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Satoshi Yamashita* and Toshinao Yoshiba**

Abstract

We evaluate expected and unexpected losses of a bank loan, taking into account the bank's strategic control of the expected return on the loan. Assuming that the bank supplies an additional loan to minimize the expected loss of the total loan, we provide analytical formulations for expected and unexpected losses with bivariate normal distribution functions.

There are two cases in which an additional loan decreases the expected loss: i) the asset/liability ratio of the firm is low but its expected growth rate is high; ii) the asset/liability ratio of the firm is high and the lending interest rate is high. With a given expected growth rate and given interest rates, the two cases are identified by two thresholds for the current asset/liability ratio. The bank maintains the current loan amount when the asset/liability ratio is between the two thresholds.

Given the bank's strategy, the bank decreases the initial expected loss of the loan. On the other hand, the bank has a greater risk of the unexpected loss.

Keywords: Probability of default (PD); Loss given default (LGD); Exposure at default (EaD); Expected loss (EL); Unexpected loss (UL); Stressed EL (SEL)
JEL classification: G21, G32, G33

***** Associate Professor, The Institute of Statistical Mathematics (E-mail: yamasita@ism.ac.jp)
****** Director and Senior Economist, Institute for Monetary and Economic Studies, Bank of Japan (E-mail: toshinao.yoshiba@boj.or.jp)

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Table of Contents

I. Introduction

An internationally active bank, adopting the advanced internal ratings-based approach under Basel II (BCBS [2005a]), has to estimate the probability of default (PD), loss given default (LGD), and exposure at default (EaD). These three factors estimated by the bank's own model determine a loss distribution of the bank's portfolio. From the viewpoint of risk management, the two most popular concepts to capture the loss distribution are the expected loss (EL) and unexpected loss (UL). UL is defined by value-at-risk (VaR) minus EL. EL should be covered by the bank's loan loss provisioning. UL should be cushioned by the bank's capital. Basel II adopts these concepts.

When measuring PD, LGD, and EaD, we have to pay attention to the relationships among them. If EaD changes, PD and LGD will also change through a structural relationship with EaD.

Some earlier studies examined EL and UL, taking into account the correlation between PD and LGD (see Frye [2000], Phytkin [2003], Peura and Jokivuolle [2005], as examples). However, most studies fix EaD to avoid dynamic development in EaD which is related to PD and LGD.

Quite recently, a number of EaD models have been proposed. Moral [2006] and Jiménez and Mencía [2007] focused on loan commitments to capture varying EaD. Kupiec [2007] proposed a model which assumes both LGD and EaD have a common systematic factor, based on the fact that both tend to increase during recession periods. They focused on the statistical correlation among PD, LGD, and EaD rather than the structural relationship of PD and LGD with varying EaD.

Following the Merton model, we provide analytical formulations for EL and UL with bivariate normal distribution functions, including changes in EaD. We assume that the bank supplies an additional loan to minimize EL at a certain time t until maturity T. An additional loan at time t decreases EL measured at time t in two cases: i) the asset/liability ratio of the firm is low but its expected growth rate is high; ii) the asset/liability ratio of the firm is high and the lending interest rate is high. With a given firm's expected growth rate and given interest rates, the three states are identified by two thresholds for the current asset/liability ratio. In the first state, where the ratio is larger than the higher of the two thresholds, the additional loan helps the firm to grow, and the bank earns a higher expected return. In the second state, where the ratio is between the

two thresholds, the bank maintains the current loan amount. In the third state, where the ratio is smaller than the lower of the two thresholds, the additional loan helps the firm to survive a temporary crisis and the bank has the benefit of a decrease in EL. Incorporating the additional loan at time *t*, both EL and UL measured at time 0 are evaluated analytically using bivariate normal distribution functions. We show some numerical examples for EL and UL. They imply that EL measured at time 0 decreases by taking the additional loan into account, but UL measured at time 0 increases by taking it into account. The larger the correlation between the firm's factor and the common systematic factor gets, the larger the increase in UL gets.

Following this introduction, Section II provides a structural model of an additional loan, and the evaluation of EL measured at time 0 under EL minimization at time *t*. Section III evaluates UL. Section IV provides numerical examples for EL and UL. Section V summarizes the findings and provides directions for extensions to this paper.

II. EL with an additional loan

A. Basic model and loss function

Following Merton [1974], we assume that the asset value A_t of the firm is assumed to have the geometric Brownian process below and the default of the firm occurs when the asset value at maturity *T* is less than the firm's liability.

$$dA_t = \mu A_t dt + \sigma A_t dW_t . \qquad (1)$$

Let the bank supply a loan of the notional amount *D* at time 0 in the form of a discount bond, that is, the bank supplies $De^{-r_{L0}T}$ amount of cash to the firm at time 0, where r_{L0} denotes the lending interest rate. We suppose that the firm's liability only consists of the bank's lending. This means that EaD is equal to *D* if the bank does not supply additional loans.

First, we assume that EaD does not change. When $A_T < D$, the default of the firm occurs and the bank recovers the lending by liquidating the firm's asset value A_T. The loss to the bank at maturity is given by:[1]

[1] $(X)^+$ denotes the positive part of a real number *X*, i.e., $(X)^+ = \max(X, 0)$.

$$L_T = D(e^{(r_{M0}-r_{L0})T} - 1) + (D - A_T)^+, \tag{2}$$

where r_{M0} denotes the bank's funding interest rate at time 0. The EL of the bank measured at time 0 is given below:

$$\begin{aligned} E_0[L_T] &= D(e^{(r_{M0}-r_{L0})T} - 1) + E_0[(D - A_T)^+] \\ &= D(e^{(r_{M0}-r_{L0})T} - 1) + D\Phi(d_0) - A_0 e^{\mu T}\Phi(d_0 - \sigma\sqrt{T}), \end{aligned} \tag{3}$$

$$d_0 = \frac{1}{\sigma\sqrt{T}}\left\{\ln\frac{D}{A_0} - \left(\mu - \frac{\sigma^2}{2}\right)T\right\}. \tag{4}$$

Here, $\Phi(\cdot)$ denotes a standard normal distribution function. In this model, PD is given by $\Phi(d_0) = \Pr[A_T < D]$ and LGD is given by $(D - A_T)^+ / \{D \times \mathrm{PD}\}$.[2]

B. EL minimization at time t

Now we introduce a change in EaD. For simplicity, we make the assumption that the bank supplies an additional loan only once at a certain time *t*. When the bank supplies an additional loan amount Δ, EaD changes from D to $D + \Delta$. The asset value of the firm changes from A_t to $A_t + \Delta e^{-r_L \tau}$, where r_L denotes the lending interest rate at time *t* and τ denotes the interval to maturity, $T - t$. Then, the loss to the bank at maturity is given by:

$$L_T(\Delta) = D(e^{(r_{M0}-r_{L0})T} - 1) + \Delta(e^{(r_M - r_L)\tau} - 1) + (D + \Delta - A_T)^+, \tag{5}$$

where r_M denotes the bank's funding interest rate at time *t*.

The EL of the bank measured at time *t* is given by:

$$\begin{aligned} EL_t(\Delta) \equiv E_t[L_T(\Delta)] &= D(e^{(r_{M0}-r_{L0})T} - 1) + \Delta(e^{(r_M - r_L)\tau} - 1) + (D + \Delta)\Phi(d_t(\Delta)) \\ &\quad - (A_t + \Delta e^{-r_L\tau})e^{\mu\tau}\Phi(d_t(\Delta) - \sigma\sqrt{\tau}), \end{aligned} \tag{6}$$

$$d_t(\Delta) = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\frac{D + \Delta}{A_t + \Delta e^{-r_L\tau}} - \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}. \tag{7}$$

[2] Most existing studies focused on a collateral value for the loan and assumed that the loss to the bank is the uncovered portion of the debt relative to the collateral (see Frye [2000], Phytkin [2003], Peura and Jokivuolle [2005], for examples). On the other hand, Altman, Resti and Sironi [2001] focused on the firm's asset value in their model and showed that the expected LGD, $E_0[(D - A_T)^+]/D$, is given by $\Phi(d_0) - (A_0/D)e^{\mu T}\Phi(d_0 - \sigma\sqrt{T})$.

We assume the bank minimizes $EL_t(\Delta)$ through supplying an additional loan amount Δ. The first derivative of $EL_t(\Delta)$ with regard to the additional loan amount Δ is given by:

$$\frac{\partial EL_t(\Delta)}{\partial \Delta} = e^{(r_M - r_L)\tau} - 1 + \Phi(d_t(\Delta)) - e^{(\mu - r_L)\tau}\Phi(d_t(\Delta) - \sigma\sqrt{\tau}). \quad (8)$$

Here, we generalize equation (8) as:

$$f(d) \equiv e^{(r_M - r_L)\tau} - 1 + \Phi(d) - e^{(\mu - r_L)\tau}\Phi(d - \sigma\sqrt{\tau}). \quad (9)$$

The function $f(d)$ is convex and reaches its maximum at $d = \bar{d} \equiv \{(r_L - \mu)/\sigma + \sigma/2\}\sqrt{\tau}$. We see that $f(-\infty) < 0$ when $r_L > r_M$ and that $f(\infty) < 0$ when $\mu > r_M$. See Appendix 1 for details.

As given in equation (7), d corresponds to the asset/liability ratio of the firm. By supplying an additional loan Δ, $d_t(\Delta)$ increases when $d_t(0) < \bar{d}$, and $d_t(\Delta)$ decreases when $d_t(0) > \bar{d}$.

Here, we assume that $\mu > r_M$, $r_L > r_M$ and $f(\bar{d}) > 0$. Then, there exist two solutions d_1^* and d_2^* where $f(d_i^*) = 0$ and $d_1^* < \bar{d} < d_2^*$. This implies that the bank supplies an additional loan under one of two conditions: i) the asset/liability ratio A_t / D is larger than the threshold ξ_1^*; or ii) the asset/liability ratio is smaller than another threshold ξ_2^*. The thresholds ξ_1^* and ξ_2^* are given by:

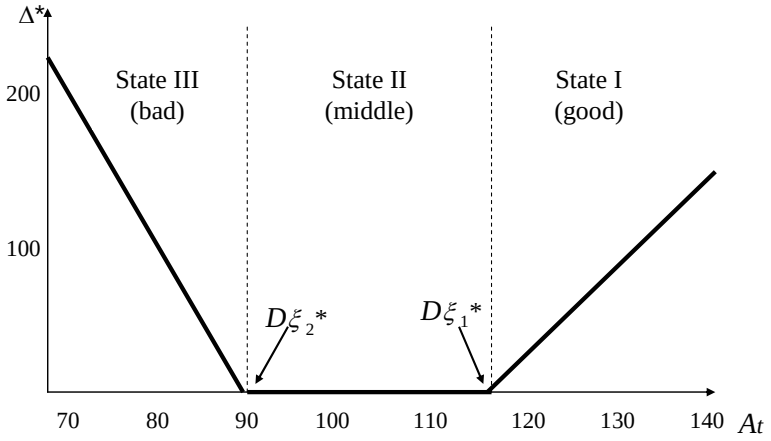
$$\xi_1^* = e^{-d_1^*\sigma\sqrt{\tau} - (\mu - \sigma^2/2)\tau}, \quad \xi_2^* = e^{-d_2^*\sigma\sqrt{\tau} - (\mu - \sigma^2/2)\tau}. \quad (10)$$

We can confirm that an additional loan reduces $EL_t(\Delta)$ in the two cases above. The optimal additional loan amount Δ^* satisfies i) $d_t(\Delta_1^*) = d_1^*$ for $A_t > D\xi_1^*$, and ii) $d_t(\Delta_2^*) = d_2^*$ for $A_t < D\xi_2^*$. Therefore,

$$\Delta_1^* = \frac{A_t - D\xi_1^*}{\xi_1^* - e^{-r_L\tau}}, \quad \Delta_2^* = \frac{A_t - D\xi_2^*}{\xi_2^* - e^{-r_L\tau}}. \quad (11)$$

Figure 1 shows the optimal additional loan amount Δ^* with respect to the asset value A_t just before supplying an additional loan.

Figure 1. Optimal additional loan at time *t*



According to the three states of A_t distinguished by the thresholds $D\xi_1^*$ and $D\xi_2^*$, the bank takes different actions to minimize $EL_t(\Delta)$.

State I ($A_t > D\xi_1^*$): The firm is in a good state. When the bank supplies the optimal additional loan amount in this state, the PD increases slightly from $\Phi(d_t(0))$ to $\Phi(d_1^*)$. The increase in PD incurs a rise in the EL. On the other hand, the additional loan increases interest earnings. Because the increase exceeds the rise in EL which results from the increase in PD, the EL decreases. The minimized EL measured at time *t* is:

$$\mathrm{EL}_t(\Delta^*) = E_t[L_T(\Delta_1^*)] = D(e^{(r_{M0}-r_{L0})T} - 1) + D\Phi(d_1^*) - A_t e^{\mu\tau}\Phi(d_1^* - \sigma\sqrt{\tau}). \quad (12)$$

State II ($D\xi_2^* \le A_t \le D\xi_1^*$): The firm is in a middle state. The bank does not supply an additional loan in this state. The EL measured at time *t* is:

$$\mathrm{EL}_t(\Delta^*) = E_t[L_T(0)] = D(e^{(r_{M0}-r_{L0})T} - 1) + D\Phi(d_t(0)) - A_t e^{\mu\tau}\Phi(d_t(0) - \sigma\sqrt{\tau}). \quad (13)$$

State III ($A_t < D\xi_2^*$): The firm is in a bad state. When the bank supplies the optimal additional loan amount in this state, the PD decreases from $\Phi(d_t(0))$ to $\Phi(d_2^*)$. The decrease in PD leads to decrease in EL. The minimized EL measured at time *t* is:

$$\mathrm{EL}_t(\Delta^*) = E_t[L_T(\Delta_2^*)] = D(e^{(r_{M0}-r_{L0})T} - 1) + D\Phi(d_2^*) - A_t e^{\mu\tau}\Phi(d_2^* - \sigma\sqrt{\tau}). \quad (14)$$

C. Numerical example of EL and PD at time t

Table 1 shows numerical examples of changes in the EL and PD caused by the optimal additional loan at time *t*. Let D = 100, T = 2, t = 1, μ = 5%, σ = 10%, $r_{L0} = r_L$ = 1%, r_{M0} =

r_M = 0.5%. In this case, $D\xi_1^* \cong 115.67$ and $D\xi_2^* \cong 89.74$ for $d_1^* \cong -1.905$ and $d_2^* \cong 0.632$. We can confirm that the PD increases with the additional loan Δ^* in state I, corresponding to the case of $A_t = 120$ and 125, and that the PD decreases with the additional loan Δ^* in state III, corresponding to the case of $A_t = 80$ and 85. In state II, corresponding to the case of $A_t = 90$ and 105, $\Delta^* = 0$ and both EL and PD do not change.

Table 1. EL and PD with/without an additional loan at time *t*

A_t	Δ^*	$\mathrm{EL}_t(\Delta^*)$	$\mathrm{EL}_t(0)$	$\mathrm{PD}_t(\Delta^*)$	$\mathrm{PD}_t(0)$
80	105.19	**13.54**	15.06	73.64%	96.26%
85	51.21	**9.85**	10.26	73.64%	88.00%
90	0.00	**6.16**	6.16	72.69%	72.69%
115	0.00	**–0.87**	–0.87	3.23%	3.23%
120	26.01	**–0.99**	–0.96	2.84%	1.15%
125	56.02	**–1.11**	–0.98	2.84%	0.37%

D. Evaluation of EL measured at time 0

The discussion above considers the bank's strategies at time *t*. Here, we derive EL measured at time 0, taking those strategies at time *t* into account. First, we obtain the probability that each of the three states occurs. Second, we obtain the component of EL on each state. Finally, we derive EL measured at time 0 by summing up those components.

State I ($A_t > D\xi_1^*$): The probability that the firm is in this state at time *t* is given by:

$$\Pr[A_t > D\xi_1^*] = \Phi(-\delta_1^*), \tag{15}$$

where

$$\delta_1^* = d_0\sqrt{T/t} - d_1^*\sqrt{\tau/t}. \tag{16}$$

The component of EL attributable to this state is given by:

$$\begin{aligned} E_0[L_T(\Delta_1^*)1_{A_t > D\xi_1^*}] = {} & D\{\Phi(d_1^*) + e^{(r_{M0}-r_{L0})T} - 1\}\Phi(-\delta_1^*) \\ & - A_0 e^{\mu T}\Phi(-\delta_1^* + \sigma\sqrt{t})\Phi(d_1^* - \sigma\sqrt{\tau}), \end{aligned} \tag{17}$$

using equation (12). See Appendix 2 for details.

State II ($D\xi_2^* \le A_t \le D\xi_1^*$): The probability that the firm is in this state at time *t* is given by:

$$\Pr[D\xi_2^* \le A_t \le D\xi_1^*] = \Phi(\delta_1^*) - \Phi(\delta_2^*), \tag{18}$$

where

$$\delta_2^* = d_0\sqrt{T/t} - d_2^*\sqrt{\tau/t}. \tag{19}$$

The component of EL attributable to this state is given by:

$$\begin{aligned} E_0[L_T(0)1_{D\xi_2^* \le A_t \le D\xi_1^*}] &= D(e^{(r_{M0}-r_{L0})T} - 1)\{\Phi(\delta_1^*) - \Phi(\delta_2^*)\} \\ &+ D\{\Phi_2(\delta_1^*, d_0; \rho) - \Phi_2(\delta_2^*, d_0; \rho)\} \\ &- A_0 e^{\mu T}\{\Phi_2(\delta_1^* - \sigma\sqrt{t}, d_0 - \sigma\sqrt{T}; \rho) - \Phi_2(\delta_2^* - \sigma\sqrt{t}, d_0 - \sigma\sqrt{T}; \rho)\}, \end{aligned} \tag{20}$$

using equation (13), where

$$\rho = \sqrt{t/T}, \tag{21}$$

and $\Phi_2(x, y; \rho)$ is a distribution function of the bivariate standard normal distribution with correlation ρ.

State III ($A_t < D\xi_2^*$): The probability that the firm is in this state at time *t* is given by:

$$\Pr[A_t < D\xi_2^*] = \Phi(\delta_2^*). \tag{22}$$

The component of EL attributable to this state is given by:

$$\begin{aligned} E_0[L_T(\Delta_2^*)1_{A_t < D\xi_2^*}] = {} & D\{\Phi(d_2^*) + e^{(r_{M0}-r_{L0})T} - 1\}\Phi(\delta_2^*) \\ & - A_0 e^{\mu T}\Phi(d_2^* - \sigma\sqrt{\tau})\Phi(\delta_2^* - \sigma\sqrt{t}), \end{aligned} \tag{23}$$

using equation(14).

EL in total

The EL measured at time 0 is evaluated as:

$$E_0[L_T(\Delta^*)] = E_0[L_T(\Delta_1^*)1_{A_t > D\xi_1^*}] + E_0[L_T(0)1_{D\xi_2^* \le A_t \le D\xi_1^*}] + E_0[L_T(\Delta_2^*)1_{A_t < D\xi_2^*}]. \tag{24}$$

By substituting equations (17), (20), and (23) for equation (24), we obtain:

$$
\begin{aligned}
E_0[L_T(\Delta^*)] &= D(e^{(r_{M0}-r_{L0})T}-1) \\
&+ D\{\Phi(d_1^*)\Phi(-\delta_1^*)+\Phi(d_2^*)\Phi(\delta_2^*)+\Phi_2(\delta_1^*,d_0;\rho)-\Phi_2(\delta_2^*,d_0;\rho)\} \\
&- A_0 e^{\mu T}\{\Phi(d_1^*-\sigma\sqrt{\tau})\Phi(-\delta_1^*+\sigma\sqrt{t})+\Phi(d_2^*-\sigma\sqrt{\tau})\Phi(\delta_2^*-\sigma\sqrt{t}) \\
&+ \Phi_2(\delta_1^*-\sigma\sqrt{t},d_0-\sigma\sqrt{T};\rho)-\Phi_2(\delta_2^*-\sigma\sqrt{t},d_0-\sigma\sqrt{T};\rho)\}.
\end{aligned}
\tag{25}
$$

We have derived an analytical formation of EL measured at time 0 taking into account the bank's strategy depending on the future value of the firm's assets. Formulation (25) is expressed by bivariate normal distributions. It can be calculated easily, because bivariate normal distribution functions can be evaluated by numerical approximation methods. See Drezner [1978] for an example of those methods.

III. UL with an additional loan

The previous section obtained EL measured at time 0 taking into account the bank's strategy at time *t*. This section derives UL measured at time 0 taking it into account. First, we show the equivalence between UL and SEL. Second, we derive SEL measured at time 0. This leads to UL measured at time 0.

A. Equivalence between UL and SEL

We evaluate the VaR to obtain the UL. For a given confidence level $\alpha \in (0,1)$, the VaR is defined as the α th quantile of the distribution of loss, and is denoted by $q_\alpha(L)$.

The VaR is equivalent to a conditional EL under these assumptions, which are adopted by Basel II (BCBS [2005b]). The first assumption is that the portfolio is well diversified, that is, it is composed of many kinds of loans to various firms. The second assumption is that all loans are correlated through the correlation between each loan and a single systematic factor, *X*. On those assumptions, the VaR of the portfolio $q_\alpha(L)$ is given by a conditional expectation of the loss $E[L \mid X = x_{1-\alpha}]$, where $x_{1-\alpha}$ is the $(1-\alpha)$ th quantile of the distribution of *X*. See Gordy [2003], Vasicek [2002], for details.

Assuming that the portfolio is well diversified, and has a single systematic factor *X*, the UL of the portfolio is given by $\mathrm{UL} = \sum_{i=1}^{M} \mathrm{UL}_i$, where:

$$\mathrm{UL}_i = E[L_i \mid X = x_{1-\alpha}] - E[L_i]. \tag{26}$$

Equation (26) implies that UL_i is the contribution of exposure *i* in the UL of the portfolio. Here, we call $E[L_i \mid X = x_{1-\alpha}]$ "stressed EL," SEL hereafter. We omit *i* hereafter for simplicity. The valuation of UL and its decomposition is equivalent to the evaluation of SEL.

We illustrate how to describe the stressed condition $X = x_{1-\alpha}$. The stochastic process of A_t is driven by one Brownian motion process W_t as in equation (1). Similar to BCBS [2005b], we suppose that the Brownian motion W_t is composed of a single systematic factor X_t and an idiosyncratic factor Y_t as follows:

$$W_t = \sqrt{R}X_t + \sqrt{1-R}Y_t. \tag{27}$$

Here, R denotes the asset correlation among bank loans. The stressed condition $X = x_{1-\alpha}$ at the time of the default corresponds to the $(1-\alpha)$ th quantile of the factor X_T. The quantile is given by $X_T = -\sqrt{T}\Phi^{-1}(\alpha)$.

SEL after supplying the optimal additional loan at time *t* is given by:

$$\begin{aligned} SEL_t(\Delta^*) &= D(e^{(r_{M0}-r_{L0})T} - 1) + \Delta^*(e^{(r_M - r_L)\tau} - 1) \\ &\quad + E_t[(D + \Delta^* - A_T)^+ \mid X_T = -\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t], \end{aligned} \tag{28}$$

where t the optimal additional loan amount Δ^* is determined by equation (11) following EL minimization at time *t* depending on the value of A_t.

B. Evaluation of SEL at time 0

We derive SEL measured at time 0 given the bank's optimal strategy at time *t*. We evaluate SEL with the constraint $X_T = -\sqrt{T}\Phi^{-1}(\alpha)$, similar to the derivation of EL measured at time 0. SEL is given by the sum of the contributions for the three states as:

$$\begin{aligned} &E_0[SEL_t(\Delta^*)] \\ &\quad = E_0[SEL_t(\Delta_1^*)1_{A_t > D\xi_1^*}] + E_0[SEL_t(0)1_{D\xi_2^* \le A_t \le D\xi_1^*}] + E_0[SEL_t(\Delta_2^*)1_{A_t < D\xi_2^*}]. \end{aligned} \tag{29}$$

State I ($A_t > D\xi_1^*$): $E_0[SEL_t(\Delta_1^*)1_{A_t > D\xi_1^*}]$ is given as below. See Appendix 3 for details.

$$
\begin{aligned}
E_0[SEL_t(\Delta_1^*)1_{A_t>D\xi_1^*}] &= D(e^{(r_{M0}-r_{L0})T}-1)\Phi(-\delta_1^*) \\
&+ \frac{1}{\xi_1^* - e^{-r_L\tau}}[(e^{(r_M-r_L)\tau}-1)\{A_0e^{\mu t}\Phi(-\delta_1^*+\sigma\sqrt{t}) - D\xi_1^*\Phi(-\delta_1^*)\} \\
&+ A_0e^{\mu t}\{\Phi_2(-\delta_1^*+\sigma\sqrt{t}, h_1^*+\sigma Rt/\sqrt{\eta};\rho^*) \\
&- e^{-\sigma h_1^*\sqrt{\eta}-\sigma^2 t/2+\sigma^2(1-R)T/2}\Phi_2(-\delta_1^*+\sigma(1-R)\sqrt{t}, h_1^*-\sigma(1-R)\tau/\sqrt{\eta};\rho^*)\} \\
&- De^{-r_L\tau}\{\Phi_2(-\delta_1^*, h_1^*;\rho^*) - e^{-\sigma h_1^*\sqrt{\eta}+\sigma^2\eta/2}\Phi_2(-\delta_1^*-\sigma R\sqrt{t}, h_1^*-\sigma\sqrt{\eta};\rho^*)\}],
\end{aligned}
\tag{30}
$$

where η, h_1^*, ρ^* are given as equations (31)–(33).

$$\eta \equiv (1-R)\tau + Rt, \tag{31}$$

$$h_i^* \equiv \frac{d_i^*\sqrt{\tau}+\Phi^{-1}(\alpha)\sqrt{RT}}{\sqrt{\eta}}, \quad i=1,2, \tag{32}$$

$$\rho^* \equiv R\sqrt{t/\eta}. \tag{33}$$

State II ($D\xi_2^* \le A_t \le D\xi_1^*$): $E_0[SEL_t(0)1_{D\xi_2^*\le A_t\le D\xi_1^*}]$ is given by:

$$
\begin{aligned}
E_0[SEL_t(0)1_{D\xi_2^*\le A_t\le D\xi_1^*}] &= D(e^{(r_{M0}-r_{L0})T}-1)\{\Phi(\delta_1^*)-\Phi(\delta_2^*)\} \\
&+ D\{\Phi_2(d_S,\delta_1^*;\rho_S)-\Phi_2(d_S,\delta_2^*;\rho_S)\} \\
&- A_0e^{(\mu-\sigma^2R/2)T-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}\{\Phi_2(d_S-\sigma_S,\delta_1^*-\rho_S\sigma_S;\rho_S) \\
&\qquad -\Phi_2(d_S-\sigma_S,\delta_2^*-\rho_S\sigma_S;\rho_S)\},
\end{aligned}
\tag{34}
$$

where d_S, ρ_S. σ_S are given as equations (35)-(37).

$$d_S \equiv \frac{d_0+\sqrt{R}\Phi^{-1}(\alpha)}{\sqrt{1-R}} = \frac{\Phi^{-1}(\mathrm{PD})+\sqrt{R}\Phi^{-1}(\alpha)}{\sqrt{1-R}}, \tag{35}$$

$$\rho_S \equiv \sqrt{(1-R)t/T}, \tag{36}$$

$$\sigma_S \equiv \sigma\sqrt{(1-R)T}. \tag{37}$$

State III ($A_t < D\xi_2^*$): $E_0[SEL_t(\Delta_2^*)1_{A_t<D\xi_2^*}]$ is given as below. h_2^* is given as equation (32).

$$
\begin{aligned}
&E_0[SEL_t(\Delta_2^*)1_{A_t<D\xi_2^*}] = D(e^{(r_{M0}-r_{L0})T}-1)\Phi(\delta_2^*)\\
&+\frac{1}{\xi_2^*-e^{-r_L\tau}}[(e^{(r_M-r_L)\tau}-1)\{A_0e^{\mu t}\Phi(\delta_2^*-\sigma\sqrt{t})-D\xi_2^*\Phi(\delta_2^*)\}\\
&+A_0e^{\mu t}\{\Phi_2(\delta_2^*-\sigma\sqrt{t},h_2^*+\sigma Rt/\sqrt{\eta};-\rho^*)\\
&\quad -e^{-\sigma h_2^*\sqrt{\eta}-\sigma^2t/2+\sigma^2(1-R)T/2}\Phi_2(\delta_2^*-\sigma(1-R)\sqrt{t},h_2^*-\sigma(1-R)\tau/\sqrt{\eta};-\rho^*)\}\\
&-De^{-r_L\tau}\{\Phi_2(\delta_2^*,h_2^*;-\rho^*)-e^{-\sigma h_2^*\sqrt{\eta}+\sigma^2\eta/2}\Phi_2(\delta_2^*+\sigma R\sqrt{t},h_2^*-\sigma\sqrt{\eta};-\rho^*)\}].
\end{aligned}
\tag{38}
$$

SEL in total

Finally, from equations (29), (30), (34), and (38), SEL measured at time 0 is evaluated as:

$$
\begin{aligned}
&E_0[SEL_t(\Delta^*)] = D(e^{(r_{M0}-r_{L0})T}-1)\\
&+\frac{1}{\xi_1^*-e^{-r_L\tau}}[(e^{(r_M-r_L)\tau}-1)\{A_0e^{\mu t}\Phi(-\delta_1^*+\sigma\sqrt{t})-D\xi_1^*\Phi(-\delta_1^*)\}\\
&+\{A_0e^{\mu t}\Phi_2(-\delta_1^*+\sigma\sqrt{t},h_1^*+\sigma Rt/\sqrt{\eta};\rho^*)-De^{-r_L\tau}\{\Phi_2(-\delta_1^*,h_1^*;\rho^*)\}\\
&-e^{-\sigma h_1^*\sqrt{\eta}}\{A_0e^{\mu t-\sigma^2t/2+\sigma_S^2/2}\Phi_2(-\delta_1^*+\sigma(1-R)\sqrt{t},h_1^*-\sigma(1-R)\tau/\sqrt{\eta};\rho^*)\\
&\quad -De^{-r_L\tau+\sigma^2\eta/2}\Phi_2(-\delta_1^*-\sigma R\sqrt{t},h_1^*-\sigma\sqrt{\eta};\rho^*)\}]\\
&+\frac{1}{\xi_2^*-e^{-r_L\tau}}[(e^{(r_M-r_L)\tau}-1)\{A_0e^{\mu t}\Phi(\delta_2^*-\sigma\sqrt{t})-D\xi_2^*\Phi(\delta_2^*)\}\\
&+\{A_0e^{\mu t}\Phi_2(\delta_2^*-\sigma\sqrt{t},h_2^*+\sigma Rt/\sqrt{\eta};-\rho^*)-De^{-r_L\tau}\Phi_2(\delta_2^*,h_2^*;-\rho^*)\}\\
&-e^{-\sigma h_2^*\sqrt{\eta}}\{A_0e^{\mu t-\sigma^2t/2+\sigma_S^2/2}\Phi_2(\delta_2^*-\sigma(1-R)\sqrt{t},h_2^*-\sigma(1-R)\tau/\sqrt{\eta};-\rho^*)\\
&\quad -De^{-r_L\tau+\sigma^2\eta/2}\Phi_2(\delta_2^*+\sigma R\sqrt{t},h_2^*-\sigma\sqrt{\eta};-\rho^*)\}]\\
&+D\{\Phi_2(d_S,\delta_1^*;\rho_S)-\Phi_2(d_S,\delta_2^*;\rho_S)\}\\
&-De^{-d_S\sigma_S+\sigma_S^2/2}\{\Phi_2(d_S-\sigma_S,\delta_1^*-\rho_S\sigma_S;\rho_S)-\Phi_2(d_S-\sigma_S,\delta_2^*-\rho_S\sigma_S;\rho_S)\}.
\end{aligned}
\tag{39}
$$

Although the expression looks complicated, it is a simple combination of certain bivariate normal distribution functions and a univariate one. It is as tractable as the EL formulation in (25). Using it, we can analyze the parameter's effects on UL.

IV. Numerical example

In this section, we show how varying EaD under the EL minimization strategy shifts EL

and UL measured at time 0 from the original values. Let D = 100, t = 1, T = 2, μ = 5%, σ = 10%, $r_{L0} = r_L$ = 1%, $r_{M0} = r_M$ = 0.5% as in Table 1. In addition, let R = 0.12, and α = 99.9% for UL valuation. Table 2 shows ELs and ULs with an optimal additional loan Δ^* for certain values of A_0. For comparison, we also show ELs and ULs without an additional loan, i.e., $\Delta^* = 0$. We also show ULs with and without an optimal additional loan in the case of R = 0.24. The table shows that the EL minimizing principle increases ULs despite the decrease in ELs. The larger the asset correlation R is, the more the UL increases.

Table 2. EL and UL with/without an additional loan

A_0	$EL(\Delta^*)$	$EL(0)$	R = 0.12				R = 0.24	
			$SEL(\Delta^*)$	$SEL(0)$	$UL(\Delta^*)$	$UL(0)$	$UL(\Delta^*)$	$UL(0)$
80	10.78	12.00	**30.16**	23.18	**19.37**	11.18	**27.40**	15.81
85	7.45	8.03	**22.26**	18.62	**14.81**	10.60	**21.30**	15.37
90	4.66	4.90	**15.97**	14.32	**11.31**	9.42	**16.82**	14.16
95	2.54	2.63	**11.18**	10.43	**8.64**	7.80	**13.56**	12.28
100	1.06	1.10	**7.69**	7.12	**6.63**	6.02	**11.17**	9.97
105	0.11	0.15	**5.32**	4.48	**5.21**	4.32	**9.59**	7.58
110	–0.47	–0.40	**3.91**	2.50	**4.38**	2.90	**8.85**	5.39
120	–1.06	–0.86	**3.16**	0.23	**4.22**	1.09	**9.63**	2.25

Note: For simplicity, let $EL(\Delta^*) \equiv E_0[L_T(\Delta^*)]$, $SEL(\Delta^*) \equiv E_0[SEL_t(\Delta^*)]$, $UL(\Delta^*) \equiv E_0[SEL_t(\Delta^*)] - E_0[L_T(\Delta^*)]$, $EL(0) \equiv E_0[L_T(0)]$, $SEL(0) \equiv E_0[SEL_t(0)]$, and $UL(0) \equiv E_0[SEL_t(0)] - E_0[L_T(0)]$.

V. Conclusions

In this paper, we developed a structural model incorporating the relationship between PD, LGD, and EaD. We assumed that a bank takes strategic control of EaD by supplying an additional loan. Our model involves dependence of EaD on the stochastic asset value of the firm. The dependence changed from EL and UL with a fixed EaD to those with stochastic EaD. We derived analytical formulations for EL and UL using bivariate normal distribution functions and provided numerical examples.

There are two cases where an additional loan decreases EL: i) the asset/liability ratio of the firm is low however the firm's expected growth rate is high; ii) the asset/liability

ratio of the firm is high and the lending interest rate is high. With a given firm's expected growth rate and given interest rates, the two cases are identified by two thresholds for the asset/liability ratio at time *t*. The bank maintains the loan amount when the asset/liability ratio is between the two thresholds. Given the bank's strategy, the bank decreases the EL measured at time 0. On the other hand, the bank has a greater risk of the UL.

Focusing on the analytical evaluation of EL and UL incorporating the change in EaD, our model examines one simple case of stochastic development in EaD. We leave the following points to be studied on more realistic assumptions.

a. Modeling the firm's demand for an additional loan
b. Use of an adjustable lending rate for the additional loan according to the firm's credit risk
c. Multiple timings for the supply of additional loans until maturity
d. Choice of an alternative bank optimization function
e. Change in the parameters of the firm's asset development, μ and σ.

As for points a, b, and c, one possible approach is an equilibrium model of loan demand and supply with a flexible lending rate, where additional loans are executable at any time during a given loan period.

As for point d, the extension from our setting requires the identification of the bank's preference about the trade-off between return and risk. A simple alternative is to minimize EL subject to the upper limit of UL, which defines risk capital allocated to the business undertaking the loan.

As for point e, we fix μ and σ to derive analytical solutions for EL and UL. A possible interpretation is that a bank chooses an appropriate loan period in which these parameters for the firm's growth are stable.

Despite the many assumptions required to obtain analytical formulations for EL and UL, our model shows the rationality of an additional loan under EL minimization because of a decrease in PD and an increase in interest earnings. It also shows the mechanics of how EaD affects PD and LGD systematically. The ideas presented here may provide a clue to estimating EaD, PD, and LGD within the advanced internal ratings-based approach in Basel II.

Appendix 1. Decision on an additional loan at time t

A. Critical value of asset at time t

The first derivative of $f(d)$ given in equation (8) is:

$$\begin{aligned} f'(d) &= \phi(d) - e^{(\mu - r_L)\tau}\phi(d - \sigma\sqrt{\tau}) \\ &= \{1 - e^{(\mu - r_L - \sigma^2/2)\tau} e^{\sigma\sqrt{\tau}d}\}\phi(d). \end{aligned} \tag{A-1}$$

It follows that:

$$f'(d)\begin{cases} > 0 & if \quad d < \bar{d}, \\ < 0 & if \quad d > \bar{d}, \end{cases} \tag{A-2}$$

where

$$\bar{d} = \frac{r_L - \mu + \sigma^2/2}{\sigma}\sqrt{\tau}\,. \tag{A-3}$$

Characteristic values of $f(d)$ are given as:

$$\lim_{d \to -\infty} f(d) = e^{(r_M - r_L)\tau} - 1, \tag{A-4}$$

$$\lim_{d \to \infty} f(d) = e^{(r_M - r_L)\tau} - e^{(\mu - r_L)\tau} = \{e^{r_M \tau} - e^{\mu\tau}\}e^{-r_L\tau}, \tag{A-5}$$

$$f(\bar{d}) = e^{(r_M - r_L)\tau} - 1 + \Phi(\bar{d}) - e^{(\mu - r_L)\tau}\Phi(\bar{d} - \sigma\sqrt{\tau})\,. \tag{A-6}$$

From equations (A-4)–(A-6), on the assumption $f(\bar{d}) > 0$, there exists d_1^* such that $f(d_1^*) = 0$ on $d_1^* < \bar{d}$ if $r_L > r_M$, and there exists d_2^* such that $f(d_2^*) = 0$ on $\bar{d} < d_2^*$ if $\mu > r_M$.

Here, we consider the level of asset A_t at which the bank supplies an additional loan for given values of t, T, r_M, r_L, μ, σ. We define $\tilde{d}(A_t)$ to be $d_t(0)$ as a function of A_t below.

$$\tilde{d}(A_t) = d_t(0) = \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\frac{D}{A_t} - \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}. \tag{A-7}$$

and define the function $h(A_t)$ as:

$$h(A_t) \equiv \lim_{\Delta \to 0}\frac{\partial EL_t(\Delta)}{\partial \Delta} = e^{(r_M - r_L)\tau} - 1 + \Phi(\tilde{d}(A_t)) - e^{(\mu - r_L)\tau}\Phi(\tilde{d}(A_t) - \sigma\sqrt{\tau})\,. \tag{A-8}$$

The bank supplies an additional loan if $h(A_t) < 0$. We can confirm that the condition $h(A_t) < 0$ is equivalent to the condition $\tilde{d}(A_t) < d_1^*$ or $\tilde{d}(A_t) > d_2^*$.

If $\tilde{d}(A_t) < d_1^*$, then $A_t > De^{-d_1^*\sigma\sqrt{\tau} - (\mu - \sigma^2/2)\tau} = D\xi_1^*$. If $\tilde{d}(A_t) > d_2^*$, then

$A_t < De^{-d_2^*\sigma\sqrt{\tau}-(\mu-\sigma^2/2)\tau} = D\xi_2^*$. $D\xi_1^*$ and $D\xi_2^*$ are the thresholds of A_t for which the bank supplies an additional loan.

Now, we derive an optimal additional loan. When $A_t > D\xi_1^*$, then the relation

$$\frac{A_t}{D} > \frac{A_t + \Delta e^{-r_L\tau}}{D+\Delta} \geq \xi_1^*, \tag{A-9}$$

holds for an additional loan amount $\Delta > 0$ using $d_1^* < \bar{d}$. It implies that the optimal loan amount Δ_1^* satisfies:

$$\frac{A_t + \Delta_1^* e^{-r_L\tau}}{D+\Delta_1^*} = \xi_1^*. \tag{A-10}$$

Similarly, when $A_t < D\xi_2^*$, the following relation holds.

$$\frac{A_t}{D} < \frac{A_t + \Delta e^{-r_L\tau}}{D+\Delta} \leq \xi_2^*. \tag{A-11}$$

It implies that the optimal loan amount Δ_2^* satisfies

$$\frac{A_t + \Delta_2^* e^{-r_L\tau}}{D+\Delta_2^*} = \xi_2^*. \tag{A-12}$$

From equations (A-10) and (A-12), we obtain equation (11).

B. Parameter relation for the optimal additional loan

We assume that the firm accepts the additional loan. If the firm maximizes the expected value of the equity after accepting an additional loan Δ, this assumption is consistent with the firm's behavior in the case of $\mu > r_L$. The expected value of the equity is evaluated as $(A_t + \Delta e^{-r_L\tau})e^{\mu\tau}\Phi(d_t(\Delta) - \sigma\sqrt{\tau}) - (D+\Delta)\Phi(d_t(\Delta))$, and the marginal expected value of the equity is given by $e^{(\mu-r_L)\tau}\Phi(d_t(\Delta) - \sigma\sqrt{\tau}) - \Phi(d_t(\Delta))$. If $\mu > r_L$, the marginal expected value at $\Delta = 0$, $e^{(\mu-r_L)\tau}\Phi(d_t(0) - \sigma\sqrt{\tau}) - \Phi(d_t(0))$, is always nonnegative. It implies that an additional loan increase the expected value of the equity.

The optimal additional loan amount may be infinite if $r_L > r_M$ and $\mu > r_M$. If $r_L \leq r_M$ or $\mu \leq r_M$, the amount is always finite. The proof is given in subsection D in this appendix.

These relations are summarized as Table A- 1.

Table A- 1 Parameter relation for an additional loan at time *t*

Parameter condition	Loan amount	Supply of additional loan by the bank	Demand of the firm
$\mu \geq r_L > r_M$	finite infinite	Done if $A_t > D\xi_1^*$ or $A_t < D\xi_2^*$ Done regardless of the level of A_t	consistent
$r_L > \mu \geq r_M$	finite infinite	Done if $A_t > D\xi_1^*$ or $A_t < D\xi_2^*$ Done regardless of the level of A_t	unknown
$\mu \geq r_M \geq r_L$	finite	Done if $A_t < D\xi_2^*$	consistent
$r_L \geq r_M > \mu$	finite	Done if $A_t > D\xi_1^*$	unknown
$r_M \geq \mu > r_L$ $r_M \geq r_L \geq \mu$	No additional loan		

C. Equivalent condition that the optimal additional loan is finite

Under the condition $\partial EL_t(\Delta)/\partial\Delta|_{\Delta=0} < 0$, if $\partial EL_t(\Delta)/\partial\Delta|_{\Delta\to\infty} < 0$ then the optimal additional loan amount is infinite. Although $\partial^2 EL_t(\Delta)/\partial\Delta^2$ is always positive[3], $\partial^2 EL_t(\Delta)/\partial\Delta^2$ converges to 0 as $\Delta \to \infty$. It implies that the marginal expected loss converges to a constant as:

$$\lim_{\Delta\to\infty} \frac{\partial EL_t(\Delta)}{\partial\Delta} = e^{(r_M - r_L)\tau} - 1 + \Phi(\bar{d}) - e^{(\mu - r_L)\tau}\Phi(\bar{d} - \sigma\sqrt{\tau}). \tag{A-13}$$

The necessary and sufficient condition that the optimal additional loan amount is finite is that the right hand side of equation (A-13) is positive. It is equivalent to:

$$f(\bar{d}) > 0. \tag{A-14}$$

D. Parameter conditions that the optimal additional loan is finite

In this subsection, we prove that the optimal additional loan is finite if $r_L \leq r_M$ or $\mu \leq r_M$.

For preparation, we show proposition A-1.

Proposition A-1 $\Phi(-\alpha + s/2) - e^{\alpha s}\Phi(-\alpha - s/2) > 0$ for any $\alpha \in R$ and $s > 0$.

(Proof) Let X be a random variable distributed as $\ln X \sim \mathrm{N}(\alpha s - s^2/2, s^2)$. Then:

---

[3] $\dfrac{\partial^2 EL_t(\Delta)}{\partial\Delta^2}$ is evaluated as $\dfrac{\partial^2 EL_t(\Delta)}{\partial\Delta^2} = \dfrac{(A_t - De^{-r_L\tau})^2}{\sigma\sqrt{\tau}(D+\Delta)(A_t + \Delta e^{-r_L\tau})^2}\phi(d_t(\Delta))$.

$$\Pr((1-X)^+ > 0) = \Pr(X < 1) = \Phi(-\alpha + s/2) > 0 . \qquad \text{(A-15)}$$

By definition, $(1-X)^+ \geq 0$ and, from (A-15), the probability that $(1-X)^+ > 0$ is positive. Therefore,

$$0 < E[(1-X)^+] = \int_{-\infty}^{-\alpha+s/2} (1 - e^{sy+\alpha s - s^2/2})\phi(y)dy = \Phi(-\alpha + s/2) - e^{\alpha s}\Phi(-\alpha - s/2) .$$

(Q.E.D.)

If $r_L \leq r_M$ then we can confirm that $e^{(r_M - r_L)\tau} - 1 \geq 0$ and $\Phi(\bar{d}) - e^{(\mu - r_L)\tau}\Phi(\bar{d} - \sigma\sqrt{\tau}) > 0$ by applying proposition A-1 with $\alpha = (\mu - r_L)\sqrt{\tau}/\sigma$ and $s = \sigma\sqrt{\tau}$. This implies that the optimal additional loan is finite if $r_L \leq r_M$.

On the other hand, if $\mu \leq r_M$, then:

$$\begin{aligned} \lim_{\Delta\to\infty} \frac{\partial EL_t(\Delta)}{\partial \Delta} &\geq e^{(\mu - r_L)\tau} - 1 + \Phi(\bar{d}) - e^{(\mu - r_L)\tau}\Phi(\bar{d} - \sigma\sqrt{\tau}) \\ &= e^{(\mu - r_L)\tau}\{\Phi(-\bar{d} + \sigma\sqrt{\tau}) - e^{(r_L - \mu)\tau}\Phi(-\bar{d})\}. \end{aligned} \qquad \text{(A-16)}$$

By applying proposition A-1 with $\alpha = (r_L - \mu)\sqrt{\tau}/\sigma$ and $s = \sigma\sqrt{\tau}$, the right hand side of equation (A-16) is positive. It implies that the optimal additional loan is finite if $\mu \leq r_M$.

Appendix 2. EL with an additional loan in each state

State I

From equation (12),

$$\begin{aligned} E_0[L_T(\Delta_1^*)1_{A_t > D\xi_1^*}] &= E_0[E_t[L_T(\Delta_1^*)]1_{A_t > D\xi_1^*}] \\ &= D\{\Phi(d_1^*) + e^{(r_{M0} - r_{L0})T} - 1\}\Pr[A_t > D\xi_1^*] - e^{\mu\tau}\Phi(d_1^* - \sigma\sqrt{\tau})E[A_t 1_{A_t > D\xi_1^*}]. \end{aligned} \qquad \text{(A-17)}$$

Here, the expectation in the second term on the right hand side of equation (A-17) is evaluated as:

$$\begin{aligned} E_0[A_t 1_{A_t > D\xi_1^*}] &= \int_{\delta_1^*}^{\infty} A_0 e^{(\mu - \sigma^2/2)t} e^{\sigma\sqrt{t}v}\phi(v)dv = A_0 e^{\mu t}\int_{\delta_1^*}^{\infty}\phi(v - \sigma\sqrt{t})dv \\ &= A_0 e^{\mu t}\{1 - \Phi(\delta_1^* - \sigma\sqrt{t})\} = A_0 e^{\mu t}\Phi(-\delta_1^* + \sigma\sqrt{t}). \end{aligned} \qquad \text{(A-18)}$$

By substituting equations (15) and (A-18) for equation (A-17), we obtain equation (17).

State II

From equation (13),

$$E_0[L_T(0)1_{D\xi_2^* \le A_t \le D\xi_1^*}] = D(e^{(r_{M0}-r_{L0})T} - 1)\Pr[D\xi_2^* \le A_t \le D\xi_1^*] + DE_0[\Phi(\tilde{d}(A_t))1_{D\xi_2^* \le A_t \le D\xi_1^*}] - e^{\mu\tau}E_0[A_t\Phi(\tilde{d}(A_t) - \sigma\sqrt{\tau})1_{D\xi_2^* \le A_t \le D\xi_1^*}], \quad \text{(A-19)}$$

where $\tilde{d}(A_t)$ is given as equation (A-7). Let:

$$\ln A_t = \ln A_0 - (\mu - \sigma^2/2)t + \sigma\sqrt{t}v,$$

using the standard normal random variable v. Then, $E_0[\Phi(\tilde{d}(A_t))1_{D\xi_2^* \le A_t \le D\xi_1^*}]$ in equation (A-19) is evaluated as:

$$E_0[\Phi(\tilde{d}(A_t))1_{D\xi_2^* \le A_t \le D\xi_1^*}] = \int_{\delta_2^*}^{\delta_1^*}\Phi(d_0\sqrt{T/\tau} - v\sqrt{t/\tau})\phi(v)dv = \Phi_2(\delta_1^*, d_0; \rho) - \Phi_2(\delta_2^*, d_0; \rho), \quad \text{(A-20)}$$

where ρ is given as equation (21). Similarly,

$$\begin{aligned} &E_0[A_t\Phi(\tilde{d}(A_t) - \sigma\sqrt{\tau})1_{D\xi_2^* \le A_t \le D\xi_1^*}] \\ &= A_0e^{(\mu-\sigma^2/2)t}\int_{\delta_2^*}^{\delta_1^*}e^{\sigma\sqrt{t}v}\Phi(d_0\sqrt{T/\tau} - \sigma\sqrt{\tau} - v\sqrt{t/\tau})\phi(v)dv \\ &= A_0e^{\mu t}\int_{\delta_2^*}^{\delta_1^*}\Phi(d_0\sqrt{T/\tau} - \sigma\sqrt{\tau} - v\sqrt{t/\tau})\phi(v - \sigma\sqrt{t})dv \\ &= A_0e^{\mu t}\{\Phi_2(\delta_1^* - \sigma\sqrt{t}, d_0 - \sigma\sqrt{T}; \rho) - \Phi_2(\delta_2^* - \sigma\sqrt{t}, d_0 - \sigma\sqrt{T}; \rho)\}. \end{aligned} \quad \text{(A-21)}$$

By substituting equations (18), (A-20) and (A-21) for equation (A-19), we obtain equation (20).

State III

From equation (14),

$$E_0[L_T(\Delta_2^*)1_{A_t < D\xi_2^*}] = D\{\Phi(d_2^*) + e^{(r_{M0}-r_{L0})T} - 1\}\Pr[A_t < D\xi_2^*] - e^{\mu\tau}\Phi(d_2^* - \sigma\sqrt{\tau})E_0[A_t1_{A_t < D\xi_2^*}]. \quad \text{(A-22)}$$

Similar to equation (A-18),

$$E_0[A_t1_{A_t < D\xi_2^*}] = A_0e^{\mu t}\Phi(\delta_2^* - \sigma\sqrt{t}). \quad \text{(A-23)}$$

By substituting equations (22) and (A-23) for equation (A-22), we obtain equation (23).

Appendix 3. SEL with an additional loan in each state

State I

From equation (28),

$$E_0[SEL_t(\Delta_1^*)1_{A_t>D\xi_1^*}] = D(e^{(r_{M0}-r_{L0})T}-1)\Pr[A_t > D\xi_1^*] + (e^{(r_M-r_L)\tau}-1)E_0[\Delta_1^* 1_{A_t>D\xi_1^*}] + E_0[E_t[(D+\Delta_1^*-A_T)^+ \mid X_T = -\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t]1_{A_t>D\xi_1^*}]. \quad \text{(A-24)}$$

The expectation in the second term on the right hand of equation (A-24) is evaluated as:

$$E_0[\Delta_1^* 1_{A_t>D\xi_1^*}] = E_0[\frac{A_t - D\xi_1^*}{\xi_1^* - e^{-r_L\tau}}1_{A_t>D\xi_1^*}] = \frac{E_0[A_t 1_{A_t>D\xi_1^*}] - D\xi_1^*\Pr[A_t > D\xi_1^*]}{\xi_1^* - e^{-r_L\tau}} = \frac{A_0 e^{\mu t}\Phi(-\delta_1^* + \sigma\sqrt{t}) - D\xi_1^*\Phi(-\delta_1^*)}{\xi_1^* - e^{-r_L\tau}}, \quad \text{(A-25)}$$

Let $X_{T-t} \equiv X_T - X_t$ and $Y_{T-t} \equiv Y_T - Y_t$. Using equation (11), $D + \Delta_1^* - A_T$ in equation (A-24) is expressed as:

$$\begin{aligned} D + \Delta_1^* - A_T &= D + \Delta_1^* - (A_t + \Delta_1^* e^{-r_L\tau})e^{(\mu-\sigma^2/2)\tau}e^{\sigma(\sqrt{R}X_{T-t}+\sqrt{1-R}Y_{T-t})} \\ &= (D+\Delta_1^*)\{1 - e^{-d_1^*\sigma\sqrt{\tau}}e^{\sigma(\sqrt{R}X_{T-t}+\sqrt{1-R}Y_{T-t})}\} \\ &= (D+\Delta_1^*)\{1 - e^{-d_1^*\sigma\sqrt{\tau}}e^{\sigma\{\sqrt{R}(-\sqrt{T}\Phi^{-1}(\alpha)-X_t)+\sqrt{1-R}Y_{T-t}\}}\}. \end{aligned} \quad \text{(A-26)}$$

Using this relation, the third term on the right hand side of equation (A-24) is evaluated as:

$$\begin{aligned} &E_0[E_t[(D+\Delta_1^*-A_T)^+ \mid X_T = -\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t]1_{A_t>D\xi_1^*}] \\ &= E_0[(D+\Delta_1^*)\{1 - e^{-d_1^*\sigma\sqrt{\tau}}e^{\sigma\{\sqrt{R}(-\sqrt{T}\Phi^{-1}(\alpha)-X_t)+\sqrt{1-R}Y_{T-t}\}}\}^+ 1_{A_t>D\xi_1^*}] \\ &= \frac{1}{\xi_1^* - e^{-r_L\tau}}E_0[(A_t - De^{-r_L\tau})(1 - e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{R}X_t}e^{\sigma\sqrt{1-R}Y_{T-t}})^+ 1_{A_t>D\xi_1^*}]. \end{aligned} \quad \text{(A-27)}$$

Here,

$$
\begin{aligned}
&E_0[(A_t - De^{-r_L\tau})(1-e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{R}X_t}e^{\sigma\sqrt{1-R}Y_{T-t}})^+ 1_{A_t>D\xi_1^*}] \\
&= E_0[(A_0 e^{(\mu-\sigma^2/2)t}e^{\sigma(\sqrt{R}X_t+\sqrt{1-R}Y_t)} - De^{-r_L\tau})(1-e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{R}X_t}e^{\sigma\sqrt{1-R}Y_{T-t}}) \\
&\qquad \times 1_{\sqrt{R}X_t+\sqrt{1-R}Y_t>\delta_1^*} 1_{\sqrt{1-R}Y_{T-t}<d_1^*\sqrt{\tau}+\sqrt{R}(\sqrt{T}\Phi^{-1}(\alpha)+X_t)}] \\
&= \int_{-\infty}^{\infty}\int_{\frac{\delta_1^*-\sqrt{R}x}{\sqrt{1-R}}}^{\infty}\int_{-\infty}^{\frac{d_1^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}} (A_0 e^{(\mu-\sigma^2/2)t}e^{\sigma\sqrt{t}(\sqrt{R}x+\sqrt{1-R}y)} - De^{-r_L\tau}) \\
&\quad (1-e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{Rt}x}e^{\sigma\sqrt{(1-R)\tau}w})\phi(w)\phi(y)\phi(x)dwdydx \\
&= A_0 e^{\mu t}\int_{-\infty}^{\infty}\Phi\left(\frac{-\delta_1^*+\sqrt{R}x}{\sqrt{1-R}}+\sigma\sqrt{(1-R)t}\right)\Phi\left(\frac{d_1^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}\right)\phi(x-\sigma\sqrt{Rt})dx \\
&\quad + A_0 e^{(\mu-\sigma^2/2)t}e^{\sigma^2(1-R)T/2}e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}\int_{-\infty}^{\infty}\Phi\left(\frac{-\delta_1^*+\sqrt{R}x}{\sqrt{1-R}}+\sigma\sqrt{(1-R)t}\right) \\
&\qquad \Phi\left(\frac{d_1^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}-\sigma\sqrt{(1-R)\tau}\right)\phi(x)dx \\
&\quad - De^{-r_L\tau}\int_{-\infty}^{\infty}\Phi\left(\frac{-\delta_1^*+\sqrt{R}x}{\sqrt{1-R}}\right)\Phi\left(\frac{d_1^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}\right)\phi(x)dx \\
&\quad + De^{-r_L\tau}e^{-d_1^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{\sigma^2(Rt+(1-R)\tau)/2}\int_{-\infty}^{\infty}\Phi\left(\frac{-\delta_1^*+\sqrt{R}x}{\sqrt{1-R}}\right) \\
&\qquad \Phi\left(\frac{d_1^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}-\sigma\sqrt{(1-R)\tau}\right)\phi(x+\sigma\sqrt{Rt})dx.
\end{aligned}
$$

Using the relation

$$
\int_{-\infty}^{\infty}\Phi\left(\frac{a+\sqrt{R}x}{\sqrt{1-R}}\right)\Phi\left(\frac{b+\sqrt{Rt/\tau}x}{\sqrt{1-R}}\right)\phi(x)dx = \Phi_2\left(a, \frac{b\sqrt{\tau}}{\sqrt{(1-R)\tau+Rt}}; \frac{R\sqrt{t}}{\sqrt{(1-R)\tau+Rt}}\right),
$$

equation (A-27) is reduced to:

$$
\begin{aligned}
&E_0[E_t[(D+\Delta_1^*-A_T)^+ \mid X_T = -\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t]1_{A_t>D\xi_1^*}] \\
&= \frac{1}{\xi_1^* - e^{-r_L\tau}}[A_0 e^{\mu t}\Phi_2(-\delta_1^*+\sigma\sqrt{t}, h_1^*+\sigma Rt/\sqrt{\eta}; \rho^*) \\
&\quad - A_0 e^{(\mu-\sigma^2/2)t}e^{\sigma^2(1-R)T/2-d_1^*\sigma\sqrt{\tau}-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}\Phi_2(-\delta_1^*+\sigma(1-R)\sqrt{t}, h_1^*-\sigma(1-R)\tau/\sqrt{\eta}; \rho^*) \\
&\quad - De^{-r_L\tau}\Phi_2(-\delta_1^*, h_1^*; \rho^*) \\
&\quad + De^{-r_L\tau}e^{-d_1^*\sigma\sqrt{\tau}-\sigma\sqrt{RT}\Phi^{-1}(\alpha)+\sigma^2(Rt+(1-R)\tau)/2}\Phi_2(-\delta_1^*-\sigma R\sqrt{t}, h_1^*-\sigma\sqrt{\eta}; \rho^*)],
\end{aligned}
\tag{A-28}
$$

where η, h_1^*, ρ^* are given as equations (31)–(33). By substituting equations (15), (A-25), and (A-28) for equation (A-24), we obtain equation (30).

State II

From equation (28),

$$E_0[SEL_t(0)1_{D\xi_2^* \le A_t \le D\xi_1^*}] = D(e^{(r_{M0}-r_{L0})T}-1)\Pr[D\xi_2^* \le A_t \le D\xi_1^*] + E_0[(D - A_0 e^{(\mu-\sigma^2/2)T} e^{\sigma\{-\sqrt{RT}\Phi^{-1}(\alpha)+\sqrt{1-R}(Y_t+Y_{T-t})\}})^+ 1_{D\xi_2^* \le A_t \le D\xi_1^*}]. \quad \text{(A-29)}$$

The second term on the right hand side of equation (A-29) is evaluated as:

$$\begin{aligned}
&E_0[(D - A_0 e^{(\mu-\sigma^2/2)T} e^{\sigma\{-\sqrt{RT}\Phi^{-1}(\alpha)+\sqrt{1-R}(Y_t+Y_{T-t})\}})^+ 1_{D\xi_2^* \le A_t \le D\xi_1^*}] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{d_S\sqrt{T}-y\sqrt{t}}{\sqrt{\tau}}} \int_{\frac{\delta_2^*-\sqrt{1-R}y}{\sqrt{R}}}^{\frac{\delta_1^*-\sqrt{1-R}y}{\sqrt{R}}} (D - A_0 e^{(\mu-\sigma^2/2)T} e^{\sigma\{-\sqrt{RT}\Phi^{-1}(\alpha)+\sqrt{1-R}(\sqrt{t}y+\sqrt{\tau}w)\}})\phi(x)\phi(w)\phi(y)dxdwdy \\
&= D\int_{-\infty}^{\infty} \Phi\left(\frac{d_S\sqrt{T}-y\sqrt{t}}{\sqrt{\tau}}\right)\left\{\Phi\left(\frac{\delta_1^*-\sqrt{1-R}y}{\sqrt{R}}\right)-\Phi\left(\frac{\delta_2^*-\sqrt{1-R}y}{\sqrt{R}}\right)\right\}\phi(y)dy \\
&\quad - A_0 e^{(\mu-\sigma^2R/2)T-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}\int_{-\infty}^{\infty}\Phi\left(\frac{d_S\sqrt{T}-y\sqrt{t}}{\sqrt{\tau}}-\sigma\sqrt{(1-R)\tau}\right) \\
&\qquad \times\left\{\Phi\left(\frac{\delta_1^*-\sqrt{1-R}y}{\sqrt{R}}\right)-\Phi\left(\frac{\delta_2^*-\sqrt{1-R}y}{\sqrt{R}}\right)\right\}\phi(y-\sigma\sqrt{(1-R)t})dy \\
&= D\{\Phi_2(d_S,\delta_1^*;\rho_S)-\Phi_2(d_S,\delta_2^*;\rho_S)\} \\
&\quad - A_0 e^{(\mu-\sigma^2R/2)T-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}\{\Phi_2(d_S-\sigma\sqrt{(1-R)T},\delta_1^*-\sigma(1-R)\sqrt{t};\rho_S) \\
&\qquad -\Phi_2(d_S-\sigma\sqrt{(1-R)T},\delta_2^*-\sigma(1-R)\sqrt{t};\rho_S)\},
\end{aligned} \quad \text{(A-30)}$$

where d_S, ρ_S are given as equations (35) and (36). By substituting equations (18) and (A-30) for equation (A-29), we obtain equation (34) using equation (37).

State III

Similar to the derivation of SEL for state I, from equation (28),

$$E_0[SEL_t(\Delta_2^*)1_{A_t<D\xi_2^*}] = D(e^{(r_{M0}-r_{L0})T}-1)\Pr[A_t < D\xi_2^*] + (e^{(r_M-r_L)\tau}-1)E_0[\Delta_2^* 1_{A_t<D\xi_2^*}] + E_0[E_t[(D+\Delta_2^*-A_T)^+ \mid X_T = -\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t]1_{A_t<D\xi_2^*}]. \quad \text{(A-31)}$$

Here,

$$E_0[\Delta_2^* 1_{A_t<D\xi_2^*}] = E_0\left[\frac{A_t - D\xi_2^*}{\xi_2^* - e^{-r_L\tau}}1_{A_t<D\xi_2^*}\right] = \frac{A_0 e^{\mu t}\Phi(\delta_2^*-\sigma\sqrt{t}) - D\xi_2^*\Phi(\delta_2^*)}{\xi_2^* - e^{-r_L\tau}}, \quad \text{(A-32)}$$

and

$$
\begin{aligned}
&E_0[E_t[(D+\Delta_2^*-A_T)^+ \mid X_T=-\sqrt{T}\Phi^{-1}(\alpha), X_t, Y_t]1_{A_t<D\xi_2^*}] \\
&=\frac{1}{\xi_2^*-e^{-r_L\tau}}E_0[(A_t-De^{-r_L\tau})(1-e^{-d_2^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{R}X_t}e^{\sigma\sqrt{1-R}Y_{T-t}})^+1_{A_t<D\xi_2^*}] \\
&=\frac{1}{\xi_2^*-e^{-r_L\tau}}\int_{-\infty}^{\infty}\int_{-\infty}^{\frac{\delta_2^*-\sqrt{R}x}{\sqrt{1-R}}}\int_{-\infty}^{\frac{d_2^*+\sqrt{RT/\tau}\Phi^{-1}(\alpha)+\sqrt{Rt/\tau}x}{\sqrt{1-R}}}(A_0e^{(\mu-\sigma^2/2)t}e^{\sigma\sqrt{t}(\sqrt{R}x+\sqrt{1-R}y)}-De^{-r_L\tau}) \\
&\qquad(1-e^{-d_2^*\sigma\sqrt{\tau}}e^{-\sigma\sqrt{RT}\Phi^{-1}(\alpha)}e^{-\sigma\sqrt{Rt}x}e^{\sigma\sqrt{(1-R)\tau}w})\phi(w)\phi(y)\phi(x)dwdydx \\
&=\frac{1}{\xi_2^*-e^{-r_L\tau}}[A_0e^{\mu t}\{\Phi_2(\delta_2^*-\sigma\sqrt{t},h_2^*+\sigma Rt/\sqrt{\eta};-\rho^*) \\
&\qquad-e^{-\sigma h_2^*\sqrt{\eta}-\sigma^2t/2+\sigma^2(1-R)T/2}\Phi_2(\delta_2^*-\sigma(1-R)\sqrt{t},h_2^*-\sigma(1-R)\tau/\sqrt{\eta};-\rho^*)\} \\
&\qquad-De^{-r_L\tau}\{\Phi_2(\delta_2^*,h_2^*;-\rho^*)-e^{-\sigma h_2^*\sqrt{\eta}+\sigma^2\eta/2}\Phi_2(\delta_2^*+\sigma R\sqrt{t},h_2^*-\sigma\sqrt{\eta};-\rho^*)\}],
\end{aligned}
\tag{A-33}
$$

where η, h_2^*, ρ^* are defined as equations (31)–(33). By substituting equations (22), (A-32) and (A-33) for equation (A-31), we obtain equation (38).

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