from (A.1) and the CMT, we have

$$\begin{aligned} T^{*-\frac{1}{2}} (x_t - \bar{x}) = &\ T^{*-\frac{1}{2}} x_t - \tfrac{T^*}{T} T^{*-1} \sum T^{*-\frac{1}{2}} x_t \\ \Rightarrow &\ \sigma W_\ell(r) - a^{-1} \sigma \int W_\ell(r) \mathrm{d}r \\ = &\ \sigma W_{\ell,h}(r), \end{aligned} \tag{A.3}$$

and so

$$\begin{aligned} T^{*-2} \sum (x_t - \bar{x})^2 = &\ T^{*-1} \sum \left[T^{*-\frac{1}{2}} (x_t - \bar{x}) \right]^2 \\ \Rightarrow &\ \sigma^2 \int W_{\ell,h}^2. \end{aligned} \tag{A.4}$$

Similarly, we have

$$T^{*-\frac{1}{2}} (y_t - \bar{y}) \Rightarrow \sigma V_{\ell,h}(r) \tag{A.5}$$

and

$$T^{*-2}\sum(y_t-\bar{y})^2 \Rightarrow \sigma^2\int V_{\ell,h}^2. \tag{A.6}$$

Multiplying both sides in (A.4) and (A.6) by $(T/T^*)^{-2}$ gives part (c).

Part (d) follows from (A.3), (A.5) and the CMT.

Proof of Theorem 1

Using Lemma 1(a), (c) and (d), we obtain

$$\widehat{\beta}_C = \frac{T^{-2}\sum(y_t-\bar{y})(x_t-\bar{x})}{T^{-2}\sum(x_t-\bar{x})^2} \Rightarrow \frac{\int W_{\ell,h}V_{\ell,h}}{\int W_{\ell,h}^2}$$

and

$$\begin{aligned} T^{-1/2}\widehat{\alpha}_C = &\ T^{-1/2}(\bar{y}-\widehat{\beta}_C\bar{x}) \\ = &\ T^{-3/2}\sum y_t - \widehat{\beta}_C T^{-3/2}\sum x_t \\ \Rightarrow &\ a^{-3/2}\sigma\left(\int V_h - \zeta_C\int W_\ell\right), \end{aligned}$$

which complete the proof of parts (a) and (b). Next, define $\widehat{\sigma}_C^2 \equiv T^{-1}\sum(y_t-\widehat{\alpha}_C-\widehat{\beta}_C x_t)^2$. Since $\widehat{\alpha}_C = \bar{y}-\widehat{\beta}_C\bar{x}$, using Lemma 1(c) and part (a), we have

$$\begin{aligned} T^{-1}\widehat{\sigma}_C^2 = &\ T^{-2}\sum\left[(y_t-\bar{y})-\widehat{\beta}_C(x_t-\bar{x})\right]^2 \\ = &\ T^{-2}\sum(y_t-\bar{y})^2 - \widehat{\beta}_C^2 T^{-2}\sum(x_t-\bar{x})^2 \\ \Rightarrow &\ a^{-2}\sigma^2\left[\int V_{\ell,h}^2 - \zeta_C^2\int W_{\ell,h}^2\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} T^{-1/2}t(\widehat{\beta}_C) &= \frac{\widehat{\beta}_C}{T^{1/2}\widehat{\sigma}_C\left[\sum(x_t-\bar{x})^2\right]^{-1/2}} \\ &= \frac{\widehat{\beta}_C\left[T^{-2}\sum(x-\bar{x})^2\right]^{1/2}}{T^{-1/2}\widehat{\sigma}_C} \\ &\Rightarrow \frac{\zeta_C a^{-1}\sigma\left[\int W_{\ell,h}^2\right]^{1/2}}{a^{-1}\sigma\left[\int V_{\ell,h}^2-\zeta_C^2\int W_{\ell,h}^2\right]^{1/2}} \end{aligned}$$

and

$$\begin{aligned} T^{-1/2}t(\widehat{\alpha}_C) &= \frac{\widehat{\alpha}_C\left[T\sum(x_t-\bar{x})^2\right]^{1/2}}{(T^{1/2}\widehat{\sigma}_C)\left(\sum x_t^2\right)^{1/2}} \\ &= \frac{(T^{-1/2}\widehat{\alpha}_C)\left[T^{-2}\sum(x_t-\bar{x})^2\right]^{1/2}}{(T^{-1/2}\widehat{\sigma}_C)\left(T^{-2}\sum x_t^2\right)^{1/2}} \\ &\Rightarrow \frac{a^{-3/2}\sigma\left(\int V_h-\zeta_C\int W_\ell\right)a^{-1}\sigma\left(\int W_{\ell,h}^2\right)^{1/2}}{a^{-1}\sigma\left(\int V_{\ell,h}^2-\zeta_C^2\int W_{\ell,h}^2\right)^{1/2}a^{-1}\sigma\left(\int W_\ell^2\right)^{1/2}}, \end{aligned}$$

as required for parts (c) and (d).

Finally, we prove part (e). Let $\widehat{y}_{C,t} \equiv \widehat{\alpha}_C + \widehat{\beta}_C x_t$. Then, from part (a) and Lemma 1(c), we have

$$R_C^2 = \frac{\sum(\widehat{y}_t-\bar{y})^2}{\sum(y_t-\bar{y})^2} = \frac{\widehat{\beta}_C^2 T^{-2}\sum(x_t-\bar{x})^2}{T^{-2}\sum(y_t-\bar{y})^2} \Rightarrow \frac{\zeta_C^2\int W_{\ell,h}^2}{\int V_{\ell,h}^2}.$$

Lemma 2

Suppose that ε_t satisfies Assumption 1 and $S/L \to 0$ as $T \to \infty$, then

$$\text{(a)} \quad T^{-1} \sum_{t=L}^{T+L-1} \mathbf{1}\left(T^{-\frac{1}{2}}x_t\right) \Rightarrow a^{-1} \int_{\ell}^{1-h} \mathbf{1}\left(W_\ell(r)\right) \mathrm{d}r \quad ,$$

$$\text{(b)} \quad T^{-1} \sum_{t=L}^{T+L-1} \mathbf{1}\left(T^{-\frac{1}{2}}x_t\right) \mathbf{1}\left(T^{-\frac{1}{2}}y_t\right) \Rightarrow a^{-1} \int_{\ell}^{1-h} \mathbf{1}\left(W_\ell(r)\right) \mathbf{1}\left(V_h(r)\right) \mathrm{d}r \quad ,$$

$$\text{(c)} \quad T^{-1} \sum_{t=L}^{T+L-1} \mathbf{1}\left(T^{-\frac{1}{2}}x_t\right) T^{-\frac{1}{2}} y_t \Rightarrow a^{-\frac{3}{2}} \sigma \int_{\ell}^{1-h} \mathbf{1}\left(W_\ell(r)\right) V_h(r) \mathrm{d}r \quad ,$$

$$\text{(d)} \quad T^{-1} \sum_{t=L}^{T+L-1} \left[\mathbf{1}\left(T^{-\frac{1}{2}}x_t\right) - T^{-1} \sum_{t=L}^{T+L-1} \mathbf{1}\left(T^{-\frac{1}{2}}x_t\right)\right]^2 \Rightarrow a^{-1} \int_{\ell}^{1-h} U_{\ell,l}^2(r) \mathrm{d}r \quad ,$$

$$\text{(e)} \quad T^{-\frac{3}{2}} \sum_{t=L}^{T+L-1} (y_t - \bar{y}) \left[\mathbf{1}\left(T^{-\frac{1}{2}}x_t\right) - T^{-1} \sum_{t=L}^{T+L-1} \mathbf{1}\left(T^{-\frac{1}{2}}x_t\right)\right] \Rightarrow a^{-\frac{3}{2}} \sigma \int_{\ell}^{1-h} V_{\ell,h}(r) U_{\ell,h}(r) \mathrm{d}r \quad ,$$

where $\mathbf{1}(x) = \mathbf{1}\{x > 0\}$ is an indicator function that takes 1 if $x > 0$ and 0 otherwise, and

$$U_{\ell,h}(r) \equiv \mathbf{1}\left(W_\ell(r)\right) - a^{-1} \int_{\ell}^{1-h} \mathbf{1}\left(W_\ell(s)\right) \mathrm{d}s.$$

Hereafter, we denote $\mathbf{1}(T^{-1/2}x_t)$ and $\mathbf{1}(T^{-1/2}y_t)$ by $\mathbf{1}_x$ and $\mathbf{1}_y$, respectively, for notational simplicity. That is, $T^{*-1}\sum \mathbf{1}(T^{-1/2}x_t)$ is denoted by $T^{*-1}\sum \mathbf{1}_x$, etc.

Proof of Lemma 2

The proof of part (a) is entirely analogue to the proof of Theorem 3.2 in Park and Phillips (1999). Recall that, ℓ and h are defined so that $\ell : h : 1 - \ell - h = L : H : T$, and so the ratio of the window length of the long moving average to the entire number of observations, i.e., L/T^* is given by $[\ell(T^* + 1)]/T^*$. Similarly, H/T^* is given by $[h(T^* + 1)]/T^*$. We denote the former by ℓ^* and the latter by h^*. Since $\ell^* \to \ell$ and $h^* \to h$ as T^* (or T) $\to \infty$, they can be asymptotically replaced by ℓ and h, respectively. We do so without mentioning each time.

To prove part (a), we define a stochastic process $W_{\ell,T}^0(r) \equiv a^{1/2} x_{[rT^*]}/(\sigma\sqrt{T})$ on $[\ell^*, 1 - h^*]$. From (A.1), we have

$$W_{\ell,T}^0 = a^{\frac{1}{2}} \frac{\sqrt{T^*}}{\sqrt{T}} \frac{x_{[rT^*]}}{\sigma\sqrt{T^*}} \Rightarrow W_\ell.$$

It then follows from the so-called Skorohod representation theorem (see Theorem 25.6, Billingsley, 1995, p. 333) that there exists $W_{\ell,T}$ such that $W_{\ell,T} \overset{d}{=} W_{\ell,T}^0$ in $D[\ell^*, 1 - h^*]$ and $W_{\ell,T} \overset{a.s.}{\to} W_\ell$ uniformly on $[\ell^*, 1 - h^*]$. Using the Skorohod representation and noting that $\mathbf{1}(x) = \mathbf{1}(cx)$ for any positive constant c, we can write

$$T^{*-1} \sum \mathbf{1}_x = T^{*-1} \sum \mathbf{1}\left(W_{\ell,T}^0(t/T^*)\right) \overset{d}{=} \int_{\ell^*}^{1-h^*} \mathbf{1}\left(W_{\ell,T}\right).$$

Define $\underline{\mathbf{1}}_\epsilon(x) \equiv \mathbf{1}(x - \epsilon)$ and $\overline{\mathbf{1}}_\epsilon(x) \equiv \mathbf{1}(x + \epsilon)$ for $\epsilon > 0$. On every compact set C, there exists, for each $\epsilon > 0$, $\delta_\epsilon > 0$ such that $\underline{\mathbf{1}}_\epsilon(x) \leq \mathbf{1}(y) \leq \overline{\mathbf{1}}_\epsilon(x)$ for all $x, y \in C$ such that $|x - y| < \delta_\epsilon$. Clearly, we have

$$\int_C (\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(x) \mathrm{d}x \to 0, \tag{A.7}$$

as $\epsilon \to 0$, where $(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(x) \equiv \overline{\mathbf{1}}_\epsilon(x) - \underline{\mathbf{1}}_\epsilon(x)$. Let $C_W = [s_{w,\min} - 1, s_{w,\max} + 1]$, where $s_{w,\min}$ and $s_{w,\max}$ are defined as $s_{w,\max} \equiv \sup_{\ell \leq r \leq 1-h} W_\ell(r)$ and $s_{w,\min} \equiv \inf_{\ell \leq r \leq 1-h} W_\ell(r)$, respectively. We may take T^* (or T) sufficiently large so that $\sup_{\ell \le r \le 1-h} |W_{\ell,T}(r) - W_\ell(r)| < \delta_\epsilon$ for any $\delta_\epsilon > 0$ almost surely, and so that both $W_{\ell,T}$ and W_ℓ are in C_W, and

$$\underline{\mathbf{1}}_\epsilon(W_\ell) \le \mathbf{1}(W_{\ell,T}) \le \overline{\mathbf{1}}_\epsilon(W_\ell), \quad a.s. \tag{A.8}$$

Note that $W_\ell(r)$ is defined only on $[\ell, 1-h]$. Since W_ℓ is a continuous semimartingale process, it satisfies the so-called occupation times formula (see Corollary 1.6, Revuz and Yor, 1999, p. 224),

$$\int_\ell^t \Phi(W_\ell(r))\mathrm{d}r = \int_{-\infty}^{\infty} \Phi(s)L(t-\ell, s)\mathrm{d}s,$$

for every $t \in [\ell, 1-h]$ and every positive Borel function Φ, where $L(t-\ell, s)$ is the local time spent by W_ℓ at the spatial point s over the interval $[\ell, t]$. Thus, we have

$$\begin{aligned} \textstyle\int_\ell^{1-h}(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(W_\ell(r))\mathrm{d}r \quad &= \textstyle\int_{-\infty}^{\infty}(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(s)L(1-h-\ell, s)\mathrm{d}s \\ &\le \textstyle\sup_s L(1-h-\ell, s)\int_{C_W}(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(x)\mathrm{d}x \\ &\overset{a.s.}{\to} 0, \end{aligned} \tag{A.9}$$

as $\epsilon \to 0$ due to (A.7). Part (a) follows from (A.8), (A.9) and multiplying both sides by $(T^*/T)^{-1}$.

For part (b), note that, on every compact set $C^2 \equiv C \times C$, there exists, for each $\epsilon > 0$, δ_ϵ such that $\underline{\mathbf{1}}_\epsilon(x_1)\underline{\mathbf{1}}_\epsilon(x_2) \le \mathbf{1}(y_1)\mathbf{1}(y_2) \le \overline{\mathbf{1}}_\epsilon(x_1)\overline{\mathbf{1}}_\epsilon(x_2)$ for all $(x_1, x_2), (y_1, y_2) \in C^2$ such that $|x_1 - y_1| < \delta_\epsilon$, and $|x_2 - y_2| < \delta_\epsilon$ because $\mathbf{1}(\cdot) \ge 0$. Define $V_{h,T}^0(r) \equiv a^{1/2}y_{[rT^*]}/(\sigma\sqrt{T})$ on $[\ell^*, 1-h^*]$. From (A.2), we have $V_{h,T}^0 \Rightarrow V_h$. Using the Skorohod representation, write

$$\begin{aligned} T^{*-1}\textstyle\sum \mathbf{1}_x\mathbf{1}_y = \quad & T^{*-1}\textstyle\sum \mathbf{1}\left(W_{\ell,T}^0(t/T^*)\right)\mathbf{1}\left(V_{h,T}^0(t/T^*)\right) \\ \overset{d}{=} \quad & \textstyle\int_{\ell^*}^{1-h^*}\mathbf{1}\left(W_{\ell,T}\right)\mathbf{1}\left(V_{h,T}\right) \end{aligned}$$

where $V_{h,T}$ satisfies $V_{h,T} \overset{d}{=} V_{h,T}^0$, and $V_{h,T} \overset{a.s.}{\to} V_h$. Let $C_V = [s_{v,\min}-1, s_{v,\max}+1]$, where $s_{v,\max}$ and $s_{v,\min}$ are defined as $s_{v,\max} \equiv \sup_{\ell \le r \le 1-h} V_h(r)$ and $s_{v,\min} \equiv \inf_{\ell \le r \le 1-h} V_h(r)$, respectively. We may take T^* sufficiently large so that $\sup_{\ell \le r \le 1-h} |W_{\ell,T}(r) - W_\ell(r)| < \delta_\epsilon$ and $\sup_{\ell \le r \le 1-h} |V_{h,T}(r) - V_h(r)| < \delta_\epsilon$ for any δ_ϵ and so that both $(W_{\ell,T}, V_{h,T})$ and (W_ℓ, V_h) are in $C_{WV} \equiv C_W \times C_V$. Therefore

$$\underline{\mathbf{1}}_\epsilon(W_\ell)\underline{\mathbf{1}}_\epsilon(V_h) \le \mathbf{1}(W_{\ell,T})\mathbf{1}(V_{h,T}) \le \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h), \tag{A.10}$$

for sufficiently large T^*. Write

$$\textstyle\int \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h) - \int \underline{\mathbf{1}}_\epsilon(W_\ell)\underline{\mathbf{1}}_\epsilon(V_h) = \int(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h) + \int(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(V_h)\underline{\mathbf{1}}_\epsilon(W_\ell). \tag{A.11}$$

Applying Schwarz's inequality to the first term of the right-hand side in (A.11), we have

$$\begin{aligned} \textstyle\int\left(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon\right)(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h) \quad &\le \left\{\textstyle\int\left[(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(W_\ell)\right]^2\right\}^{\frac{1}{2}}\left\{\textstyle\int[\overline{\mathbf{1}}_\epsilon(V_h)]^2\right\}^{\frac{1}{2}} \\ &= \left\{\textstyle\int(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(W_\ell)\right\}^{\frac{1}{2}}\left\{\textstyle\int\overline{\mathbf{1}}_\epsilon(V_h)\right\}^{\frac{1}{2}} \\ &\overset{a.s.}{\to} 0, \end{aligned}$$

as $\epsilon \to 0$ due to (A.9). For the second term, similar arguments lead to

$$\int(\overline{\mathbf{1}}_\epsilon - \underline{\mathbf{1}}_\epsilon)(V_h)\underline{\mathbf{1}}_\epsilon(W_\ell) \overset{a.s.}{\to} 0,$$

and hence we have

$$\int \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h) - \int \underline{\mathbf{1}}_\epsilon(W_\ell)\underline{\mathbf{1}}_\epsilon(V_h) \overset{a.s.}{\to} 0, \tag{A.12}$$

as $\epsilon \to 0$. Part (b) follows from (A.10) and (A.12).

For part (c), using the Skorohod representation, write

$$
\begin{aligned}
T^{*-1}\sum \mathbf{1}_x T^{-\frac{1}{2}} y_t = &\ 2T^{*-1}\sum \mathbf{1}_x\mathbf{1}_y|T^{-\frac{1}{2}}y_t| - T^{*-1}\sum \mathbf{1}_x|T^{-\frac{1}{2}}y_t| \\
= &\ a^{-\frac{1}{2}}\sigma\Big[2T^{*-1}\sum \mathbf{1}(W^0_{\ell,T}(t/T^*))\mathbf{1}(V^0_{h,T}(t/T^*))|V^0_{h,T}(t/T^*)| \\
&\ -T^{*-1}\sum \mathbf{1}(W^0_{\ell,T}(t/T^*))|V^0_{h,T}(t/T^*)|\Big]. \\
\stackrel{d}{=} &\ a^{-\frac{1}{2}}\sigma\Big[2\int_{\ell^*}^{h^*}\mathbf{1}(W_{\ell,T})\mathbf{1}(V_{h,T})|V_{h,T}| - \int_{\ell^*}^{h^*}\mathbf{1}(W_{\ell,T})|V_{h,T}|\Big]
\end{aligned}
$$

Define $\overline{A}_\epsilon(x) \equiv |x| + \epsilon$ and $\underline{A}_\epsilon(x) \equiv \max\{0, |x| - \epsilon\}$ for $\epsilon > 0$. Noting that $|\cdot| \geq 0$, $\underline{A}_\epsilon(\cdot) \geq 0$, and $\overline{A}_\epsilon(\cdot) \geq 0$, we can, by the same arguments as used in (A.10), show that

$$
\underline{\mathbf{1}}_\epsilon(W_\ell)\underline{\mathbf{1}}_\epsilon(V_h)\underline{A}_\epsilon(V_h) \leq \mathbf{1}(W_{\ell,T})\mathbf{1}(V_{h,T})|V_{h,T}| \leq \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h)\overline{A}_\epsilon(V_h), \tag{A.13}
$$

and

$$
\underline{\mathbf{1}}_\epsilon(W_\ell)\underline{A}_\epsilon(V_h) \leq \mathbf{1}(W_{\ell,T})|V_{h,T}| \leq \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{A}_\epsilon(V_h), \tag{A.14}
$$

for sufficiently large T^*. Again, we can use the same arguments as used in (A.12) to show

$$
\int \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{\mathbf{1}}_\epsilon(V_h)\overline{A}_\epsilon(V_h) - \int \underline{\mathbf{1}}_\epsilon(W_\ell)\underline{\mathbf{1}}_\epsilon(V_h)\underline{A}_\epsilon(V_h) \stackrel{a.s.}{\to} 0, \tag{A.15}
$$

and

$$
\int \overline{\mathbf{1}}_\epsilon(W_\ell)\overline{A}_\epsilon(V_h) - \int \underline{\mathbf{1}}_\epsilon(W_\ell)\underline{A}_\epsilon(V_h) \stackrel{a.s.}{\to} 0. \tag{A.16}
$$

From (A.13), (A.14), (A.15) and (A.16), we have

$$
\begin{aligned}
T^{*-1}\sum \mathbf{1}_x T^{-\frac{1}{2}} y_t \Rightarrow &\ a^{-\frac{1}{2}}\sigma\left[2\int \mathbf{1}(W_\ell)\mathbf{1}(V_h)|V_h| - \int \mathbf{1}(W_\ell)|V_h|\right] \\
= &\ a^{-\frac{1}{2}}\sigma\int \mathbf{1}(W_\ell)V_h.
\end{aligned}
$$

Multiplying both sides by (T^*/T) $(\to a^{-1})$ gives part (c).

For part (d), we have

$$
\begin{aligned}
T^{-1}\sum\left[\mathbf{1}_x - T^{-1}\sum \mathbf{1}_x\right]^2 = &\ T^{-1}\sum \mathbf{1}_x - \left(T^{-1}\sum \mathbf{1}_x\right)^2 \\
\Rightarrow &\ a^{-1}\int \mathbf{1}(W_\ell) - a^{-2}\left[\int \mathbf{1}(W_\ell)\right]^2,
\end{aligned}
$$

from part (a). Also, we have

$$
\begin{aligned}
\int U^2_{\ell,h} = &\ \int\left\{\mathbf{1}(W_\ell)^2 - 2a^{-1}\mathbf{1}(W_\ell)\int \mathbf{1}(W_\ell) + a^{-2}\left[\int \mathbf{1}(W_\ell)\right]^2\right\} \\
= &\ \int \mathbf{1}(W_\ell) - a^{-1}\left[\int \mathbf{1}(W_\ell)\,\mathrm{d}s\right]^2,
\end{aligned}
$$

which completes the proof of part (d).

For part (e), we have

$$
\begin{aligned}
T^{-\frac{3}{2}}\sum(y_t - \bar{y})\left(\mathbf{1}_x - T^{-1}\sum \mathbf{1}_x\right) = &\ T^{-\frac{3}{2}}\sum y_t\mathbf{1}_x - \bar{y}T^{-\frac{3}{2}}\sum \mathbf{1}_x \\
\Rightarrow &\ a^{-3/2}\sigma\int \mathbf{1}(W_\ell)V_h - \left[a^{-\frac{3}{2}}\sigma\int V_h\right]\left[a^{-1}\int \mathbf{1}(W_\ell)\right],
\end{aligned}
$$

from Lemma 1(a), parts (a) and (c). Also, we have

$$
\begin{aligned}
\int V_{\ell,h}U_{\ell,h} = &\ \int\left(V_h - a^{-1}\int V_h\right)\left(\mathbf{1}(W_\ell) - a^{-1}\int \mathbf{1}(W_\ell)\right) \\
= &\ \int \mathbf{1}(W_\ell)V_h - a^{-1}\left[\int V_h\right]\left[\int \mathbf{1}(W_\ell)\right],
\end{aligned}
$$

which completes the proof of part (d).

Proof of Theorem 2

From Lemma 2(d) and (e), we have

$$\begin{aligned} T^{-1/2}\widehat{\beta}_D = & \frac{T^{-3/2}\sum(y_t-\bar{y})(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)}{T^{-1}\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2} \\ \Rightarrow & \frac{a^{-3/2}\sigma\int V_{\ell,h}U_{\ell,h}}{a^{-1}\int U_{\ell,h}^2}, \end{aligned}$$

as required for part (a). From Lemma 1(a), Lemma 2(a) and Theorem 2(a), we have

$$\begin{aligned} T^{-1/2}\widehat{\alpha}_D = & \; T^{-3/2}\sum y_t - (T^{-1/2}\widehat{\beta}_D)(T^{-1}\sum\mathbf{1}_x) \\ \Rightarrow & \; a^{-3/2}\sigma\int V_h - a^{-1/2}\sigma\zeta_D a^{-1}\int\mathbf{1}\,(W_\ell)\,, \end{aligned}$$

as required for part (b). Next, define $\widehat{\sigma}_D^2 \equiv T^{-1}\sum(y_t-\widehat{\alpha}_D-\widehat{\beta}_D\mathbf{1}_x)^2$. Since $\widehat{\alpha}_D = \bar{y}-\widehat{\beta}_D T^{-1}\sum\mathbf{1}_x$, using Lemma 1(c), Lemma 2(d) and Theorem 2(a) and (b), we have

$$\begin{aligned} T^{-1}\widehat{\sigma}_D^2 = & \; T^{-2}\sum\left[(y_t-\bar{y}) - \widehat{\beta}_D(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)\right]^2 \\ = & \; T^{-2}\sum(y_t-\bar{y})^2 - (T^{-1/2}\widehat{\beta}_D)^2 T^{-1}\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2 \\ \Rightarrow & \; a^{-2}\sigma^2\int V_{\ell,h}^2 - a^{-1}\sigma^2\zeta_D^2 a^{-1}\int U_{\ell,h}^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} T^{-1/2}t(\widehat{\beta}_D) = & \frac{\widehat{\beta}_D}{T^{1/2}s(\widehat{\beta}_D)} \\ = & \frac{\widehat{\beta}_D}{T^{1/2}\widehat{\sigma}_D\left[\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2\right]^{-1/2}} \\ = & \frac{T^{-1/2}\widehat{\beta}_D\left[T^{-1}\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2\right]^{1/2}}{T^{-1/2}\widehat{\sigma}_D} \\ \Rightarrow & \frac{a^{-1/2}\sigma\zeta_D\left[a^{-1}\int U_{\ell,h}^2\right]^{1/2}}{a^{-1}\sigma\left[\int V_{\ell,h}^2 - \zeta_D^2\int U_{\ell,h}^2\right]^{1/2}}, \end{aligned}$$

as required for part (c) . Noting that $\mathbf{1}_x^2 = \mathbf{1}_x$, we have

$$\begin{aligned} T^{-1/2}t(\widehat{\alpha}_D) & = \frac{\widehat{\alpha}_D}{T^{1/2}s(\widehat{\alpha}_D)} = \frac{\widehat{\alpha}_D\left[T\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2\right]^{1/2}}{(T^{1/2}\widehat{\sigma}_D)\left(\sum\mathbf{1}_x^2\right)^{1/2}} \\ & = \frac{T^{-1/2}\widehat{\alpha}_D\left[T^{-1}\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2\right]^{1/2}}{(T^{-1/2}\widehat{\sigma}_D)\left[T^{-1}\sum\mathbf{1}_x\right]^{1/2}} \\ & \Rightarrow \frac{a^{-3/2}\sigma\left[\int V_h - \zeta_D\int\mathbf{1}(W_\ell)\right]a^{-1/2}\left(\int U_{\ell,h}^2\right)^{1/2}}{a^{-1}\sigma\left(\int V_{\ell,h}^2 - \zeta_D^2\int U_{\ell,h}^2\right)^{1/2}a^{-1/2}\left[\int\mathbf{1}\,(W_\ell)\right]^{1/2}}, \end{aligned}$$

as required for part (d). Lastly, we prove part (e). Define $\widehat{y}_{D,t} \equiv \widehat{\alpha}_D + \widehat{\beta}_D x_t$, then we have

$$\begin{aligned} R_D^2 = \frac{\sum(\widehat{y}_{D,t}-\bar{y})^2}{\sum(y_t-\bar{y})^2} = & \frac{(T^{-1/2}\widehat{\beta}_D)^2 T^{-1}\sum(\mathbf{1}_x - T^{-1}\sum\mathbf{1}_x)^2}{T^{-2}\sum(y_t-\bar{y})^2} \\ \Rightarrow & \frac{a^{-1}\sigma^2\zeta_D^2 a^{-1}\int U_{\ell,h}^2}{a^{-2}\sigma^2\int V_{\ell,h}^2}, \end{aligned}$$

which completes the proof of part (e).

Proof of Theorem 3

From Lemma 2(a) and (b), we immediately have

$$\begin{aligned}(\widehat{p}_+, \widehat{p}_-) &= \left(\frac{T^{-1}\sum \mathbf{1}_y \mathbf{1}_x}{T^{-1}\sum \mathbf{1}_x}, \frac{T^{-1}\sum(1-\mathbf{1}_y)(1-\mathbf{1}_x)}{T^{-1}\sum(1-\mathbf{1}_x)}\right) \\ &\Rightarrow \left(\frac{\int \mathbf{1}(V_h)\mathbf{1}(W_\ell)}{\int \mathbf{1}(W_\ell)}, \frac{\int[1-\mathbf{1}(V_h)][1-\mathbf{1}(W_\ell)]}{\int[1-\mathbf{1}(W_\ell)]}\right),\end{aligned}$$

and

$$\begin{aligned}(T^{-1/2}t_+, T^{-1/2}t_-) &= \left(\frac{(\widehat{p}_+ - 0.5)\left[T^{-1}\sum \mathbf{1}_x\right]^{1/2}}{0.5}, \frac{(\widehat{p}_- - 0.5)\left[T^{-1}\sum(1-\mathbf{1}_x)\right]^{1/2}}{0.5}\right) \\ &\Rightarrow \left(\tfrac{\kappa_+ - 0.5}{0.5a^{1/2}}\left[\int \mathbf{1}\left(W_\ell\right)\right]^{\frac{1}{2}}, \tfrac{\kappa_- - 0.5}{0.5a^{1/2}}\left[\int[1-\mathbf{1}\left(W_\ell\right)]\right]^{\frac{1}{2}}\right),\end{aligned}$$

which complete the proofs of parts (a) and (b). Since $-W$ is also a standard Brownian motion, the two stochastic processes W_ℓ and $-W_\ell$, which are constructed from W and $-W$ in the same manner, respectively, are equivalent, or have the same finite dimensional distributions. Similarly, the two processes, V_h and $-V_h$, are equivalent. Clearly, these hold jointly, i.e., (W_ℓ, V_h) and $(-W_\ell, -V_h)$ are equivalent. Thus, noting that $\mathbf{1}\{x > 0\} = 1 - \mathbf{1}\{-x > 0\}$, we have

$$\begin{aligned}\left(\int \mathbf{1}(W_\ell), \int \mathbf{1}(V_h)\mathbf{1}(W_\ell)\right) &= \left(\int[1-\mathbf{1}(-W_\ell)], \int[1-\mathbf{1}(-V_h)][1-\mathbf{1}(-W_\ell)]\right) \\ &\stackrel{d}{=} \left(\int[1-\mathbf{1}(W_\ell)], \int[1-\mathbf{1}(V_h)][1-\mathbf{1}(W_\ell)]\right),\end{aligned}$$

thereby establishing part (c).

Proof of Theorem 4

From (A.1), (A.2), Theorem 1(a) and (b), we have

$$\begin{aligned}T^{*-1/2}\widehat{u}_{[Tr]} &= T^{*-1/2}y_t - T^{*-1/2}T^{1/2}T^{-1/2}\widehat{\alpha}_C - \widehat{\beta}_C T^{*-1/2}x_t \\ &\Rightarrow \sigma V_h(r) - a^{1/2}a^{-3/2}\sigma\left(\int V_h - \zeta_C \int W_\ell\right) - \sigma\zeta W_\ell(r) \\ &\equiv \sigma Q_{\ell,h}(r).\end{aligned} \tag{A.17}$$

From (A.3), (A.4), (A.17), and applying the CMT, we have

$$\begin{aligned}s^2_{HAC}(\widehat{\beta}_C) &= \left(\sum(x_t-\bar{x})^2\right)^{-2}\sum_{t=L}^{T+L-1}\sum_{\tau=L}^{T+L-1}(x_t-\bar{x})\widehat{u}_t k\left(\tfrac{t-\tau}{bT}\right)\widehat{u}_\tau(x_\tau-\bar{x}) \\ &= \left(\tfrac{1}{T^{*2}}\sum(x_t-\bar{x})^2\right)^{-2}\tfrac{1}{T^{*2}}\sum_{t=L}^{T+L-1}\sum_{\tau=L}^{T+L-1}\frac{x_t-\bar{x}}{T^{*1/2}}\frac{\widehat{u}_t}{T^{*1/2}}k\left(\tfrac{t-\tau}{bT}\right)\frac{\widehat{u}_\tau}{T^{*1/2}}\frac{x_\tau-\bar{x}}{T^{*1/2}} \\ &\Rightarrow \left(\sigma^2\int W^2_{\ell,h}\right)^{-2}\sigma^4\int\int W_{\ell,h}(r)Q_{\ell,h}(r)k\left(\tfrac{r-s}{b}\right)W_{\ell,h}(s)Q_{\ell,h}(s)\mathrm{d}r\mathrm{d}s.\end{aligned}$$

In view of Theorem 1(a), we have Theorem 4.

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Table 1: Finite Sample Properties of the Continuous Technical Trading Regression

		$\lvert\widehat{\beta}_C\rvert$			$\lvert t(\widehat{\beta}_C)\rvert$			R_C^2			$Freq(t(\widehat{\beta}_C) > 1.64)$			$Freq(t(\widehat{\beta}_C) < -1.64)$		
h	ℓ	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000
0.1	0.1	0.40	0.38	0.37	2.46	5.45	7.73	0.08	0.08	0.08	0.10	0.18	0.21	0.49	0.62	0.66
	0.2	0.41	0.41	0.41	3.34	7.67	10.80	0.12	0.13	0.13	0.09	0.14	0.16	0.60	0.72	0.74
	0.3	0.48	0.47	0.47	4.48	9.81	14.01	0.19	0.18	0.18	0.07	0.11	0.12	0.72	0.79	0.82
0.2	0.1	0.59	0.58	0.57	3.14	7.07	10.09	0.11	0.11	0.11	0.07	0.12	0.14	0.61	0.74	0.76
	0.2	0.65	0.65	0.64	4.95	11.05	16.06	0.21	0.21	0.21	0.08	0.12	0.12	0.74	0.80	0.83
	0.3	0.75	0.76	0.76	7.05	15.78	22.05	0.31	0.31	0.31	0.07	0.10	0.10	0.81	0.84	0.86
0.3	0.1	0.78	0.75	0.75	3.98	8.93	12.59	0.16	0.16	0.15	0.06	0.09	0.11	0.73	0.82	0.83
	0.2	0.86	0.84	0.85	6.64	15.03	21.09	0.29	0.30	0.29	0.07	0.10	0.10	0.79	0.84	0.85
	0.3	0.94	0.93	0.93	9.62	21.86	30.40	0.40	0.41	0.40	0.07	0.10	0.10	0.82	0.86	0.87

Note: For the first three blocks of columns, the means of $\lvert\widehat{\beta}_C\rvert$, $\lvert t(\widehat{\beta}_C)\rvert$, and R_C^2 respectively, are reported. For the fourth and fifth blocks, frequencies of rejection for the one-sided tests based on 5 percent significance level are reported. All the numbers are based on 10,000 replications.

Table 2: Finite Sample Properties of the Discrete Technical Trading Regression

		$\lvert\widehat{\beta}_D\rvert$			$\lvert t(\widehat{\beta}_D)\rvert$			R_D^2			$Freq(t(\widehat{\beta}_D) > 1.64)$			$Freq(t(\widehat{\beta}_D) < -1.64)$		
h	ℓ	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000
0.1	0.1	1.27	2.82	4.01	1.99	4.54	6.48	0.05	0.05	0.06	0.10	0.20	0.22	0.41	0.58	0.61
	0.2	1.79	4.06	5.86	2.64	5.94	8.52	0.09	0.09	0.09	0.11	0.19	0.21	0.51	0.64	0.68
	0.3	2.49	5.61	7.78	3.18	7.17	9.97	0.11	0.11	0.11	0.10	0.15	0.16	0.60	0.72	0.74
0.2	0.1	2.01	4.61	6.59	2.48	5.75	8.24	0.08	0.08	0.08	0.08	0.14	0.15	0.53	0.69	0.73
	0.2	3.20	7.17	10.22	3.69	8.27	11.71	0.14	0.14	0.14	0.09	0.14	0.14	0.67	0.75	0.78
	0.3	4.42	9.90	13.93	4.63	10.42	14.58	0.19	0.19	0.19	0.09	0.12	0.14	0.72	0.79	0.80
0.3	0.1	2.76	6.27	8.89	3.00	6.90	9.75	0.10	0.11	0.11	0.06	0.10	0.12	0.64	0.77	0.80
	0.2	4.47	10.15	14.38	4.62	10.46	14.64	0.19	0.19	0.19	0.07	0.11	0.12	0.74	0.81	0.82
	0.3	5.77	13.12	18.65	5.72	12.76	18.05	0.25	0.25	0.25	0.07	0.10	0.11	0.76	0.82	0.83

Note: For the first three blocks of columns, the means of $\lvert\widehat{\beta}_D\rvert$, $\lvert t(\widehat{\beta}_D)\rvert$, and R_D^2 respectively, are reported. For the fourth and fifth blocks, frequencies of rejection for the one-sided tests based on 5 percent significance level are reported. All the numbers are based on 10,000 replications.

Table 3: Finite Sample Properties of the Proportion tests

		t_+								
		$\|t_+\|$			$Freq(t_+ > 1.64)$			$Freq(t_+ < -1.64)$		
h	ℓ	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000
0.1	0.1	2.06	4.63	6.50	0.23	0.34	0.37	0.30	0.45	0.48
	0.2	2.31	5.17	7.43	0.23	0.33	0.36	0.36	0.49	0.51
	0.3	2.54	5.65	8.02	0.24	0.33	0.35	0.41	0.52	0.55
0.2	0.1	2.86	6.39	9.01	0.29	0.37	0.40	0.39	0.49	0.50
	0.2	3.21	7.20	10.18	0.28	0.34	0.35	0.45	0.54	0.56
	0.3	3.56	7.98	11.44	0.27	0.32	0.33	0.50	0.58	0.61
0.3	0.1	3.58	8.10	11.39	0.31	0.37	0.38	0.44	0.52	0.54
	0.2	4.08	9.12	12.86	0.30	0.35	0.36	0.51	0.56	0.59
	0.3	4.50	9.94	14.07	0.31	0.34	0.34	0.54	0.59	0.61

		t_-								
		$\|t_-\|$			$Freq(t_- > 1.64)$			$Freq(t_- < -1.64)$		
h	ℓ	$T = 100$	500	1000	$T = 100$	500	1000	$T = 100$	500	1000
0.1	0.1	2.08	4.56	6.45	0.23	0.34	0.37	0.31	0.44	0.48
	0.2	2.31	5.28	7.33	0.24	0.34	0.36	0.37	0.49	0.51
	0.3	2.58	5.68	8.04	0.24	0.33	0.35	0.42	0.52	0.54
0.2	0.1	2.85	6.39	9.05	0.29	0.36	0.38	0.39	0.49	0.52
	0.2	3.20	7.16	10.14	0.28	0.34	0.36	0.45	0.53	0.55
	0.3	3.57	8.11	11.40	0.26	0.33	0.34	0.51	0.58	0.59
0.3	0.1	3.58	8.03	11.45	0.31	0.37	0.39	0.44	0.52	0.54
	0.2	4.09	9.26	12.91	0.30	0.36	0.37	0.51	0.56	0.58
	0.3	4.50	10.05	14.04	0.30	0.34	0.36	0.55	0.60	0.60

Note: For the first block of columns, the means of $|t_+|(|t_-|)$ is reported. For the second and third blocks, frequencies of rejection for the one-sided tests based on 5 percent significance level are reported. All the numbers are based on 10,000 replications.

Table 4: Asymptotic Distribution of $\mathbf{T}^{-1/2}t(\widehat{\beta}_C)$

		Percentiles								
h	ℓ	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
0.1	0.1	-0.836	-0.714	-0.611	-0.505	-0.150	0.171	0.263	0.339	0.436
	0.2	-1.106	-0.968	-0.843	-0.707	-0.236	0.142	0.248	0.347	0.477
	0.3	-1.401	-1.212	-1.071	-0.907	-0.382	0.083	0.219	0.349	0.494
0.2	0.1	-1.008	-0.878	-0.76	-0.635	-0.251	0.109	0.209	0.305	0.401
	0.2	-1.638	-1.363	-1.158	-0.976	-0.426	0.082	0.244	0.398	0.584
	0.3	-2.215	-1.852	-1.609	-1.359	-0.626	0.068	0.262	0.428	0.591
0.3	0.1	-1.198	-1.042	-0.907	-0.765	-0.355	0.058	0.180	0.297	0.429
	0.2	-2.166	-1.817	-1.564	-1.298	-0.596	0.082	0.264	0.418	0.579
	0.3	-3.400	-2.867	-2.428	-1.985	-0.803	0.038	0.242	0.411	0.624

Table 5: Asymptotic Distribution of $\mathbf{T}^{-1/2}t(\widehat{\beta}_D)$

		Percentiles								
h	ℓ	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
0.1	0.1	-0.635	-0.561	-0.485	-0.404	-0.113	0.166	0.257	0.334	0.426
	0.2	-0.826	-0.715	-0.634	-0.534	-0191	0.189	0.304	0.432	0.580
	0.3	-0.914	-0.820	-0.728	-0.625	-0.245	0.160	0.302	0.431	0.562
0.2	0.1	-0.776	-0.682	-0.602	-0.512	-0.206	0.120	0.218	0.312	0.413
	0.2	-1.103	-0.956	-0.859	-0.721	-0.283	0.134	0.273	0.415	0.555
	0.3	-1.353	-1.175	-1.042	-0.888	-0.375	0.120	0.278	0.422	0.590
0.3	0.1	-0.887	-0.779	-0.683	-0.592	-0.269	0.082	0.193	0.301	0.408
	0.2	-1.334	-1.175	-1.035	-0.891	-0.417	0.093	0.235	0.359	0.526
	0.3	-1.869	-1.583	-1.373	-1.140	-0.458	0.080	0.240	0.364	0.547

Table 6: Asymptotic Distribution of $\mathbf{T}^{-1/2}t_+$ ($\mathbf{T}^{-1/2}t_-$)

		Percentiles								
h	ℓ	1.0%	2.5%	5.0%	10.0%	50.0%	90.0%	95.0%	97.5%	99.0%
0.1	0.1	-0.500	-0.460	-0.412	-0.350	-0.047	0.309	0.406	0.491	0.582
	0.2	-0.519	-0.482	-0.445	-0.390	-0.063	0.341	0.449	0.551	0.673
	0.3	-0.542	-0.501	-0.461	-0.411	-0.091	0.366	0.495	0.603	0.711
0.2	0.1	-0.604	-0.566	-0.527	-0.471	-0.052	0.445	0.593	0.700	0.802
	0.2	-0.622	-0.583	-0.543	-0.493	-0.116	0.500	0.663	0.780	0.882
	0.3	-0.652	-0.610	-0.575	-0.529	-0.185	0.574	0.743	0.843	0.916
0.3	0.1	-0.663	-0.631	-0.600	-0.553	-0.109	0.586	0.739	0.820	0.875
	0.2	-0.683	-0.651	-0.620	-0.576	-0.180	0.687	0.815	0.897	0.950
	0.3	-0.759	-0.727	-0.691	-0.632	-0.214	0.741	0.872	0.940	0.979

Table 7: Stock Price Index (TOPIX)

	(1)	(2)	(3)	(4)	(5)
(S, L)	$(1, 50)$	$(1, 150)$	$(5, 150)$	$(1, 200)$	$(2, 200)$
T	400	525	525	400	400
h	0.1	0.1	0.1	0.1	0.1
ℓ	0.1	0.2	0.2	0.3	0.3
	(i) Continuous Technical Trading Regression				
$\widehat{\beta}_C$	0.24	-0.02	-0.03	-0.17	-0.17
$t(\widehat{\beta}_C)$	2.64**	-0.39	-0.50	-3.25**	-3.32**
R_C^2	0.02	0.00	0.00	0.03	0.03
$t(\widehat{\beta}_C)/\sqrt{T}$	0.13	-0.02	-0.02	-0.16	-0.17
$t_{HAC}(\widehat{\beta}_C)$	1.20	-0.22	-0.28	-2.12	-2.16
	(ii) Discrete Technical Trading Regression				
$\widehat{\beta}_D$	0.38	0.36	0.35	0.16	0.12
$t(\widehat{\beta}_D)$	0.98	0.95	0.93	0.30	0.22
R_D^2	0.00	0.00	0.00	0.00	0.00
$t(\widehat{\beta}_D)/\sqrt{T}$	0.05	0.04	0.04	0.01	0.01
$t_{HAC}(\widehat{\beta}_D)$	0.47	0.47	0.48	0.18	0.14
	(iii) Proportion Test (+)				
$\widehat{p}_+$	0.68	0.57	0.59	0.63	0.63
t_+	6.07**	2.68**	3.30**	4.67**	4.62**
$t_+/\sqrt{T}$	0.30	0.12	0.14	0.23	0.23

Note: Sample periods are from 2005/3/8 to 2006/10/19 for (1), from 2004/7/27 to 2006/9/12 for (2) and (3), and from 2005/2/10 to 2006/9/25 for (4) and (5). ** signifies statistically significant at the 5% level, and * at the 10% level.

Table 8: Yen/Dollar Exchange Rate

	(1)	(2)	(3)	(4)	(5)
(S, L)	$(1, 50)$	$(1, 150)$	$(5, 150)$	$(1, 200)$	$(2, 200)$
T	400	525	525	400	400
h	0.1	0.1	0.1	0.1	0.1
ℓ	0.1	0.2	0.2	0.3	0.3
	(i) Continuous Technical Trading Regression				
$\widehat{\beta}_C$	-0.59	-0.23	-0.23	-0.38	-0.38
$t(\widehat{\beta}_C)$	-7.95**	-4.42**	-4.31**	-8.17**	-8.05**
R^2_C	0.13	0.03	0.03	0.14	0.14
$t(\widehat{\beta}_C)/\sqrt{T}$	-0.39	-0.19	-0.18	-0.41	-0.40
$t_{HAC}(\widehat{\beta}_C)$	-7.27***	-2.16	-2.15	-5.28**	-5.15**
	(ii) Discrete Technical Trading Regression				
$\widehat{\beta}_D$	-0.96	-1.07	-1.17	-0.87	-0.86
$t(\widehat{\beta}_D)$	-6.99**	-7.22**	-7.93**	-5.82**	-5.78**
R^2_D	0.10	0.09	0.11	0.08	0.08
$t(\widehat{\beta}_D)/\sqrt{T}$	-0.34	-0.32	-0.34	-0.29	-0.29
$t_{HAC}(\widehat{\beta}_D)$	-6.81**	-3.47	-3.67	-2.91	-2.86
	(iii) Proportion Test (+)				
$\widehat{p}_+$	0.59	0.53	0.52	0.65	0.64
t_+	3.29**	1.39	0.96	4.98**	4.85**
$t_+/\sqrt{T}$	0.16	0.06	0.04	0.25	0.24

Note: Sample periods are from 2005/3/8 to 2006/10/19 for (1), from 2004/7/27 to 2006/9/12 for (2) and (3), and from 2005/2/10 to 2006/9/25 for (4) and (5). ** signifies statistically significant at the 5% level, and * at the 10% level.

Figure 1: Distributions of t-statistics and rescaled t-statistics

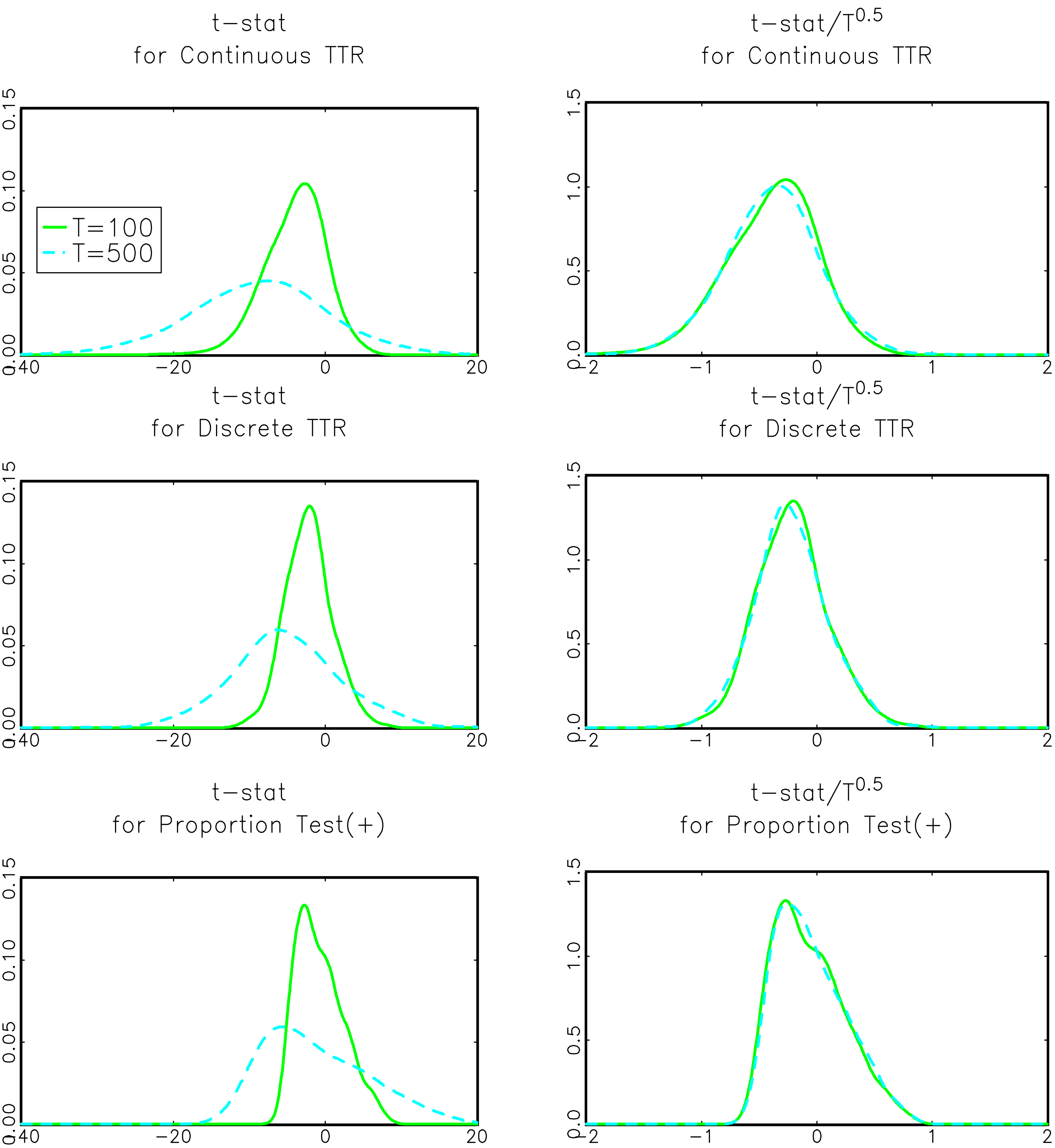
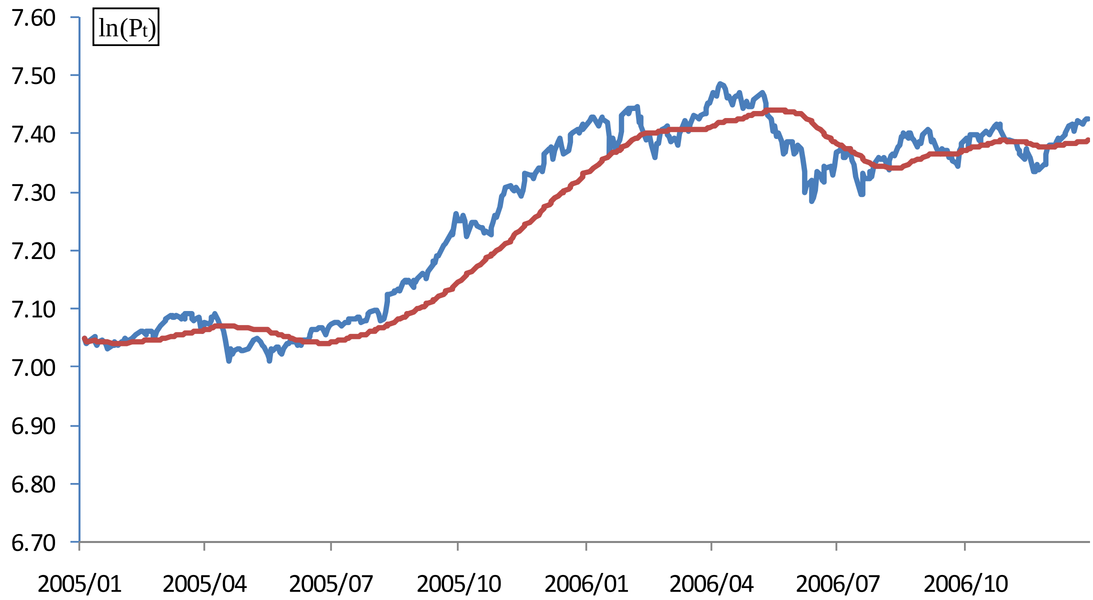


Figure 2: Stock Price Index (TOPIX)

TOPIX and its moving average of past 50 days



TTR(1,50)

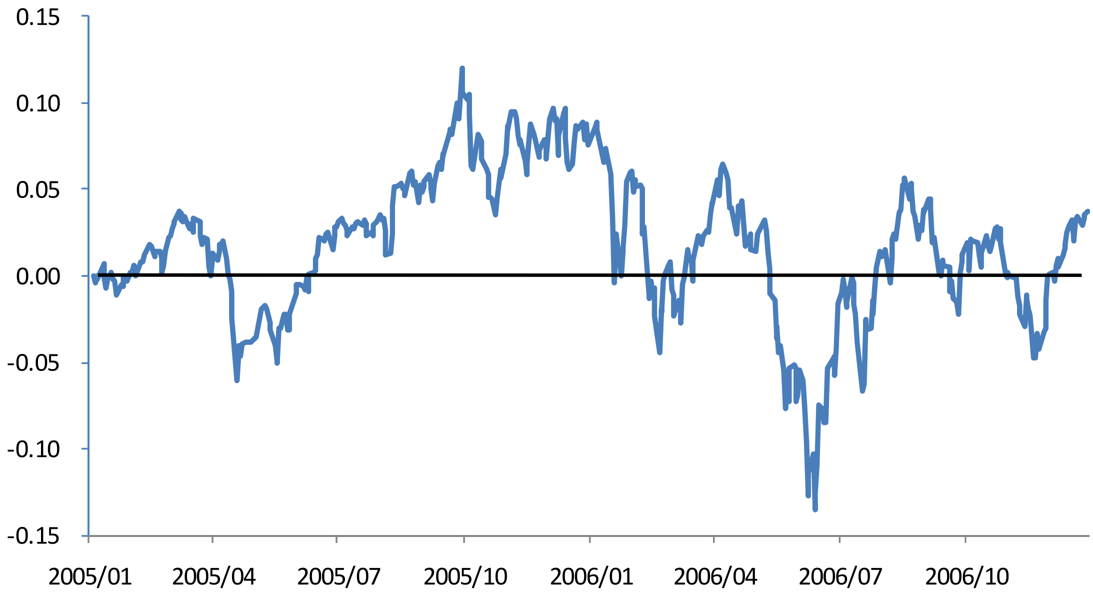


Figure 3: Yen/Dollar Exchange Rate

Yen/Dollar rate and its moving average of past 50 days

TTR(1,50)