IMES DISCUSSION PAPER SERIES

# Testing for Coefficient Stability of AR(1) Model When the Null is an Integrated or a Stationary Process

Daisuke Nagakura

**Discussion Paper No. 2007-E-20**

IMES

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

2-1-1 NIHONBASHI-HONGOKUCHO
CHUO-KU,TOKYO 103-8660
JAPAN

**IMES Discussion Paper Series 2007-E-20**
**November 2007**

# Testing for Coefficient Stability of AR(1) Model When the Null is an Integrated or a Stationary Process

**Daisuke Nagakura***

**Abstract**

In this paper, we propose a test for coefficient stability of an AR(1) model against the random coefficient autoregressive model of order 1 or RCA(1) model without assuming a stationary nor a non-stationary process under the null hypothesis of constant coefficient. The proposed test is obtained as a modification of the locally best invariant (LBI) test by Lee (1998). We examine finite sample properties of the proposed test by Monte Carlo experiments comparing with other existing tests including the LBI test by McCabe and Tremayne (1995), which is for the null of unit root against the alternative of stochastic unit root.

**Keywords:** Random Coefficient Autoregressive Model; Stability; Constancy
**JEL classification:** C12, C22

***Economist, Institute for Monetary and Economic Studies, Bank of Japan (E-mail: daisuke.nagakura@boj.or.jp)

The author would like to thank Saraswata Chaudhuri, Eric Zivot, Don Percival, Richard Startz, Charles Nelson, Masahito Kobayashi, Keiji Nagai, Yoshihiko Nishiyama, Kohtaro Hitomi, Tatsuyoshi Okimoto, as well as the staff of the Institute for Monetary and Economic Studies (IMES), the Bank of Japan, and seminar participants at the University of Washington, Yokohama National University, and Kyoto University for their helpful comments. Any remaining errors are mine. Views expressed in this paper are those of the author and do not necessarily reflect the official views of the Bank of Japan.

# 1 Introduction

In this paper, we propose a test for coefficient stability of an AR(1) model against the alternative known as a random coefficient autoregressive model with order 1 or RCA(1) model. Specifically, we consider the following model as the alternative model:

$$\begin{array}{l} y_t = (\phi + b_t) y_{t-1} + \epsilon_t, \quad \text{for} \quad t = 1, 2, ...., \\ E(b_t) = E(\epsilon_t) = 0, \ E(b_t^2) = \omega^2, \ E(\epsilon_t^2) = \sigma^2, \ \text{cov}(b_t, \epsilon_t) = \psi\omega, \end{array} \tag{1}$$

where $(b_t, \ \epsilon_t)'$ is an iid random vector and $|\psi| \leq \sigma$. We set $y_0 = O_p(1)$ as the initial condition so that $y_0$ is allowed to be a constant or a certain specified distribution. The model reduces to the ordinary AR(1) model under the null hypothesis $H_0 : \omega^2 = 0$.

Note that the process defined in (1) is a one-sided process. It is known that the two-sided version of (1), i.e., $y_t^* = (\phi + b_t) y_{t-1}^* + \epsilon_t$, for $t = 0 \pm 1, \pm 2, ....$, has a strictly stationary solution (almost surely) if $\eta \equiv E(\log|\phi + b_t|) < 0$ and only if $\eta \leq 0$ (Quinn, 1982). It follows further that $y_t^*$ has the strictly stationary solution with a finite second-order stationary moment in the sense of mean-square convergence as well as almost sure convergence if and only if $\phi^2 + \omega^2 < 1$ (Nicholls and Quinn, 1982). Note that the latter condition is more restrictive than the former (see Hwang and Basawa, 2006, p.810); if the former condition is satisfied but the latter is not, then $y_t^*$ is strictly stationary with an infinite variance. These results on the two-sided RCA(1) model carry over to the one-sided RCA(1) model in (1) if $y_t$ starts with $y_0$ having the stationary solution of the two-sided RCA(1) model. One important feature of the RCA(1) model is that it exhibits conditional heteroskedasticity, which is often observed in financial time series. It is easy to show that $\text{var}(y_t|y_{t-1}) = y_{t-1}^2 \omega^2 + 2 y_{t-1} \psi\omega + \sigma^2$. Note that if $\psi = 0$, then the conditional variance structure is the same as that of the well-known autoregressive conditional heterosckedastic (ARCH) model. See Hwang and Basawa (1998); Aue *et al.* (2006); Hwang and Basawa (2006); Hwang *et al.* (2006) and references therein for more details on the properties and estimation of RCA models.

Testing the null hypothesis of constant coefficient, i.e., $\omega^2 = 0$, in the RCA(1) model has been considered by Nicholls and Quinn (1982), Akharif and Hallin (2003), Ramanathan and Rajarshi (1994), Leybourne *et al.* (1996),[1] McCabe and Tremayne (1995), and Lee (1998). In these papers, except for McCabe and Tremayne (1995) and Leybourne *et al.* (1996), a stationary process is assumed under the null hypothesis. The first three papers derive test statistics under the assumption that $b_t$ and $\epsilon_t$ are independent. In particular, the test by Nicholls and Quinn (1982) is not necessarily consistent without this assumption (see Lee, 1998, p.94). The latter three papers do not assume the independence of $b_t$ and $\epsilon_t$.

The last two papers are particularly relevant to our paper. The test statistics in McCabe and Tremayne (1995) and Lee (1998) are both derived as locally

[1] Actually, Nicholls and Quinn (1982), Akharif and Hallin (2003), and Leybourne *et al.* (1996) consider testing coefficient stability of AR($p$) models against RCA($p$) models.

best invariant (LBI) tests.[2] However, the null and alternative hypotheses of Lee (1998)'s LBI test are different from those of McCabe and Tremayne (1995)'s LBI test. Lee (1998) assumes that $\phi^2 + \omega^2 < 1$, and hence, the null hypothesis is a stationary AR(1) model while the alternative hypothesis is a stationary RCA(1) model. McCabe and Tremayne (1995) assume that $\phi = 1$ under both the null and the alternative hypotheses. When $\phi = 1$, the RCA(1) model is called a *stochastic unit root* process (Granger and Swanson, 1997; Leybourne *et al.*, 1996), which has recently been applied in several empirical finance literatures (Bleaney *et al.*, 1999; Sollis *et al.*, 2000; Bleaney and Leybourne, 2003). Thus, the null hypothesis for McCabe and Tremayne (1995)'s LBI test is a unit root process and the alternative hypothesis is a stochastic unit root process. The limit distributions of these two LBI tests are different and we have to refer to two different distribution table depending on the maintained assumption on the value of $\phi$ under the null hypothesis, which limits the usefulness of these two tests because, in practice, we rarely know whether the true process is stationary or non-stationary in advance.

The test statistic proposed in this paper assumes neither $\phi = 1$ nor $|\phi| < 1$ under the null hypothesis of $\omega^2 = 0$. More specifically, it follows the standard normal distribution asymptotically under the null regardless of $|\phi| < 1$ or $\phi = 1$ so that we can use the same distribution table. In other words, we can purely test coefficient stability without the maintained assumptions on $\phi$. If the test rejects the null hypothesis, then we can go to the estimation of RCA(1) model and if the test accepts the null hypothesis, then we can utilize the usual unit root tests for examining non-stationarity of the series.

We obtain the test statistic as a modification of Lee (1998)'s LBI test. Lee (1998) prove that his LBI test follows the standard normal distribution under the null when $|\phi| < 1$. We show that Lee's LBI test follows the standard normal distribution even when $\phi = 1$ if the correlation between $\epsilon_t$ and $\epsilon_t^2$ is zero. Note that if $\epsilon_t$ has a symmetric distribution, then $E(\epsilon_t^3) = 0$ and so the correlation, $\rho \equiv E(\epsilon_t^3)/[\sqrt{\text{var}(\epsilon_t^2)}\sqrt{\text{var}(\epsilon_t)}]$ is zero. It is shown that if the correlation is not zero, the limit distribution of Lee's LBI test is represented by a weighted sum of the standard normal and a non-standard distribution with weights depending on the value of the correlation. Based on the result, we construct our test statistic by subtracting the latter part and re-weighting the remaining part with an estimate of the correlation so that the asymptotic null distribution is the standard normal regardless of the value of the correlation when $\phi = 1$. We also show that the proposed test statistic follows the standard normal distribution even when $|\phi| < 1$ under the null. Furthermore, it is proved that the test is consistent against stationary RCA(1) models with a finite fourth moment. Interestingly, our Monte Carlo experiments show that Lee's LBI and the proposed tests are more powerful than McCabe and Tremayne (1995)'s LBI test even when $\phi = 1$.

The rest of the paper is organized as follows. In Section 2, we derive the limit

[2] Leybourne *et al.* (1996) derive their test statistic as a score test, which takes the same form as the LBI test by McCabe and Tremayne (1995) in the case of RCA(1) model.

distribution of Lee's LBI test under the null hypothesis of $\omega^2 = 0$ even when $\phi = 1$. A new test is proposed in this section. In Section 3, we conduct several Monte Carlo experiments to check the finite sample performances of the proposed test comparing with Lee (1998) and McCabe and Tremayne (1995)'s LBI tests. The Appendix provides proofs for the theorems in the text.

# 2 Test Statistics

## 2.1 LBI Tests of Lee (1998) and McCabe and Tremayne (1995)

Lee (1998) derive a locally best invariant (LBI) test (see Ferguson, 1967, p.235) for the null hypothesis $H_0 : \omega^2 = 0$ against the alternative hypothesis $H_1 : \omega^2 > 0$ under the assumption that $(b_t, \epsilon_t)$ are jointly normal. It is assumed that $|\psi| < \sigma$ in deriving the test statistic but not assumed in deriving the limit distribution. We do not assume this condition in this paper. Hence, the model defined in (1) covers the Markovian bilinear model as a special case. See Lee (1998, p.98).

We consider the test statistic defined in Theorem 3.2 in Lee (1998, p.96). Below in (2), we give a slightly simplified form of the original test statistic ignoring terms that do not affect the asymptotic distribution.

**(Lee test)**

$$\widetilde{Z}_T \equiv [\widehat{\tau}_T \kappa_T(\widehat{\phi})]^{-1} T^{-1/2} Z_T(\widehat{\phi}), \tag{2}$$

where

$$\begin{aligned} Z_T(\phi) &\equiv \sum_{t=1}^{T} [\epsilon_t^2(\phi) - \sigma_T^2(\phi)] y_{t-1}^2 \quad \sigma_T(\phi) \equiv \left[ T^{-1} \sum_{t=1}^{T} \epsilon_t^2(\phi) \right]^{1/2}, \\ \epsilon_t(\phi) &\equiv y_t - \phi y_{t-1}, \quad \kappa_T(\phi) \equiv \left[ T^{-1} \sum_{t=1}^{T} \epsilon_t^4(\phi) - \sigma_T^4(\phi) \right]^{1/2}, \\ \widehat{\tau}_T &\equiv \left[ T^{-1} \sum_{t=1}^{T} y_{t-1}^4 - \left( T^{-1} \sum_{t=1}^{T} y_{t-1}^2 \right)^2 \right]^{1/2}, \end{aligned} \tag{3}$$

and $\widehat{\phi}$ is a $\sqrt{T}$ consistent estimator for $\phi$ (e.g., OLS).

Hereafter, we call this test statistic the **Lee test**. We abbreviate $\kappa_T(\widehat{\phi})$, $\sigma_T(\widehat{\phi})$, and $\epsilon_t(\widehat{\phi})$, to $\widehat{\kappa}_T$, $\widehat{\sigma}_T$, and $\widehat{\epsilon}_t$, respectively, for notational convenience. We can see that this test statistic is essentially an estimate for the correlation between $\epsilon_t^2$ and $y_{t-1}^2$ (notice that $\widehat{\tau}_T^2$ is an estimate for $\text{var}(y_t^2)$ if the process is stationary and ergodic with a finite fourth moment). It is easy to check that the correlation between $\epsilon_t^2$ and $y_{t-1}^2$ is zero under the null hypothesis and the Lee test examines whether the correlation is significantly different from zero. From the conditional variance structure of RCA(1) models, we can also see that the Lee test examines a certain type of conditional heteroskedasticity. We expect that the test would have a non-trivial power against a similar type of conditional heteroskedasticity such as that of bilinear models (we examine this point by Monte Carlo experiment in Section 3).

Lee (1998) proves, without the normality assumption of $(b_t, \epsilon_t)$, but assuming the existence of finite fourth moments for $\epsilon_t$ and $b_t$ and $\phi^2+\omega^2 < 1$, that if $T^{1/2}(\widehat{\phi}-\phi) = O_p(1)$ under both $H_0$ and $H_1$, then $\widetilde{Z}_T$ asymptotically follows the standard normal distribution under the null hypothesis and is a consistent test against stationary RCA(1) models with a finite forth moment. Notice that the assumption $\phi^2+\omega^2 < 1$ excludes the case where $\phi = 1$. Thus, Lee (1998)'s result cannot be directly applied when the null model is unit root non-stationary, i.e., $\phi = 1$.

McCabe and Tremayne (1995) derive a LBI test (Hereafter the MT test) for the null hypothesis $H_0 : \omega^2 = 0$ assuming that $(b_t, \epsilon_t)$ are jointly normal and $\phi = 1$. We consider the test statistic proposed in Corollary 3 in McCabe and Tremayne (1995). The test statistic takes the following form:

**(MT test)**

$$Z_T^* \equiv [\kappa_T(1)\sigma_T^2(1)]^{-1}T^{-3/2}Z_T(1). \tag{4}$$

Note that, here $\phi$ is not estimated since it is assumed to be one. The asymptotic distribution of $Z_T^*$ is non-standard and its critical values are tabulated in Table 1 in McCabe and Tremayne (1995). Similarly to the Lee test, McCabe and Tremayne (1995) remove the normality assumption in deriving the asymptotic distribution of $Z_T^*$ under the null hypothesis. One important drawback of the MT test is that it converges in probability to zero if the true process is stationary and ergodic with a finite fourth moment, hence has no power against stationary RCA(1) models with a finite fourth moment.

In this paper, $\widehat{\phi}$ is supposed to satisfy the following property under the null hypothesis: $T(\widehat{\phi} - 1) = O_p(1)$ when $\phi = 1$, and $T^{1/2}(\widehat{\phi} - \phi) = O_p(1)$ when $|\phi| < 1$. For example, the OLS estimator, $\widehat{\phi}_{ols} \equiv (\sum_{t=1}^T y_t y_{t-1})/(\sum_{t=1}^T y_{t-1}^2)$, satisfies the property.

The following theorem derives the asymptotic distribution of the Lee test when the true process is a unit root process. The notation " $\Rightarrow$" denotes weak convergence in the space D[0, 1] under the Skorohod metric,

**Theorem 1** Assume that $y_t$ is generated by $y_t = y_{t-1} + \epsilon_t$ for $t = 1, ...., T$ with $y_0 = O_p(1)$, where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with $E(\epsilon_t) = 0$, $E(\epsilon_t^2) = \sigma^2$, $Var(\epsilon_t^2) = \kappa^2$, and $Var(\epsilon_t^3) < \infty$.

(i) If $T(\widehat{\phi} - 1) = O_p(1)$, then ,as $T \to \infty$, we have $\widehat{\kappa}_T^2 \xrightarrow{p} \kappa^2$,

$$T^{-2}\widehat{\tau}_T^2 \Rightarrow \sigma^4 \int_0^1 \theta(r)^2 dr, \tag{5}$$

$$T^{-3/2}Z_T(\widehat{\phi}) \Rightarrow \sigma^2\kappa \int_0^1 \theta(r)dW^*(r), \tag{6}$$

and

$$\widetilde{Z}_T \Rightarrow \frac{\displaystyle\int_0^1 \theta(r) dW^*(r)}{\left(\displaystyle\int_0^1 \theta(r)^2 dr\right)^{1/2}}, \tag{7}$$

where $\theta(r) \equiv W_1(r)^2 - \int_0^1 W_1(s)^2 ds$, $W^*(r) \equiv \rho W_1(r) + (1-\rho^2)^{1/2} W_2(r)$, $W_1(r)$ and $W_2(r)$ are mutually independent standard Wiener processes, and $\rho \equiv corr(\epsilon_i, \epsilon_i^2 - \sigma^2)$.

(ii) The distribution of the random variable in (7) reduces to the standard normal distribution when $\rho = 0$.

**Proof**: See the Appendix.

The correlation between $\epsilon_t^2 - \sigma^2$ and $\epsilon_t$, which is equivalent to the correlation between $\epsilon_t^2$ and $\epsilon_t$ (since $E(\epsilon_t) = 0$), is zero if and only if $E(\epsilon_t^3) = 0$. Note that symmetricity of the distribution of $\epsilon_t$ is a sufficient, but not a necessary, condition for $E(\epsilon_t^3)$ to be zero.[3] The assumption that $\text{var}(\epsilon_t^3) < \infty$ is needed only for proving that $\widehat{\kappa}_T^2$ is a consistent estimator for $\kappa^2$.

Theorem 1 shows that the limit distribution of the Lee test is in fact the standard normal distribution under the null hypothesis even when $\phi = 1$ if $E(\epsilon_t^3) = 0$. However, this assumption is restrictive for some applications. We propose a modified version of the Lee test in the next section to deal with this problem.

## 2.2 Modified Lee Test

In this section, we propose a modified version of the Lee test (hereafter modified Lee test). It is shown that the modified Lee test asymptotically follows the standard normal distribution regardless of the value of the correlation $\rho$ between $\epsilon_t$ and $\epsilon_t^2$, and when $\rho = 0$, it is asymptotically equivalent to the Lee test.

Basic idea of the modified Lee test is as follows. First, notice that the numerator of the right-hand side in (7) is alternatively represented as

$$\int_0^1 \theta(r) dW^*(r) = (1-\rho^2)^{1/2} \int_0^1 \theta(r) dW_2(r) + \rho \int_0^1 \theta(r) dW_1(r). \tag{8}$$

The result (ii) in Theorem 1 implies that if we subtract its second term from (8) and divide the remaining part, i.e., the first term, by $(1-\rho^2)^{1/2} \times [\int_0^1 \theta(r)^2 dr]^{1/2}$, we recover the standard normal distribution. We obtain the modified Lee test by making essentially this correction to the Lee test. To make this correction, we need a sequence that converges to the second term in (8). For constructing such a sequence, we define

$$G_T(\widehat{\phi}) \equiv \frac{1}{3} y_T^3 - \sum_{t=1}^{T} y_{t-1} \widehat{\epsilon}_t^2 - T^{-1} y_T \sum_{t=1}^{T} y_{t-1}^2. \tag{9}$$

[3] I thank Don Percival for pointing this point.

In the Appendix, we show that $T^{-3/2}G_T(\widehat{\phi}) \Rightarrow \sigma^3 \int_0^1 \theta(r)dW_1(r)$. Thus, utilizing $T^{-3/2}G_T(\widehat{\phi})$ with consistent estimators for $\rho$ and $\sigma$, we can obtain such a correction term.

It is shown in the Appendix that the following estimator is consistent for $\rho$ under the assumptions in Theorem 1:

$$\widehat{\rho}_T \equiv (\widehat{\sigma}_T \widehat{\kappa}_T)^{-1} T^{-1} \sum_{t=1}^{T} \widehat{\epsilon}_t^3. \tag{10}$$

This estimator is also consistent even when the underlying process is a stationary AR(1) process. To obtain Theorem 3 below, which shows the asymptotic normality of our test statistic defined in (13) when $|\phi| < 1$; however, we need an estimator that is consistent for $\rho$ when the underlying process is the unit root process but converges in probability to zero when the underlying process is a stationary AR(1) process.

One way to obtain such an estimator is to multiply $\widehat{\rho}_T$ by a sequence $s_T$ that satisfies the following condition:

$$s_T \xrightarrow{p} 1 \text{ if } \phi = 1 \text{ and } s_T \xrightarrow{p} 0 \text{ if } |\phi| < 1. \tag{11}$$

In this paper, we use

$$s_T \equiv 1 - \exp\left[ - \left( T^{-3/2} \widehat{\sigma}_T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \right)^{\delta} \right], \tag{12}$$

where $\delta > 0$ is a constant that satisfies the condition in Theorem 3. It is easy to see that $s_T$ satisfies the condition in (11) since $T^{-2}\widehat{\sigma}_T^{-2} \sum_{t=1}^{T} y_{t-1}^2 = O_p(1)$ when $\phi = 1$ and $T^{-1}\widehat{\sigma}_T^{-2} \sum_{t=1}^{T} y_{t-1}^2 = O_p(1)$ when $|\phi| < 1$. The value of $\delta$ controls the speed of convergence of $s_t$. If $\delta$ is large, then the convergence of $s_T$ to 0 or 1 is very fast; in that case, virtually $s_T$ takes only 0 or 1 even when $T$ is not very large.

It should be noted that this is not the only possible choice for $s_T$; another possible choice is, for example, $s_T = 1$ if $|\widehat{\phi} - 1| \leq T^{-1/2}$ and $s_T = 0$ if $|\widehat{\phi} - 1| > T^{-1/2}$. This choice of $s_T$ is motivated by recent papers by Perron and Yabu (2006a,b), who consider testing for trend coefficient with an integrated or a stationary component. Based on their results, it is easy to show that $s_T$ satisfies the condition in (11). How we should select $s_T$ would be one of the topics of subsequent research.

We are now ready to give our test statistic. Given $y_t$, $t = 0, ..., T$, the modified Lee test is defined as follows:

**(Modified Lee test)**

$$\widetilde{G}_{T,\delta} \equiv (1 - \rho_T^{*2})^{-1/2} \widehat{\tau}_T^{-1} T^{-1/2} \left[ \widehat{\kappa}_T^{-1} Z_T(\widehat{\phi}) - \rho_T^* \widehat{\sigma}_T^{-1} G_T(\widehat{\phi}) \right], \tag{13}$$

where $\rho_T^* \equiv \widehat{\rho}_T s_T$. Note that when $\rho_T^* = 0$, $\widetilde{G}_{T,\delta}$ reduces to $\widetilde{Z}_T$.

Theorem 2 shows that the asymptotic distribution of $\widetilde{G}_{T,\delta}$ is the standard normal distribution when the true data generating process is the unit root process described in Theorem 1, regardless of the value of the correlation.

**Theorem 2** Assume the same data generating process and conditions as in Theorem 1 for a series $\{y_t\}_{t=0}^{T}$. If $T(\widehat{\phi}-1)=O_p(1)$ and $\delta>0$, then as $T\to\infty$, we have $\rho_T^* \xrightarrow{p} \rho$,

$$T^{-3/2}G_T(\widehat{\phi}) \Rightarrow \sigma^3 \int_0^1 \theta(r)dW_1(r), \tag{14}$$

and

$$\widetilde{G}_{T,\delta} \xrightarrow{d} N(0,1), \tag{15}$$

where $\rho$, $\theta(r)$, and $W_1(r)$ are defined as in Theorem 1.

**Proof**. See the Appendix.

If we assume that $E(|\phi+b_t|^4)<1$ and the existence of finite fourth moments for $\epsilon_t$ and $b_t$ for a two sided RCA(1) model $y_t^*$, then we have $E(|y_t^*|^4)<\infty$ (Lemma 3 in Aue et al., 2006). Theorem 3 shows the asymptotic normality of $\widetilde{G}_{T,\delta}$ under the null and $|\phi|<1$. Theorem 3 also shows that the test is a consistent test against the alternative of RCA(1) model with $E(|\phi+b_t|^4)<1$.

**Theorem 3** Assume that a series $\{y_t\}_{t=0}^{T}$ follows the RCA(1) model defined in (1) and $E(b_t^4)<\infty$, $E(\epsilon_t^4)<\infty$, $E(|\phi+b_t|^4)<1$. If $T^{1/2}(\widehat{\phi}-\phi)=O_p(1)$ under both $H_0:\omega^2=0$ and $H_1:\omega^2>0$, then as $T\to\infty$, we have

(a) for $\delta>0$ $\widetilde{G}_{T,\delta} \xrightarrow{d} N(0,1)$ under $H_0$.

(b) for $\delta\geq 1$ $\widetilde{G}_{T,\delta} \xrightarrow{p} \infty$ under $H_1$.

**Proof**. See the Appendix.

In Theorem 3, we do not need to assume that $\mathrm{var}(\epsilon_t^3)<\infty$. The condition that $T^{1/2}(\widehat{\phi}-\phi)=O_p(1)$ under both $H_0$ and $H_1$ in Theorem 4 is satisfied, for example, by the OLS estimator $\widehat{\phi}_{ols}$ (see Hwang and Basawa, 1998, 2006).

We expect that the proposed test is also consistent against RCA(1) models with $E(|\phi+b_t|^4)\geq 1$; however, we have not succeeded in proving it mathematically. Instead, we examine this issue by Monte Carlo experiments in the next section.

# 3 Monte Carlo experiment

In this section we conduct Monte Carlo (MC) experiments to compare empirical sizes and powers of the Lee test defined in (2), the MT test defined in (4), and the modified Lee test defined in (13). We examine two values for $\delta$ to see how the value of $\delta$ affects the finite sample propertiles of the modified Lee test. Throughout the experiments in this section, we calculate the test statistics with the OLS estimator, $\widehat{\phi}_{ols}$, the initial value is fixed at $y_0=0$, the number of MC replications is $10,000$, and the sample size is $T=50$, 100, 200, or 1000.

## 3.1 Size property

We generate the null model of AR(1) with various values of $\phi$. Specifically, we examine nine values: $\phi = -0.9$, $-0.6$, $-0.3$, $0.0$, $0.3$, $0.6$, $0.9$, $0.95$, $0.98$, and $1.0$. As to the distribution of $\epsilon_t$, we examine three distributions: (1) $\epsilon_t \sim N(0,1)$ $(\rho = 0.0)$, (2) $\epsilon_t \sim (1/\sqrt{2})(\chi^2(1) - 1)$ $(\rho = 2/\sqrt{7} \approx 0.7559)$, and (3) $\epsilon_t \sim (1/\sqrt{20})(\chi^2(10) - 10)$ $(\rho = 0.5)$, where $\chi^2(k)$ denotes the chi-square distribution with $k$ degrees of freedom and inside the parentheses is the correlation between $\epsilon_t$ and $\epsilon_t^2$. In all cases, the mean and variance of $\epsilon_t$ are zero and one, respectively. Cases (2) and (3) are for addressing how the value of $\rho$ affects the size. Figure (1) draws the pdfs of these three distributions. We see that the distribution of $\epsilon_t$ is symmetric in Case (1), heavily skewed to the right in Case (2), moderately skewed to the right in Case (3). Case (3) is an intermediate case between Case (2) and (3).

We conduct tests of 5% nominal level. The critical value of the Lee and modified Lee tests are set to 1.6449, i.e., the 95% point of the standard normal distribution. We reject the null when the test statistic is above the critical value. Note that the rejection region is in the upper tail area since the statistic diverges to $\infty$ under the alternative of RCA(1) models. For the MT test, we use the critical values tabulated in Table 1 in McCabe and Tremayne (1995, p.1022); it is 0.77 for $T = 50$, 0.79 for $T = 100$, 0.80 for $T = 200$, and 0.81 for $T = 1000$.

Table 1 reports the results of Monte Carlo experiment for Case (1), where the correlation $\rho$ is zero. In this case, according to Theorem 1, the statistic $\widetilde{Z}_T$ asymptotically follows the standard normal distribution regardless of $\phi = 1$ or $|\phi| < 1$. The asymptotic normal approximation works reasonably well for any value of $\phi$ as $T$ increases although the empirical size of Lee test tends to be smaller than the 5 % nominal level when $T$ is small. The empirical size of MT test is almost zero when $|\phi| < 1$ i.e., the underlying process is a stationary AR(1). This is expected since the statistic converges in probability to zero when the process is stationary and ergodic with a finite fourth moment. The empirical sizes of the modified Lee tests are very similar to that of the Lee test although there is a tendency of over-rejection. The value of $\delta$ does not seem to give much differences.

Table 2(a) reports the results for Cases (2). Similarly to Case (1), the empirical size of MT test is almost zero when $|\phi| < 1$ in Case (2). We see that when $\phi = 1$ the empirical sizes of Lee and MT tests are severely distorted upward, that is, these tests reject the true null hypothesis too often (when $T = 1000$, the actual sizes of the Lee and MT tests are 0.153 and 0.104 for $\phi = 1$, respectively). The modified Lee test performs much better than the Lee and MT test for $\phi = 1$ (and $\phi$ close to one); when $T = 1000$, the empirical sizes of the modified Lee tests with $\delta = 10$ and with $\delta = 1$ are 0.048 and 0.052, respectively. This shows that our modification works quite well for reducing the size distortion. Figure 2 draws the histograms of $100,000$ samples of $\widetilde{Z}_T$ and $\widehat{G}_{T,1}$ in Case (2) when $T = 1000$ for $\phi = 1$. We can see that the distribution of $\widetilde{Z}_T$ is almost bimodal. This is because, as Theorem 1 shows, when $\rho \neq 0$, the distribution of $\widetilde{Z}_T$ is a mixture of the standard normal and a non-standard distribution with weights depending on the value of $\rho$. The value of

$\delta$ seems to affect the actual rejection percentages for $\phi$ close to 1. It is seen that the modified Lee test with $\delta = 10$ tends to reject the null hypothesis more often than that with $\delta = 1$ does although the difference disappears quickly as $T$ increases. The results for Case (3), which is shown in Table 2(b), parallel those for Case (2) except that the size distortions of Lee and MT tests are smaller.

## 3.2 Power

We examine two data generating processes as the alternative. The first one is the RCA(1) model with $b_t \sim N(0, \omega^2)$ and various values of $\phi$ and $\omega^2$. Specifically, we set $\phi$ = 0.6, 0.9, or 1.0, $\omega^2$ = 0.01, 0.05, 0.1, or 0.5. The error term $\epsilon_t$ is set as Case (a): $\epsilon_t \sim N(0, 1)$ or Case (b): $\epsilon_t \sim (1/\sqrt{2})[\chi^2(1) - 1]$. The RCA(1) model is stationary and ergodic with a finite fourth moment if $E(|\phi + b_t|^4) < 1$. This condition reduces to $\phi^4 + 6\phi^2\omega^2 + 3\omega^4 < 1$ when $b_t \sim N(0, \omega^2)$. This condition is not satisfied, for example, when $\phi = 0.9$ and $\omega^2 = 0.1$. We indicate those cases that satisfy this condition by asterisks in Table 3.

The second alternative data generating process is a bilinear process, which is obtained by replacing $b_t$ with $b\epsilon_{t-1}$. As noted in Section 2.1, the Lee test in effect examines a certain type of conditional heteroskedasticity. Although the bilinear process is not included explicitly as the alternative hypothesis, we expect that our tests would have non-trivial power against the bilinear process because the bilinear process also exhibits a conditional heteroskedasticity similar to that of the RCA(1) model; for the bilinear model, we have $\text{var}(y_t|y_{t-1}) = (y_{t-1}^2 b^2 + 1)\sigma^2$. We examine four cases: $b = 0.05$, 0.10, $\sqrt{0.05} \approx 0.224$, and $\sqrt{0.10} \approx 0.316$. The values of $\phi$ and $\omega^2$ are the same as the cases of RCA(1) models.

When $\phi = 1$, the bilinear process is called a unit root biliner (URB) process and has been recently applied for analyzing stock market indices in Charemza et al. (2005). They have proposed several tests for $b = 0$ in the URB process. Results of the Monte Carlo experiments here may be compared with those in Charemza et al. (2005) since some of the values of $b$ examined here are the same as those examined in Charemza et al. (2005).[4]

Table 3 reports the results for the alternative of RCA(1) model. First, we will see the results for Case (a), where $\epsilon_t \sim N(0, 1)$ and $\rho = \text{corr}(\epsilon_t, \epsilon_t^2) = 0$. As is expected, the MT test has no power against stationary RCA(1) models with a finite fourth moment. This means that the MT test cannot distinguish a unit root process and a stationary RCA(1) model with a finite fourth moment. Notice that, even when $\phi = 1$, the power of MT test is much lower than those of the Lee and modified Lee tests. Furthermore, when $\phi = 1$, the power of the MT test does not monotonically increase as $\omega^2$ increases, which is also seen in Table 2 in McCabe and Tremayne (1995, p.1022) (they examined three values: $\omega^2 = 0.001$, 0.01, and 0.1 with other settings equal to ours). For example, when $T = 1000$, the power against $\omega^2 = 0.1$

[4] Although Charemza et al. (2005) do not state which distribution they generated $\epsilon_t$ from, we have confirmed with the authors that they generated $\epsilon_t$ from the standard normal distribution.

is 0.555, whereas the power against $\omega^2 = 0.5$ is 0.148. It is also seen that the power against $\omega^2 = 0.5$ does not increase as $T$ increases. This suggests the possibility that the MT test is not a consistent test against large values of $\omega^2$.

The power properties of the Lee and modified Lee tests are virtually the same, and the same comments apply to both tests. First, we observe that the value of $\phi$ greatly affects the power. In general, the larger the value of $\phi$ is, the higher the power is. For example, when $T = 1,000$, the power of the Lee test against $\omega^2 = 0.01$ is 0.117 for $\phi = 0.6$, whereas it is 0.456 for $\phi = 0.9$ and 1.000 for $\phi = 1.0$. Second, in contrast to the MT test, the power of the Lee and modified Lee tests increases monotonically as $\omega^2$ and $T$ increase. This supports our conjecture that these tests are consistent against RCA(1) models for any value of $\omega^2$.

Next, we check the results for Case (b), where $\epsilon_t \sim (1/\sqrt{2})[\chi^2(1) - 1]$ and $\mathrm{corr}(\epsilon_t, \epsilon_t^2) \approx 0.7559$. The powers of the three tests are lowered compared with Case (a). Again the power of the modified Lee test is almost the same as that of the Lee test, and their powers are much higher than that of the MT test. In an unreported experiment, we also examined the case where $\epsilon_t \sim -(1/\sqrt{2})[\chi_(1) - 1]$ and $\mathrm{corr}(\epsilon_t, \epsilon_t^2) \approx -0.7559$. The results are qualitatively similar to those in Case (b). It seems that the powers of these three tests reduce as the value of $\rho$ is away from zero.

Table 4 reports the results for the alternative of biliner models. The results parallel those for the alternative of RCA(1) models, and thus our comments are brief. The performances of the Lee and modified Lee tests are virtually the same. The power of the MT test is much lower than those of the Lee and modified Lee tests. The power of the MT test does not increase monotonically as $b$ increases.

# 4 Concluding Remarks

In this paper we propose a new test statistic for coefficient stability of AR(1) model. It is obtained as a modification of Lee (1998)'s LBI test. The proposed test statistic assumes neither a stationary AR(1) model nor a unit root process under the null hypothesis of constant coefficient and have the same limit distribution in both cases. We prove that the test is consistent against the alternative of RCA(1) models that is stationary and ergodic with a finite fourth moment. Although we have not succeeded in proving that the proposed test is also consistent against the alternative of RCA(1) models that do not belong to this class, our Monte Carlo experiments show that the proposed test has high power even against those RCA(1) models. It is our conjecture that the test is consistent against any RCA(1) models. Our Monte Carlo experiments also show that the proposed test has high power against bilinear processes. Lastly, neither this article nor McCabe and Tremayne (1995) proves consistency of the LBI test proposed by McCabe and Tremayne (1995). Our Monte Carlo experiments suggest the possibility that the test is inconsistent against certain RCA(1) models.

# Appendix

For simplifying the proofs, we assume that $y_0 = 0$ throughout the appendix, then $y_1 = \epsilon_1$. Extensions of the proofs to the case $y_0 = O_p(1)$ are straightforward. The following lemmas are repeatedly used in the appendix. Lemma 1 is a simple extension of Lemma A1 in McCabe and Tremayne (1995).

**Lemma 1** (Joint convergence) Suppose that $\{\epsilon_t\}_{t=1}^T$ is a sequence of i.i.d. random variables with $E(\epsilon_t^p) = \mu_p$, $\text{var}(\epsilon_t^p) = \sigma_p^2$, and $0 < \sigma_p^2 < \infty$ for a positive integer $p$. Define the partial sum process of $\{\epsilon_t^p - \mu_p\}$ as $W_T^{(p)}(r) \equiv (\sigma_p\sqrt{T})^{-1}\sum_{t=1}^{[rT]}(\epsilon_t^p - \mu_p)$ for $1/T \le r \le 1$ and $W_T(r) \equiv 0$ for $0 \le r < 1/T$, where $[rT]$ denotes the integer part of $rT$. Let $\rho \equiv \text{corr}(\epsilon_t^p, \epsilon_t^q)$ for $q \le p$. Then, the stochastic functions $W_T^{(p)}(r)$ and $W_T^{(q)}(r)$ jointly converge so that

$$[W_T^{(p)}(r), W_T^{(q)}(r)]' \Rightarrow [W_1(r), W^*(r)]',$$

where $\Rightarrow$ denotes weak convergence in the space D[0, 1] under the Skorohod metric, $W^*(r) = \rho W_1(r) + (1-\rho^2)^{1/2}W_2(r)$, $W_1(r)$ and $W_2(r)$ are mutually independent standard Wiener processes on $[0,1]$.

**Proof**. When $q = p$, obviously $\rho = 1$ and the result follows immediately. Suppose that $q < p$. Then $|\rho| < 1$ and from Theorem 7.27 in White (2001, p.188), we have

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1/2} [W_T^{(p)}(r), W_T^{(q)}(r)]' \Rightarrow [W_1(r), W_2(r)]',$$

By applying the continuous mapping theorem (see van der Vaart, 1998, p.7), we have

$$\begin{aligned} [W_T^{(p)}(r), W_T^{(q)}(r)]' &\Rightarrow \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{1/2} [W_T^{(p)}(r), W_T^{(q)}(r)]' \\ &= [W_1(r), \rho W_1(r) + (1-\rho^2)^{1/2}W_2(r)]' \\ &= [W_1(r), W^*(r)]', \end{aligned}$$

which completes the proof. ¤

**Lemma 2** Suppose that $y_t = y_{t-1} + \epsilon_t$ for $t = 1, ....T$ with $y_0 = 0$, where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with $E(\epsilon_t) = 0$, $\text{var}(\epsilon_t^2) = \sigma^2$, $0 < \sigma^2 < \infty$, $E(\epsilon_t^p) = \mu_p$ and $\text{var}(\epsilon_t^p) = \sigma_p^2 < \infty$, $0 < \sigma_p^2 < \infty$ for a positive integer $p$ such that

$2 \leq p$. Then, we have

$$\begin{aligned}
&(a) \quad T^{-(k+2)/2}\sum_{t=1}^{T} y_{t-1}^{k} \Rightarrow \sigma^{k}\int_0^1 W_1(r)^k dr,\\
&(b) \quad T^{-(k+1)/2}\sum_{t=1}^{T} y_{t-1}^{k}(\epsilon_t^p-\mu_p) \Rightarrow \sigma^{k}\sigma_p\int_0^1 W_1(r)^k dW^*(r),\\
&\qquad \text{in particular, } T^{-(k+1)/2}\sum_{t=1}^{T} y_{t-1}^{k}\epsilon_t \Rightarrow \sigma^{k+1}\int_0^1 W_1(r)^k dW_1(r),\\
&(c) \quad T^{-(k+2)/2}\sum_{t=1}^{T} y_{t-1}^{k}\epsilon_t^p \Rightarrow \sigma^{k}\mu_p\int_0^1 W_1(r)^k dW^*(r),
\end{aligned} \tag{16}$$

where $\Rightarrow$ denotes weak convergence in the space D[0, 1] under the Skorohod metric, $W^*(r) = \rho W_1(r) + (1-\rho^2)^{1/2}W_2(r)$, $\rho \equiv corr(\epsilon_t, \epsilon_t^p)$, $W_1(r)$ and $W_2(r)$ are mutually independent standard Wiener processes on $[0,1]$.

**Proof**. We use the same notation as in Lemma 1 for the partial sum process of $\{\epsilon_t^p - \mu_p\}$ except that $W_T^{(1)}(r)$ is abbreviated to $W_T(r)$.

(a) This is an immediate result from the functional central limit theorem and the continuous mapping theorem. ¤
(b) Set $U_{T,t} = W_T(t/T)^k$ and $Y_{T,t} = W_T^{(p)}(t/T)$ in Theorem 2.1 in Hansen (1992). From Lemma 1 and the continuous mapping theorem, it follows that $[U_T(r), Y_T(r)] \equiv [U_{T,[Tr]}, Y_{T,[Tr]}] \Rightarrow [W_1(r)^k, W^*(r)]$. Following the notation of Hansen (1992), define $\epsilon_{T,t} \equiv W_T^{(p)}(t/T) - W_T^{(p)}((t-1)/T) = (\sigma_p\sqrt{T})^{-1}(\epsilon_t - \mu_p)$. Under the assumptions, it is easily verified that $\sup_T \sum_{t=1}^{T} E(\epsilon_{T,t}^2) = 1 < \infty$. Thus, conditions in Theorem 2.1 in Hansen (1992) are all satisfied and we have (b). ¤
(c) Write

$$T^{-(k+2)/2}\sum_{t=1}^{T} y_{t-1}^{k}\epsilon_t^p = T^{-1/2}T^{-(k+1)/2}\sum_{t=1}^{T} y_{t-1}^{k}(\epsilon_t^p-\mu_p) + \mu_p T^{-(k+2)/2}\sum_{t=1}^{T} y_{t-1}^{k}. \tag{17}$$

Then, it follows immediately from Lemma 2(a) and (b). ¤

Hereafter, for example, $W_1(r)$'s that appear in the representations of limit distributions indicate the same Wiener process.

**Proof of Theorem 1**. Write $\widetilde{Z}_T = Z_1 \times Z_2$, where $Z_1 = T(\widehat{\tau}_T\widehat{\kappa}_T)^{-1}$ and $Z_2 = T^{-3/2}Z_T(\widehat{\phi})$. First, we derive the limit distribution of $Z_1$.

From Lemma 2(a), we have

$$
\begin{aligned}
T^{-2}\widehat{\tau}_T^2 &\Rightarrow \int_0^1 \sigma^4 W_1(r)^4 dr - \left(\int_0^1 \sigma^2 W_1(r)^2 dr\right)^2 \\
&= \sigma^4 \left(\int_0^1 \theta(r)^2 dr\right).
\end{aligned} \tag{18}
$$

Also, we have

$$
\begin{aligned}
\widehat{\sigma}_T^2 = &\ T^{-1}\sum_{t=1}^T \widehat{\epsilon}_t^2 \\
= &\ T^{-1}\sum_{t=1}^T \epsilon_t^2 - 2[T(\widehat{\phi}-1)]T^{-2}\sum_{t=1}^T \epsilon_t y_{t-1} + [T(\widehat{\phi}-1)]^2 T^{-3}\sum_{t=1}^T y_{t-1}^2, \\
\xrightarrow{p} &\ \sigma^2
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
\widehat{\kappa}_T^2 = &\ T^{-1}\sum_{t=1}^T \widehat{\epsilon}_t^4 - \widehat{\sigma}_T^4 \\
= &\ T^{-1}\sum_{t=1}^T \epsilon_t^4 - 4[T(\widehat{\phi}-1)]T^{-2}\sum_{t=1}^T \epsilon_t^3 y_{t-1} + 6[T(\widehat{\phi}-1)]^2 T^{-3}\sum_{t=1}^T \epsilon_t^2 y_{t-1}^2 \\
&-4[T(\widehat{\phi}-1)]^3 T^{-4}\sum_{t=1}^T \epsilon_t y_{t-1}^3 + [T(\widehat{\phi}-1)]^4 T^{-5}\sum_{t=1}^T y_{t-1}^4 - \widehat{\sigma}_T^4 \\
\xrightarrow{p} &\ E(\epsilon_t^4) - \sigma^4 = \kappa^2,
\end{aligned} \tag{20}
$$

by noting that $T(\widehat{\phi}-1) = O_p(1)$ and applying Lemma 2. Note that here, we used the assumption that $Var(\epsilon_t^3) < \infty$ to ensure that $T^{-3/2}\sum_{t=1}^T \epsilon_t^3 y_{t-1} = O_p(1)$ by Lemma 2(c). Thus, eventually we have

$$
Z_1 \Rightarrow \sigma^{-2}\kappa^{-1}\left(\int_0^1 \theta(r)^2 dr\right)^{-1/2}. \tag{21}
$$

Next, we derive the limit distribution of $Z_2$. Similarly to (19), we have

$$
\begin{aligned}
Z_2 = &\ T^{-3/2}\left(\sum_{t=1}^T \widehat{\epsilon}_t^2 y_{t-1}^2 - \widehat{\sigma}_T^2 \sum_{t=1}^T y_{t-1}^2\right) \\
= &\ T^{-3/2}\left(\sum_{t=1}^T \epsilon_t^2 y_{t-1}^2 - 2(\widehat{\phi}-1)\sum_{t=1}^T \epsilon_t y_{t-1}^3 + (\widehat{\phi}-1)^2\sum_{t=1}^T y_{t-1}^4 \right.\\
&\left. -\widetilde{\sigma}_T^2\sum_{t=1}^T y_{t-1}^2 + \widetilde{\sigma}_T^2\sum_{t=1}^T y_{t-1}^2 - \widehat{\sigma}_T^2\sum_{t=1}^T y_{t-1}^2\right) \\
= &\ T^{-3/2}\sum_{t=1}^T (\epsilon_t^2 - \widetilde{\sigma}_T^2) y_{t-1}^2 - 2T(\widehat{\phi}-1)T^{-5/2}\sum_{t=1}^T \epsilon_t y_{t-1}^3 \\
&+[T(\widehat{\phi}-1)]^2 T^{-7/2}\sum_{t=1}^T y_{t-1}^4 + T^{1/2}(\widetilde{\sigma}_T^2 - \widehat{\sigma}_T^2)T^{-2}\sum_{t=1}^T y_{t-1}^2,
\end{aligned} \tag{22}
$$

where $\widetilde{\sigma}_T^2 \equiv T^{-1}\sum_{t=1}^T \epsilon_t^2$. Noting that $T(\widehat{\phi}-1) = O_p(1)$ and applying Lemma 2, the second and third terms of the right-hand side in the last equality in (22) converge in probability to zero. Similarly, the fourth term converges in probability to zero since $T^{-2}\sum_{t=1}^T y_{t-1}^2 = O_p(1)$ and

$$\begin{aligned} T^{1/2}(\widetilde{\sigma}_T^2 - \widehat{\sigma}_T^2) &= T^{1/2}\left(\frac{2(\widehat{\phi}-1)\sum_{t=1}^T \epsilon_t y_{t-1}}{T} - \frac{(\widehat{\phi}_T - 1)^2 \sum_{t=1}^T y_{t-1}^2}{T}\right) \\ &= T^{-1/2}\left(2T(\widehat{\phi}-1)\frac{\sum_{t=1}^T \epsilon_t y_{t-1}}{T} - [T(\widehat{\phi}-1)]^2 \frac{\sum_{t=1}^T y_{i-1}^2}{T^2}\right) \\ &= o(1)[O_p(1)O_p(1) - O_p(1)O_p(1)] \xrightarrow{p} 0. \end{aligned} \tag{23}$$

Thus, the limit distribution of $Z_2$ is equivalent to that of the first term. From the proof of Theorem 2 in McCabe and Tremayne (1995), it follows that the first term converges weakly so that

$$T^{-3/2}\sum_{t=1}^T (\epsilon_t^2 - \widetilde{\sigma}_T^2)y_{t-1}^2 \Rightarrow \sigma^2\kappa \int_0^1 \theta(r)dW^*(r), \tag{24}$$

where $W^*(r) \equiv \rho W_1(r) + (1-\rho^2)^{1/2}W_2(r)$, $W_1(r)$ and $W_2(r)$ are mutually independent standard Wiener processes, $\sigma^2 = E(\epsilon_t^2)$, $\kappa^2 = Var(\epsilon_t^2)$, and $\rho = corr(\epsilon_t, \epsilon_t^2)$.[5] Thus, we have

$$Z_2 \Rightarrow \sigma^2\kappa \int_0^1 \theta(r)dW^*(r). \tag{25}$$

From (21) and (25), we eventually have

$$\widetilde{Z}_T \Rightarrow \frac{\int_0^1 \theta(r)dW^*(r)}{\left(\int_0^1 \theta(r)^2 dr\right)^{1/2}}. \tag{26}$$

Lastly, we prove that the limit distribution shown in (26) is reduced to the standard normal distribution when $\rho = 0$. Let $\mathcal{F}_1$ be the $\sigma$-algebra generated by the Wiener process $\{W_1(r); 0 \le r \le 1\}$. Since $W_2(r)$ and $W_1(r)$ are mutually independent, so are $\mathcal{F}_1$ and $\{W_2(r); 0 \le r \le 1\}$. Conditioned on the $\sigma$-algebra $\mathcal{F}_1$, the function $\theta(r)$ is a deterministic function and $\int_0^1 \theta(r)dW_2(r) \sim N(0, \int_0^1 \theta(r)^2 dr)$. Therefore, we have that $\widetilde{Z}_T|\mathcal{F}_1 \sim N(0,1)$. Since the conditional distribution does not depend any other random variables, the unconditional distribution of $\widetilde{Z}_T$ is also $N(0,1)$. ¤

[5] Actualy, McCabe and Tremayne (1995) proved (24) assuming a more general process for $\epsilon_t$.

**Proof of Theorem 2** First, we prove that $\rho_T^* \xrightarrow{p} \rho$. Since $s_T \xrightarrow{p} 1$, we just have to prove $\widehat{\rho}_T \xrightarrow{p} \rho$. Similarly to (19) and (20), we have

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\widehat{\epsilon}_t^3 = &\ T^{-1}\sum_{t=1}^{T}\epsilon_t^3 - 3T(\widehat{\phi}-1)T^{-2}\sum_{t=1}^{T}\epsilon_t^2 y_{t-1} \\ &+ 3[T(\widehat{\phi}-1)]^2T^{-3}\sum_{t=1}^{T}\epsilon_t y_{t-1}^2 - [T(\widehat{\phi}-1)]^3T^{-4}\sum_{t=1}^{T}y_{t-1}^3 \\ \xrightarrow{p} &\ E(\epsilon_t^3). \end{aligned}$$

Thus, $\widehat{\rho}_T \xrightarrow{p} \rho$. Write (13) as

$$\widetilde{G}_{T,\delta} = G_1 \times [G_2 - G_3], \tag{27}$$

where $G_1 \equiv T(1-\widehat{\rho}_T^{*2})^{-1/2}\widehat{\tau}_T^{-1}$, $G_2 \equiv \widehat{\kappa}_T^{-1}T^{-3/2}Z_T(\widehat{\phi})$, and $G_3 \equiv \widehat{\rho}_T^*\widehat{\sigma}_T^{-1}T^{-3/2}G_T(\widehat{\phi})$. Since $\widehat{\rho}_T^* \xrightarrow{p} \rho$, and $\widehat{\kappa}_T^2 \xrightarrow{p} \kappa^2$, it follows from (18) and (25) that

$$G_1 \Rightarrow (1-\rho^2)^{-1/2}\sigma^{-2}\left(\int_0^1 \theta(r)^2 dr\right)^{-1/2}, \tag{28}$$

and

$$G_2 \Rightarrow \sigma^2(1-\rho^2)^{1/2}\int_0^1 \theta(r)dW_2(r) + \sigma^2\rho\int_0^1 \theta(r)dW_1(r), \tag{29}$$

as $T \to \infty$. Lastly, we prove that $G_3 \Rightarrow \rho\sigma^2\int_0^1 \theta(r)dW_1(r)$. Noting that $y_t^3 = y_{t-1}^3 + 3y_{t-1}^2\epsilon_t + 3y_{t-1}\epsilon_t^2 + \epsilon_t^3$, we have

$$\frac{1}{3}y_T^3 = \frac{1}{3}\sum_{t=1}^{T}\Delta(y_t^3) = \sum_{t=1}^{T}y_{t-1}^2\epsilon_t + \sum_{t=1}^{T}y_{t-1}\epsilon_t^2 + \frac{1}{3}\sum_{t=1}^{T}\epsilon_t^3,$$

where $\Delta(y_t^3) \equiv y_t^3 - y_{t-1}^3$. Thus, we can write $T^{-3/2}G_T(\widehat{\phi})$ as

$$T^{-3/2}G_T(\widehat{\phi}) = g_1 + g_2 - g_3, \tag{30}$$

where

$$\begin{aligned} g_1 &\equiv T^{-3/2}\sum_{t=1}^{T}y_{t-1}^2\epsilon_t, \\ g_2 &\equiv T^{-3/2}\sum_{t=1}^{T}y_{t-1}\epsilon_t^2 + \frac{1}{3}T^{-3/2}\sum_{t=1}^{T}\epsilon_t^3 - T^{-3/2}\sum_{t=1}^{T}y_{t-1}\widehat{\epsilon}_t^2, \text{ and} \\ g_3 &\equiv \left(T^{-1/2}y_T\right)\left(T^{-2}\sum_{t=1}^{T}y_{t-1}^2\right). \end{aligned}$$

Applying Lemma 2(a), we have $g_1 \Rightarrow \sigma^3\int_0^1 W_1(r)^2 dW_1(r)$ and $g_3 \Rightarrow \sigma^3 W_1(1)\int_0^1 W_1(r)^2 dr$. By a similar argument to that used in (23), we can easily show that $g_2 \xrightarrow{p} 0$. Thus, we have $T^{-3/2}G_T(\widehat{\phi}) \Rightarrow \sigma^3\int_0^1 \theta(r)dW_1(r)$ and

$$G_3 = \widehat{\rho}_T^*\widehat{\sigma}_T^{-1}T^{-3/2}G_T(\widehat{\phi}) \Rightarrow \rho\sigma^2\int_0^1 \theta(r)dW_1(r). \tag{31}$$

From (27), (28), (29), and (31), we obtain

$$\widetilde{G}_{T,\delta} \Rightarrow \frac{\displaystyle\int_0^1 \theta(r)dW_2(r)}{\left(\displaystyle\int_0^1 \theta(r)^2 dr\right)^{1/2}} \overset{d}{=} N(0,1), \tag{32}$$

where the notation $\overset{d}{=}$ denotes equivalence in distribution. This completes the proof of Theorem 2.

**Proof of Theorem 3** Write $\widetilde{G}_{T,\delta} = (1-\rho_T^{*2})^{-1/2}\widetilde{Z}_T + I_T$, where

$$I_T \equiv (1-\rho_T^{*2})^{-1/2}\rho_T^* \widehat{\tau}_T^{-1}\widehat{\sigma}_T^{-1}T^{-1/2}G_T(\widehat{\phi}).$$

We will show that $I_T$ is $O_p(T^{-\delta/2})$ under $H_0$ and is $O_p(T^{-(\delta-1)/2})$ under $H_1$. Then, Theorem 3 follows immediately from Theorem 3.2 in Lee (1998, p.100).

**Proof of Part(a)**. First note that $y_t$ is stationary and ergodic with a finite fourth moment. Since $\widehat{\epsilon}_t = y_t - \widehat{\phi}y_{t-1}$, we have

$$\begin{aligned} \widehat{\sigma}_T^2 = \;& T^{-1}\sum_{t=1}^T \widehat{\epsilon}_t^2 \\ = \;& T^{-1}\sum_{t=1}^T y_t^2 - 2\widehat{\phi}T^{-1}\sum_{t=1}^T y_t y_{t-1} + \widehat{\phi}^2 T^{-1}\sum_{t=1}^T y_{t-1}^2 \\ \overset{p}{\to} \;& (1+\phi^2)E(y_t^2) - 2\phi E(y_t y_{t-1}), \end{aligned} \tag{33}$$

and hence $\widehat{\sigma}_T^2 = O_p(1)$. By similar arguments, it is easy to show that $T^{-1}\sum_{t=1}^T \widehat{\epsilon}_t^3 = O_p(1)$ and $T^{-1}\sum_{t=1}^T \widehat{\epsilon}_t^4 = O_p(1)$, which ensures that $\widehat{\rho}_T = O_p(1)$ and $\widehat{\tau}_T = O_p(1)$. Let $v_T \equiv T^{-1}\widehat{\sigma}^{-2}\sum_{t-1}^T y_{t-1}^2$. Since $v_T = O_p(1)$, we have

$$\begin{aligned} s_T &= 1 - \left[1 - (T^{-1/2}v_T)^\delta + \tfrac{1}{2}(T^{-1/2}v_T)^{2\delta} - \cdots\right] \\ &= T^{-\delta/2}v_T^\delta + o_p(T^{-\delta/2}) \\ &= O_p(T^{-\delta/2}). \end{aligned} \tag{34}$$

Therefore, we have $\rho_T^* = O_p(T^{-\delta/2})$. Next, note that

$$T^{-1/2}G_T(\widehat{\phi}) = T^{-1/2}\frac{1}{3}y_T^3 - T^{-1/2}\sum_{t=1}^T y_{t-1}\widehat{\epsilon}_t^2 - T^{-1/2}y_T T^{-1}\sum_{t=1}^T y_{t-1}^2. \tag{35}$$

It is easy to see that the first and third terms converge in probability to zero. We will show that the second term is $O_p(1)$ under $H_0$. Let $\theta_t = y_t - \phi y_{t-1}$. Because

$$\begin{aligned} \theta_t^2 - \widehat{\epsilon}_t^2 &= 2(\widehat{\phi}-\phi)y_t y_{t-1} + \phi^2 y_{t-1}^2 - \widehat{\phi}^2 y_{t-1}^2 \\ &= 2(\widehat{\phi}-\phi)(\theta_t + \phi y_{t-1})y_{t-1} + \phi^2 y_{t-1}^2 - \widehat{\phi}^2 y_{t-1}^2 \\ &= 2(\widehat{\phi}-\phi)\theta_t y_{t-1} - (\widehat{\phi}-\phi)^2 y_{t-1}^2, \end{aligned} \tag{36}$$

we can write

$$T^{-1/2}\sum_{t=1}^{T}(\theta_t^2-\widehat{\epsilon}_t^2)y_{t-1} = 2T^{1/2}(\widehat{\phi}-\phi)T^{-1}\sum_{t=1}^{T}y_{t-1}^2\theta_t + T^{-1/2}[T^{1/2}(\widehat{\phi}-\phi)]^2T^{-1}\sum_{t=1}^{T}y_{t-1}^3. \quad (37)$$

By the ergodicity of $y_t$, we have $T^{-1}\sum_{t=1}^{T}y_{t-1}^3 \xrightarrow{p} E(y_{t-1}^3)$, hence the second term in (37) converges in probability to zero. Similarly, noting that when the true process is a stationary AR(1), $\theta_t = \epsilon_t$, the first term converges in probability to zero since $y_{t-1}$ is independent from $(b_t, \epsilon_t)$ and $T^{-1}\sum_{t=1}^{T}y_{t-1}^2\epsilon_t \xrightarrow{p} E(y_{t-1}^2)E(\epsilon_t) = 0$ under $H_0$. This yields

$$T^{-1/2}\sum_{t=1}^{T}y_{t-1}\widehat{\epsilon}_t^2 = T^{-1/2}\sum_{t=1}^{T}y_{t-1}\theta_t^2 + o_p(1). \quad (38)$$

Let $\mathcal{F}_t$ be the $\sigma$-field generated by $\{y_t, \epsilon_t, y_{t-1}, \epsilon_{t-1}, ...., y_0\}$. Write $y_{t-1}\epsilon_t^2 = x_t + y_{t-1}\sigma^2$, where $x_t \equiv y_{t-1}(\epsilon_t^2 - \sigma^2)$. Then, $\{x_t, \mathcal{F}_t\}$ is a martingale difference sequence with variance $Var(x_t) = \sigma^2\kappa^2/(1-\phi^2)$ for all $t$. It follows from the stationarity that the Lindeberg condition is satisfied (see White, 2001, p.118, and p.135). By the ergodicity we have $T^{-1}\sum_{t=1}^{T}x_t^2 \xrightarrow{p} \sigma^2\kappa^2/(1-\phi^2)$. Thus, we can apply a martingale central limit theorem (White, 2001, Theorem 5.24, p.133) to obtain $T^{-1/2}\sum_{t=1}^{T}x_t \xrightarrow{d} N(0, \sigma^2\kappa^2/(1-\phi^2))$. It can be shown that $T^{-1/2}\sum_{t=1}^{T}y_t \xrightarrow{d} N(0, \sigma^2/(1-\phi)^2)$ by a central limit theorem for a stationary sequence (Hamilton, 1994, Proposition 7.11, p.195). Hence, the first term in (38) is $O_p(1)$, which implies that the second term in (35) is $O_p(1)$ under $H_0$. From these arguments, we have $I_T = O_p(T^{-\delta/2})$ under $H_0$, which completes the proof of part (a).

**Proof of Part(b)** The same arguments as in the proof of Part (a) can be applied to show that $\widehat{\sigma}_T^2 = O_p(1)$, $\widehat{\rho}_T^* = O_p(T^{-\delta/2})$, and the first and third terms in (35) converge in probability to zero, since these arguments use only that $y_t$ is stationary and ergodic with a finite fourth moments. We will show that the second term in (35) is $O_p(T^{1/2})$ under $H_1$. Note that $\theta_t = b_t y_{t-1} + \epsilon_t$ under $H_1$. The first and second terms in (37) converges in probability to zero, since $T^{-1}\sum_{t=1}^{T}y_{t-1}^2\theta_t = T^{-1}\sum_{t=1}^{T}y_{t-1}^3 b_t + T^{-1}\sum_{t=1}^{T}y_{t-1}^2\epsilon_t \xrightarrow{p} E(y_{t-1}^3)E(b_t) + E(y_{t-1}^2)E(\epsilon_t) = 0$. Thus, the equation in (38) holds under $H_1$ too. Furthermore, we have

$$\begin{aligned} T^{-1}\sum_{t-1}^{T}y_{t-1}\theta_t^2 &\xrightarrow{p} \omega^2E(y_t^3) + 2E(b_t\epsilon_t)E(y_{t-1}^2) + \sigma^2E(y_{t-1}) \\ &= \omega^2E(y_t^3) + 2\psi\omega E(y_{t-1}^2), \end{aligned} \quad (39)$$

which implies that the second term in (35) is $O_p(T^{1/2})$, and thus we have $I_T = O_p(T^{-(\delta-1)/2})$ under $H_1$. This completes the proof of part (b). ¤

## References


Akharif, A., Hallin, M., 2003. Efficient detection of random coefficients in autoregressive models. Ann. Statist. 31(2), 675-704.

Aue, A., Horvath, L., Steinebach, J., 2006. Estimation in random coefficient autoregressive models. J. Time Ser. Anal. 27(1), 61-76.

Bleaney, M.F., Leybourne, S.J., 2003. Real exchange rate dynamics under the current float: a re-examination. Manchester School 71, 156-171.

Bleaney, M.F., Leybourne, S.J., Mizen, P., 1999. Mean reversion of real exchange rates in high-inflation countries. Southern Econom. J. 65, 839-854.

Charemza, W.W., Lifshits, M., Makarova, S., 2005. Conditional testing for unit-root bilinearity in financial time series: some theoretical and empirical results. J. Econom. Dyna. Cont. 29, 63-96.

Ferguson, T.S., 1967. Mathematical Statistics: A Decision Theoretic Approach. Academic Press, New York.

Granger, C. W. J. and N. R. Swanson (1997) An introduction to stochastic unit-root processes, Journal of Econometrics, 80, 35-62.

Hamilton, D.J., 1994. Time Series Analysis. Princeton University Press. Princeton, NJ.

Hansen, E.B., 1992. Convergence to stochastic integrals for dependent heterogeneous processes. Econometric Theory 8, 489-500.

Hwang, S.Y., Basawa, I.V., 1998. Parameter estimation for generalized random coefficient autoregressive processes. J. Statist. Plann. Inference, 68, 323-337.

Hwang, S.Y., Basawa, I.V., 2006. Explosive random-coefficient AR(1) processes and related asymptotics for least-squares estimation. J. Time Ser. Anal. 26(6), 807-824.

Hwang, S.Y., Basawa, I.V., Kim, T.Y., 2005. Least squares estimation for critical random coefficient first-order autoregressive processes. Statist. Probab. Lett. 76, 310-317.

Kobayashi, M., McAleer, M., 1999. Tests of linear and logarithmic transformations for integrated processes. J. Amer. Statist. Assoc. 94, 860-868.

Lee, S., 1998. Coefficient constancy test in a random coefficient autoregressive model. J. Statist. Plann. Inference, 74, 93-101.

Leybourne, S.J., McCabe, B.P.M., Tremayne, A.R., 1996. Can economic time series be differenced to stationarity? J. Busi. Econom. Statist. 14, 435-446.

McCabe, B.P.M., Tremayne, A.R., 1995. Testing a time series for difference stationary. Ann. Statist. 23(3), 1015-1028.

McCabe, B.P.M., and Smith, R.J., 1998. The power of some tests for difference stationary under local heteroscedastic integration. J. Amer. Statist. Assoc. 93, 751-761.

Nicholls, D.F., Quinn, B.G., 1982. Random Coefficient Autoregressive Models: An Introduction. Springer, New York.

Quinn, B.G., 1982. A note on the existence of strictly stationary solutions to bilinear equations. J. Time Ser. Anal. 3. 249-252.

Perron, P., Yabu, T. 2006a. Estimating deterministic trends with an integrated or stationary noise component, IMES discussion paper series, no. 2006-E-1, available at http://www.imes,boj.or.jp/english/publication.html

Perron, P., Yabu, T. 2006b. Testing for shifts in trend with an integrated or stationary noise component, IMES discussion paper series, no. 2006-E-2, available at http://www.imes,boj.or.jp/english/publication.html

Ramanathan, T.V., Rajarshi, M.B., 1994. Rank tests for testing the randomness of autoregressive coefficients. Statist. Probab. Lett. 21, 115-120.

Sollis, R., Leybourne, S.J., Newbold, P., 2000. Stochastic unit root modeling of stock price indices. Appl. Finan. Econom. 10(3), 311-315.

van der Vaart, A.W.(1996). Asymptotic Statistics, Cambridge University Press, Cambridge

White, H., 2001. Asymptotic Theory for Econometricians: Revised Ed. Academic Press, California.

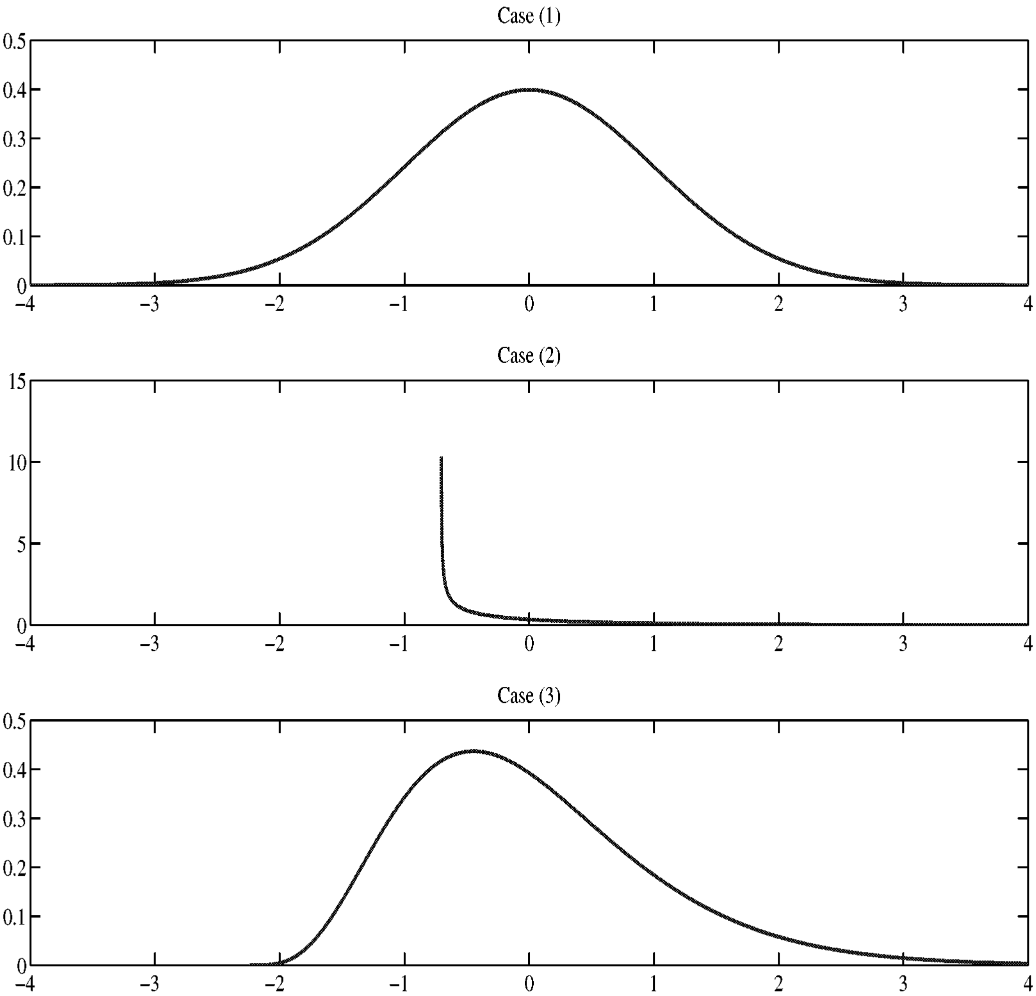
Table 1: Empirical size at 5 % nominal level

Case (1) The true model: $y_t = \phi y_{t-1} + \epsilon_t$, $\epsilon_t \sim i.i.d.N(0,1)$.

| | | $\phi$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | | $-0.9$ | $-0.6$ | $-0.3$ | 0.0 | 0.3 | 0.6 | 0.9 | 0.95 | 0.98 | 1.0 |
| 50 | $\widetilde{Z}_T$ | 0.035 | 0.027 | 0.027 | 0.031 | 0.027 | 0.027 | 0.032 | 0.035 | 0.037 | 0.039 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.007 | 0.023 | 0.043 |
| | $\widetilde{G}_{T,1}$ | 0.042 | 0.031 | 0.033 | 0.033 | 0.031 | 0.032 | 0.050 | 0.059 | 0.062 | 0.071 |
| | $\widetilde{G}_{T,10}$ | 0.036 | 0.029 | 0.028 | 0.027 | 0.028 | 0.025 | 0.038 | 0.053 | 0065 | 0.067 |
| 100 | $\widetilde{Z}_T$ | 0.037 | 0.032 | 0.034 | 0.037 | 0.038 | 0.034 | 0.039 | 0.037 | 0.042 | 0.039 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.009 | 0.043 |
| | $\widetilde{G}_{T,1}$ | 0.038 | 0.033 | 0.036 | 0.038 | 0.037 | 0.034 | 0.042 | 0.049 | 0.062 | 0.064 |
| | $\widetilde{G}_{T,10}$ | 0.042 | 0.035 | 0.035 | 0.034 | 0.034 | 0.030 | 0.036 | 0.049 | 0065 | 0.065 |
| 200 | $\widetilde{Z}_T$ | 0.038 | 0.036 | 0.040 | 0.043 | 0.041 | 0.040 | 0.040 | 0.041 | 0.045 | 0.043 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 | 0.049 |
| | $\widetilde{G}_{T,1}$ | 0.037 | 0.037 | 0.040 | 0.040 | 0.040 | 0.039 | 0.044 | 0.051 | 0.058 | 0.061 |
| | $\widetilde{G}_{T,10}$ | 0.036 | 0.036 | 0.037 | 0.041 | 0.039 | 0.037 | 0.038 | 0.043 | 0059 | 0.066 |
| 1000 | $\widetilde{Z}_T$ | 0.044 | 0.044 | 0.046 | 0.046 | 0.049 | 0.047 | 0.044 | 0.045 | 0.048 | 0.048 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.052 |
| | $\widetilde{G}_{T,1}$ | 0.040 | 0.044 | 0.048 | 0.051 | 0.051 | 0.046 | 0.044 | 0.044 | 0.053 | 0.055 |
| | $\widetilde{G}_{T,10}$ | 0.042 | 0.043 | 0.046 | 0.047 | 0.046 | 0.043 | 0.045 | 0.043 | 0050 | 0.052 |

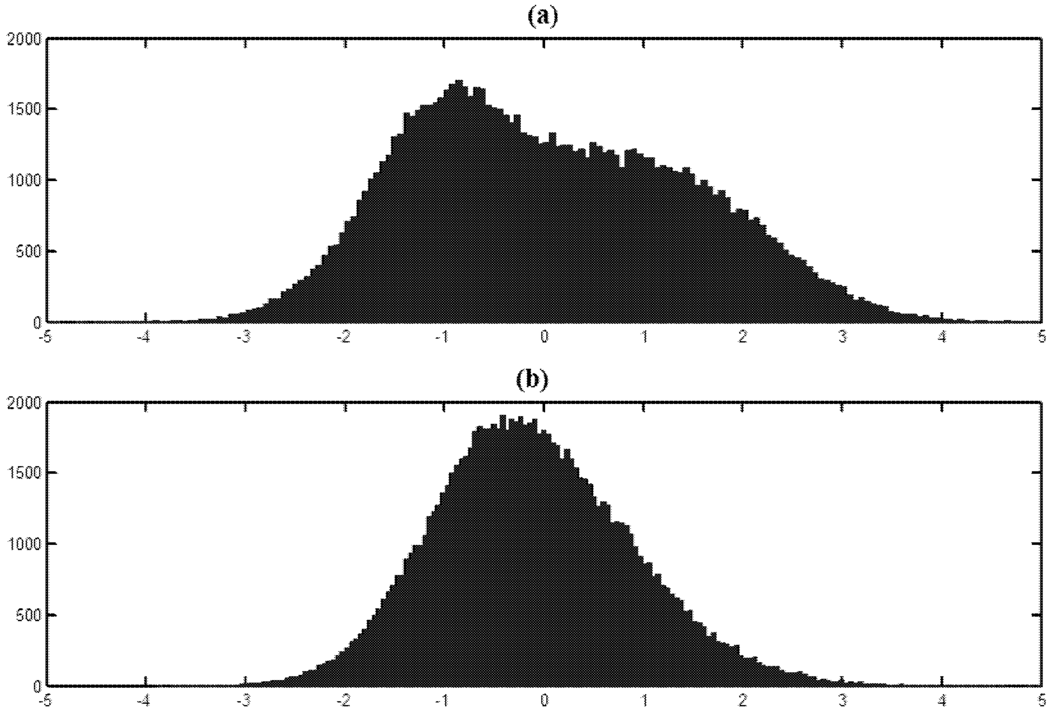
Note: The second column indicates the test statistics, where $\widetilde{Z}_T$ is the Lee test defined in (2), $Z_T^*$ is the MT test defined in (4), and $\widetilde{G}_{T,\delta}$ is the modified Lee test defined in (13). The critical values of the MT test are taken from the Table 1 in McCabe and Tremayne (1995).

Figure 1: Pdfs of the three distributions




Note: these pdfs are of the following three distributions: (1) $\epsilon_t \sim N(0,1)$ $(\rho = 0)$; (2) $\epsilon_t \sim (1/\sqrt{2})(\chi^2(1) - 1)$ $(\rho = 2/\sqrt{7} \approx 0.7559)$; (3) $\epsilon_t \sim (1/\sqrt{20})(\chi^2(10) - 10)$ $(\rho = 0.5)$, where $\rho = \text{corr}(\epsilon_t, \epsilon_t^2)$.

Figure 2: Histograms of $\widetilde{Z}_T$ and $\widetilde{G}_{T,1}$




Note: histograms of 100,000 samples of (a) $\widetilde{Z}_T$ and (b) $\widetilde{G}_{T,1}$ with $T = 1000$ and $\phi = 1$ in Case (2), where $\widetilde{Z}_T$ is the Lee test defined in (2), and $\widetilde{G}_{T,1}$ is the proposed test defined in (13).

Table 2: Empirical size at 5 % nominal level

Case (2) The true model: $y_t = \phi y_{t-1} + \epsilon_t,\ \ \epsilon_t \sim (1/\sqrt{2})[\chi^2(1) - 1]$, $\text{corr}(\epsilon_t^2, \epsilon_t) \approx 0.7559$.

| | | $\phi$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | | -0.9 | -0.6 | -0.3 | 0.0 | 0.3 | 0.6 | 0.9 | 0.95 | 0.98 | 1.0 |
| 50 | $\widetilde{Z}_T$ | 0.041 | 0.035 | 0.037 | 0.028 | 0.018 | 0.013 | 0.080 | 0.122 | 0.161 | 0.196 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.014 | 0.069 | 0.140 |
| | $\widetilde{G}_{T,1}$ | 0.057 | 0.040 | 0.042 | 0.036 | 0.020 | 0.010 | 0.027 | 0.038 | 0.053 | 0.059 |
| | $\widetilde{G}_{T,10}$ | 0.039 | 0.033 | 0.035 | 0.028 | 0.015 | 0.013 | 0.073 | 0.088 | 0083 | 0.077 |
| 100 | $\widetilde{Z}_T$ | 0.044 | 0.042 | 0.043 | 0.036 | 0.024 | 0.014 | 0.045 | 0.079 | 0.125 | 0.187 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.018 | 0.129 |
| | $\widetilde{G}_{T,1}$ | 0.057 | 0.048 | 0.046 | 0.038 | 0.028 | 0.016 | 0.021 | 0.031 | 0.045 | 0.058 |
| | $\widetilde{G}_{T,10}$ | 0.040 | 0.039 | 0.040 | 0.032 | 0.021 | 0.010 | 0.048 | 0.081 | 0077 | 0.061 |
| 200 | $\widetilde{Z}_T$ | 0.040 | 0.042 | 0.041 | 0.037 | 0.029 | 0.018 | 0.024 | 0.046 | 0.086 | 0.173 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.123 |
| | $\widetilde{G}_{T,1}$ | 0.051 | 0.045 | 0.047 | 0.041 | 0.033 | 0.024 | 0.017 | 0.026 | 0.037 | 0.051 |
| | $\widetilde{G}_{T,10}$ | 0.041 | 0.047 | 0.045 | 0.038 | 0.027 | 0.016 | 0.021 | 0.062 | 0078 | 0.057 |
| 1000 | $\widetilde{Z}_T$ | 0.045 | 0.053 | 0.054 | 0.050 | 0.046 | 0.039 | 0.024 | 0.024 | 0.033 | 0.153 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.104 |
| | $\widetilde{G}_{T,1}$ | 0.050 | 0.052 | 0.053 | 0.050 | 0.046 | 0.037 | 0.025 | 0.028 | 0.039 | 0.048 |
| | $\widetilde{G}_{T,10}$ | 0.046 | 0.049 | 0.050 | 0.048 | 0.042 | 0.032 | 0.022 | 0.024 | 0.064 | 0.052 |

Case(3) The true model: $y_t = \phi y_{t-1} + \epsilon_t,\ \ \epsilon_t \sim (1/\sqrt{20})[\chi^2(10) - 10]$, $\text{corr}(\epsilon_t^2, \epsilon_t) = 0.5$.

| | | $\phi$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | | -0.9 | -0.6 | -0.3 | 0.0 | 0.3 | 0.6 | 0.9 | 0.95 | 0.98 | 1.0 |
| 50 | $\widetilde{Z}_T$ | 0.039 | 0.030 | 0.032 | 0.027 | 0.024 | 0.025 | 0.048 | 0.064 | 0.072 | 0.086 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.007 | 0.032 | 0.065 |
| | $\widetilde{G}_{T,1}$ | 0.043 | 0.035 | 0.036 | 0.033 | 0.024 | 0.025 | 0.042 | 0.053 | 0.064 | 0.067 |
| | $\widetilde{G}_{T,10}$ | 0.037 | 0.031 | 0.033 | 0.028 | 0.024 | 0.027 | 0.053 | 0.069 | 0073 | 0.074 |
| 100 | $\widetilde{Z}_T$ | 0.038 | 0.038 | 0.038 | 0.034 | 0.028 | 0.026 | 0.042 | 0.055 | 0.066 | 0.088 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.012 | 0.073 |
| | $\widetilde{G}_{T,1}$ | 0.045 | 0.038 | 0.041 | 0.037 | 0.029 | 0.025 | 0.039 | 0.050 | 0.058 | 0.069 |
| | $\widetilde{G}_{T,10}$ | 0.037 | 0.039 | 0.043 | 0.037 | 0.026 | 0.025 | 0.043 | 0.065 | 0067 | 0.070 |
| 200 | $\widetilde{Z}_T$ | 0.038 | 0.043 | 0.045 | 0.044 | 0.037 | 0.030 | 0.037 | 0.046 | 0.059 | 0.097 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.075 |
| | $\widetilde{G}_{T,1}$ | 0.047 | 0.047 | 0.048 | 0.046 | 0.036 | 0.030 | 0.035 | 0.043 | 0.054 | 0.064 |
| | $\widetilde{G}_{T,10}$ | 0.042 | 0.040 | 0.045 | 0.039 | 0.035 | 0.028 | 0.037 | 0.055 | 0071 | 0.067 |
| 1000 | $\widetilde{Z}_T$ | 0.046 | 0.048 | 0.051 | 0.052 | 0.047 | 0.042 | 0.039 | 0.040 | 0.045 | 0.093 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.069 |
| | $\widetilde{G}_{T,1}$ | 0.046 | 0.048 | 0.051 | 0.051 | 0.048 | 0.043 | 0.043 | 0.045 | 0.049 | 0.055 |
| | $\widetilde{G}_{T,10}$ | 0.045 | 0.049 | 0.054 | 0.054 | 0.046 | 0.042 | 0.038 | 0.040 | 0056 | 0.056 |

Note: The notation $\chi^2(k)$ denotes the random variable distributed chi-square with $k$ degrees of freedom. In both Cases (a) and (b), the mean and variance of $\epsilon_t$ are 0 and 1, respectively. The second column indicates the test statistics, where $\widetilde{Z}_T$ is the Lee test defined in (2), $Z_T^*$ is the MT test defined in (4), $\widetilde{G}_{T,\delta}$ is the modified Lee test defined in (13).

Table 3: Power at 5 % nominal level against RCA(1) models

Case (a) The true model: $y_t = (\phi + b_t)y_{t-1} + \epsilon_t$, $b_t \sim i.i.d.N(0, \omega^2)$, $\epsilon_t \sim i.i.d.N(0, 1)$.

| $T$ | | $\omega^2(\phi = 0.6)$ | | | | $\omega^2(\phi = 0.9)$ | | | | $\omega^2(\phi = 1.0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.010* | 0.050* | 0.100* | 0.500* | 0.010* | 0.050* | 0.100 | 0.500 | 0.010 | 0.050 | 0.100 | 0.500 |
| 50 | $\widetilde{Z}_T$ | 0.037 | 0.079 | 0.150 | 0.644 | 0.069 | 0.263 | 0.463 | 0.849 | 0.209 | 0.564 | 0.715 | 0.906 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.011 | 0.005 | 0.046 | 0.087 | 0.092 | 0.194 | 0.347 | 0.353 | 0.153 |
| | $\widetilde{G}_{T,1}$ | 0.040 | 0.085 | 0.162 | 0.654 | 0.088 | 0.290 | 0.480 | 0.850 | 0.254 | 0.585 | 0.724 | 0.898 |
| | $\widetilde{G}_{T,10}$ | 0.036 | 0.078 | 0.150 | 0.640 | 0.081 | 0.270 | 0.460 | 0.836 | 0.257 | 0.581 | 0.712 | 0.887 |
| 100 | $\widetilde{Z}_T$ | 0.051 | 0.141 | 0.283 | 0.911 | 0.101 | 0.477 | 0.747 | 0.989 | 0.491 | 0.870 | 0.947 | 0.996 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.008 | 0.001 | 0.031 | 0.079 | 0.090 | 0.343 | 0.479 | 0.445 | 0.154 |
| | $\widetilde{G}_{T,1}$ | 0.054 | 0.148 | 0.293 | 0.916 | 0.117 | 0.498 | 0.759 | 0.988 | 0.532 | 0.876 | 0.946 | 0.991 |
| | $\widetilde{G}_{T,10}$ | 0.051 | 0.141 | 0.283 | 0.911 | 0.104 | 0.477 | 0.745 | 0.987 | 0.536 | 0.869 | 0.942 | 0.991 |
| 200 | $\widetilde{Z}_T$ | 0.059 | 0.221 | 0.497 | 0.996 | 0.149 | 0.738 | 0.952 | 1.000 | 0.811 | 0.988 | 0.998 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.005 | 0.000 | 0.017 | 0.063 | 0.089 | 0.517 | 0.587 | 0.502 | 0.150 |
| | $\widetilde{G}_{T,1}$ | 0.061 | 0.226 | 0.503 | 0.997 | 0.160 | 0.750 | 0.954 | 0.999 | 0.824 | 0.989 | 0.998 | 0.998 |
| | $\widetilde{G}_{T,10}$ | 0.059 | 0.222 | 0.498 | 0.996 | 0.149 | 0.738 | 0.952 | 0.999 | 0.826 | 0.988 | 0.998 | 0.999 |
| 1000 | $\widetilde{Z}_T$ | 0.117 | 0.700 | 0.984 | 1.000 | 0.456 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.002 | 0.000 | 0.003 | 0.029 | 0.082 | 0.850 | 0.738 | 0.555 | 0.148 |
| | $\widetilde{G}_{T,1}$ | 0.118 | 0.701 | 0.984 | 1.000 | 0.461 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | $\widetilde{G}_{T,10}$ | 0.117 | 0.700 | 0.984 | 1.000 | 0.456 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Case (b) The true model: $y_t = (\phi + b_t)y_{t-1} + \epsilon_t$, $b_t \sim i.i.d.N(0, \omega^2)$, $\epsilon_t \sim (1/\sqrt{2})[\chi^2(1) - 1]$.

| $T$ | | $\omega^2(\phi = 0.6)$ | | | | $\omega^2(\phi = 0.9)$ | | | | $\omega^2(\phi = 1.0)$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.010* | 0.050* | 0.100* | 0.500* | 0.010* | 0.050* | 0.100 | 0.500 | 0.010 | 0.050 | 0.100 | 0.500 |
| 50 | $\widetilde{Z}_T$ | 0.020 | 0.051 | 0.110 | 0.605 | 0.088 | 0.223 | 0.400 | 0.840 | 0.282 | 0.532 | 0.682 | 0.904 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.001 | 0.013 | 0.003 | 0.042 | 0.088 | 0.099 | 0.210 | 0.333 | 0.357 | 0.159 |
| | $\widetilde{G}_{T,1}$ | 0.018 | 0.052 | 0.117 | 0.614 | 0.051 | 0.213 | 0.399 | 0.835 | 0.216 | 0.528 | 0.677 | 0.895 |
| | $\widetilde{G}_{T,10}$ | 0.019 | 0.050 | 0.108 | 0.598 | 0.085 | 0.225 | 0.399 | 0.825 | 0.245 | 0.549 | 0.680 | 0.886 |
| 100 | $\widetilde{Z}_T$ | 0.019 | 0.069 | 0.173 | 0.862 | 0.069 | 0.317 | 0.617 | 0.980 | 0.426 | 0.798 | 0.912 | 0.992 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.010 | 0.001 | 0.028 | 0.073 | 0.093 | 0.291 | 0.443 | 0.430 | 0.150 |
| | $\widetilde{G}_{T,1}$ | 0.024 | 0.077 | 0.187 | 0.868 | 0.052 | 0.318 | 0.621 | 0.979 | 0.419 | 0.798 | 0.905 | 0.988 |
| | $\widetilde{G}_{T,10}$ | 0.019 | 0.069 | 0.172 | 0.860 | 0.071 | 0.315 | 0.616 | 0.978 | 0.450 | 0.802 | 0.907 | 0.988 |
| 200 | $\widetilde{Z}_T$ | 0.028 | 0.117 | 0.312 | 0.982 | 0.067 | 0.485 | 0.845 | 1.000 | 0.672 | 0.962 | 0.993 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.005 | 0.000 | 0.016 | 0.058 | 0.085 | 0.423 | 0.550 | 0.486 | 0.145 |
| | $\widetilde{G}_{T,1}$ | 0.032 | 0.126 | 0.328 | 0.983 | 0.065 | 0.487 | 0.846 | 0.999 | 0.689 | 0.962 | 0.992 | 0.999 |
| | $\widetilde{G}_{T,10}$ | 0.027 | 0.116 | 0.312 | 0.982 | 0.067 | 0.484 | 0.844 | 0.999 | 0.716 | 0.962 | 0.992 | 0.999 |
| 1000 | $\widetilde{Z}_T$ | 0.067 | 0.441 | 0.863 | 1.000 | 0.153 | 0.972 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 0.003 | 0.031 | 0.080 | 0.777 | 0.736 | 0.555 | 0.148 |
| | $\widetilde{G}_{T,1}$ | 0.070 | 0.455 | 0.868 | 1.000 | 0.167 | 0.972 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| | $\widetilde{G}_{T,10}$ | 0.067 | 0.442 | 0.863 | 1.000 | 0.153 | 0.972 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |

Note: The second column indicates the test statistics, where $\widetilde{Z}_T$ is the Lee test defined in (2), $Z_T^*$ is the MT test defined in (4), and $\widetilde{G}_{T,\delta}$ is the proposed test or Modified Lee test defined in (13). Asterisks indicate the cases of stationary RCA(1) models with a finite fourth moment.

Table 4: Power at 5 % nominal level against bilinear models

The true model: $y_t = (\phi + b\epsilon_{t-1})y_{t-1} + \epsilon_t$, $\epsilon_t \sim i.i.d.N(0,1)$

| | | ($\phi = 0.6$) | | | | ($\phi = 0.9$) | | | | ($\phi = 1.0$) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $b$ | | | | $b$ | | | | $b$ | | | |
| $T$ | | 0.050 | 0.100 | $\sqrt{0.05}$ | $\sqrt{0.1}$ | 0.050 | 0.100 | $\sqrt{0.05}$ | $\sqrt{0.1}$ | 0.050 | 0.100 | $\sqrt{0.05}$ | $\sqrt{0.1}$ |
| 50 | $\widetilde{Z}_T$ | 0.028 | 0.034 | 0.124 | 0.299 | 0.039 | 0.072 | 0.388 | 0.639 | 0.081 | 0.237 | 0.619 | 0.792 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.002 | 0.013 | 0.003 | 0.023 | 0.205 | 0.320 | 0.101 | 0.237 | 0.500 | 0.570 |
| | $\widetilde{G}_{T,1}$ | 0.032 | 0.039 | 0.141 | 0.328 | 0.051 | 0.091 | 0.410 | 0.656 | 0.124 | 0.276 | 0.625 | 0.787 |
| | $\widetilde{G}_{T,10}$ | 0.028 | 0.033 | 0.122 | 0.294 | 0.047 | 0.084 | 0.401 | 0.639 | 0.126 | 0.278 | 0.619 | 0.782 |
| 100 | $\widetilde{Z}_T$ | 0.038 | 0.062 | 0.267 | 0.571 | 0.048 | 0.128 | 0.690 | 0.913 | 0.204 | 0.508 | 0.899 | 0.976 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.005 | 0.001 | 0.017 | 0.245 | 0.362 | 0.205 | 0.407 | 0.707 | 0.697 |
| | $\widetilde{G}_{T,1}$ | 0.042 | 0.068 | 0.296 | 0.608 | 0.058 | 0.148 | 0.710 | 0.920 | 0.244 | 0.538 | 0.902 | 0.975 |
| | $\widetilde{G}_{T,10}$ | 0.038 | 0.062 | 0.267 | 0.569 | 0.051 | 0.139 | 0.692 | 0.911 | 0.248 | 0.538 | 0.898 | 0.974 |
| 200 | $\widetilde{Z}_T$ | 0.053 | 0.114 | 0.537 | 0.870 | 0.058 | 0.226 | 0.920 | 0.995 | 0.435 | 0.787 | 0.994 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.001 | 0.000 | 0.006 | 0.235 | 0.332 | 0.339 | 0.614 | 0.848 | 0.735 |
| | $\widetilde{G}_{T,1}$ | 0.055 | 0.120 | 0.569 | 0.891 | 0.066 | 0.248 | 0.929 | 0.996 | 0.468 | 0.799 | 0.994 | 0.999 |
| | $\widetilde{G}_{T,10}$ | 0.053 | 0.114 | 0.537 | 0.870 | 0.059 | 0.230 | 0.920 | 0.995 | 0.471 | 0.797 | 0.993 | 0.999 |
| 1000 | $\widetilde{Z}_T$ | 0.116 | 0.435 | 0.995 | 1.000 | 0.159 | 0.721 | 1.000 | 1.000 | 0.959 | 1.000 | 1.000 | 1.000 |
| | $Z_T^*$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.093 | 0.172 | 0.773 | 0.964 | 0.915 | 0.663 |
| | $\widetilde{G}_{T,1}$ | 0.118 | 0.446 | 0.996 | 1.000 | 0.167 | 0.740 | 1.000 | 1.000 | 0.964 | 1.000 | 1.000 | 1.000 |
| | $\widetilde{G}_{T,10}$ | 0.116 | 0.435 | 0.995 | 1.000 | 0.159 | 0.721 | 1.000 | 1.000 | 0.964 | 1.000 | 1.000 | 1.000 |

Note: The second column indicates the test statistics, where $\widetilde{Z}_T$ is the Lee test defined in (2), $Z_T^*$ is MT test defined in (4), and $\widetilde{G}_{T,\delta}$ is the proposed test or Modified Lee test defined in (13).