IMES DISCUSSION PAPER SERIES

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Discussion Paper No. 2006-E-13

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

C.P.O BOX 203 TOKYO

100-8630 JAPAN

IMES Discussion Paper Series 2006-E-13
July 2006

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Abstract

In this article, the prices of credit derivatives within multiple defaultable entities are evaluated using the asymptotic expansion approach. The theoretical prices of credit derivatives, such as credit default swaptions are often analytically intractable to get. However, recent developments in the asymptotic expansion method more easily enable the evaluation of these contingent claims. This article provides the prices of credit default swaps and swaptions taking account of counterparty credit risks. The validity of the asymptotic expansion method is also discussed.

Keywords: Defaultable bond, counterparty credit risk, credit default swap, swaption, asymptotic expansion method, basket swap, Malliavin calculus
JEL classification: G13

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Views expressed in this paper are those of the author and do not necessarily reflect the official views of the Bank of Japan. This paper is written while I was at Institute for Monetary and Economic Studies, Bank of Japan as an economist. The author really thanks many comments from the members of the Institute for Monetary and Economic Studies, especially from Mr. Yoshiba and Mr. Ieda. I am also grateful for suggestions from Professor Nakagawa, Tokyo Institute of Technology. Of course, the remained mistakes are mine.

Contents

1 Introduction

Credit derivative valuation methods have developed quickly in the last two decades alongside rapid growth in the market for credit derivatives. Among credit derivatives, credit default swaps are one of the more important instruments used to hedge credit risk. It is then necessary to develop more efficient numerical methods to evaluate these contingent claims. For this purpose, in this article the prices of credit default swaps and their swaptions with counterparty credit risks are evaluated using the asymptotic expansion method. A general model based on multiple defaultable entities is established to evaluate credit derivatives in the model with counterparty credit risks. This model is similar to one for basket-type credit default swaps and their swaptions. The article also discusses the problem of pricing basket-type credit default swaps.

Two major approaches are used to evaluate credit derivatives, namely, reduced modeling and structural modeling. In the structural approach, the default time is regarded as an exogenous variable that depends on the characteristics of the specific firm. These models were originally proposed by Merton (1974). On the other hand, the default time in the reduced form approach, which is exploited in this paper, is an unpredictable and endogenous stopping time governed by a default intensity process. Although the reduced modeling approach has fewer economic implications than has structural modeling, the prices of credit derivatives are evaluated efficiently and systematically. These approaches have been employed to evaluate credit derivatives in Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Muroi (2002, 2005).

Several studies have investigated the pricing of credit derivatives in a market including counterparty credit risks. Huge and Lando (1999) considered the pricing methods of credit default swaps in credit migration models with counterparty credit risks. Jarrow and Yu (2001) discussed the pricing of credit default swaps in a model with dependent default intensities. In their model, the default intensity of one firm jumps at the default time of another entity. Chen and Filipović (2003) investigated the pricing methods of credit default swaps and their swaptions using the partial differential equation approach in the affine structure model. The asymptotic expansion method of Kunitomo and Takahashi (2001, 2003) is exploited in Muroi (2005) to evaluate credit derivatives such as options on defaultable bonds. This method enables us to evaluate credit default swaptions more easily because closed-form approximation formulas are available. The valuation of credit default swaps of the basket type, which are briefly discussed in this article, have been analyzed by, for example, Kijima (2000) and Kijima and Muromachi (2000).

This paper is organized as follows. Section 2 explains the basic model, and the expectation form of the pricing formulas for the credit default swaps are shown. The approximation formulas for the prices of the credit default swaps and their swaptions with the asymptotic expansion approach are given in Section 3. Section 4 discusses the pricing of basket swaps and their swaptions. The numerical results are presented in Section 5 and some concluding remarks are given in Section 6. The validity of the asymptotic expansion approach is discussed in the appendix.

2 A Model with Multiple Defaultable Entities

In this section, a model with multiple defaultable firms is constructed. This model is useful for evaluating the credit default swaps with counterparty credit risks discussed in the second half of this section, and the credit default swaps of the basket type that are considered in Section 4. These problems are considered in a frictionless economy with a trading horizon, $[0, U]$. Two kinds of risks are found in this model, interest rate risk and default risk. A probability space $(\Omega, \mathcal{F}, P)$ is fixed and the probability measure P is regarded as a real-world probability measure. A risk-neutral probability measure Q, which is equivalent to the real-world probability measure P, is assumed to exist. This probability space is formally large enough to support the positively valued stochastic processes, $r(t)$ and $h_i(t)$ $(i = 1, \ldots, I)$, and the stopping time, τ_i $(i = 1, \ldots, I)$. The stochastic process $r(t)$ is regarded as the spot interest rate process, and the stochastic process $h_i(t)$ is regarded as the default intensity process for the i-th firm. Stopping time τ_i $(i = 1, \ldots, I)$ represents the default time of the i-th firm. The following assumptions for the stochastic processes $r(t)$ and $h_i(t)$ $(i = 1, \ldots, I)$ are imposed.

Assumption 1 *The spot interest rate process $\{r(t)\}$ and the default intensity rate process $\{h_i(t)\}$ $(i = 1, \ldots, I)$ are positive and predictable processes that satisfy the conditions*

$$\int_0^t r(s)ds < \infty, \quad \int_0^t h_i(s)ds < \infty \ (i = 1, \ldots, I)$$

for any $t \in [0, T]$.

The information sets at time t are given by

$$\begin{aligned} \mathcal{G}_t &= \sigma\{(r(s), h_1(s), \ldots, h_I(s)) : 0 \le s \le t\} \\ \mathcal{H}_t^i &= \sigma\{1_{\{\tau_i \le s\}} : 0 \le s \le t\} \ (i = 1, \ldots, I) \\ \mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^I \ . \end{aligned}$$

A model with the structure of conditional independence is introduced to characterize the multiple defaultable firms. Conditional independence is often used to evaluate credit default swaps of the basket type as discussed by Kijima (2000) and Kijima and Muromachi (2000). Conditional independence also plays an important role in the evaluation of the credit default swaps with counterparty credit risks in this article.

Definition 1 *The stopping times $\tau_1, \ldots, \tau_I$ are conditionally independent with respect to the filtration $\{\mathcal{G}_t\}_{t \in [0,T]}$ under Q, if the relation*

$$Q[\tau_1 > t_1, \ldots, \tau_I > t_I | \mathcal{G}_T] = \prod_{i=1}^{I} Q[\tau_i > t_i | \mathcal{G}_T]$$

is satisfied for arbitrary $t_1, \ldots, t_I \in [0, T]$.

As pointed out by Kijima (2000), conditional independence does not imply ordinal independence and vice versa. The stronger condition, dynamically conditional independence, is also imposed, because the joint survival probability at the future date T_0 is also needed to evaluate the value of swaption.

Definition 2 *The stopping times $\tau_1, \ldots, \tau_I$ are dynamically conditionally independent with respect to the filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$ under the risk-neutral probability measure Q, if the relation*

$$Q[\tau_1 > t_1, \ldots, \tau_I > t_I | \mathcal{G}_T \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^I] = \prod_{i=1}^{I} Q[\tau_i > t_i | \mathcal{G}_T \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^I]$$

is satisfied for arbitrary $t_1, \ldots, t_I \in [0,T]$.

Conditional independence and dynamically conditional independence are introduced in Section 9 of Bielecki and Rutkowski (2002) to evaluate credit default swaps of the basket type. Exponentially distributed independent random variables $E_1, \ldots, E_I$ that are independent of $(r(t), h_1(t), \ldots, h_I(t))$ are introduced. The stopping times $\tau_1, \ldots, \tau_I$ defined by

$$\tau_i = \inf\{t : \int_0^t h_i(s)ds \geq E_i\} \quad (i = 1, \ldots, I) \tag{1}$$

are conditionally independent stopping times. The default time of the i-th firm is modeled with the stopping times in (1). These stopping times are also dynamically conditionally independent. This model can be found in Example 9.1.5 in Bielecki and Rutkowski (2002).

Lemma 1 *The stopping times $\tau_1, \ldots, \tau_I$ defined by (1) satisfies the following properties.*
(i)The combined survival probability at $t_1, \ldots, t_I$ is given by

$$Q[\tau_1 > t_1, \ldots, \tau_I > t_I | \mathcal{G}_T] = \prod_{i=1}^{I} e^{-\int_0^{t_i} h_i(s)ds} = e^{-\sum_{i=1}^{I}\int_0^{t_i} h_i(s)ds}$$

(ii)The stopping times $\tau_1, \ldots, \tau_I$ are conditionally independent.

The proof can be found in Lemma 9.1.1 in Bielecki and Rutkowski (2002). Although the hazard rate processes $h_1(t), \ldots, h_I(t)$ are correlated, the stopping times, $\tau_1, \ldots, \tau_I$, are conditionally independent. Because the pricing of swaptions is also considered in this article, the dynamically conditional independence in our framework must be checked. Fortunately, the following lemma is satisfied.

Lemma 2 *The default time of the firm i $(i = 1, \ldots, I)$ is defined by (1). The following facts must be concluded in our model.*
(i)The stopping times $\tau_1, \ldots, \tau_I$ are conditionally independent.
(ii)The combined survival probability of the firms $i = 1, \ldots, I$ at time $t_1, \ldots, t_I$ is given by

$$Q[\tau_1 > t_1, \ldots, \tau_I > t_I | \mathcal{G}_T \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^I] = e^{-\sum_{i=1}^{I}\int_t^{t_i} h_i(s)ds} 1_{\{\tau_1 > t, \ldots, \tau_I > t\}} . \tag{2}$$

The proof of this lemma can be found in Lemma 9.1.3 in Bielecki and Rutkowski (2002). The following lemma is also important for evaluating credit default swaps of the basket type.

Lemma 3 *The stopping times $\tau_1, \ldots, \tau_I$ satisfy*

$$Q[s \leq \tau_i < s + ds, \tau_k > s\ (k \neq i) | \mathcal{G}_T \vee \mathcal{H}_t^1 \vee \ldots \vee \mathcal{H}_t^I] = h_i(s) e^{-\sum_{i=1}^{I}\int_t^{s} h_i(u)du} ds 1_{\{\tau_1 > t, \ldots, \tau_I > t\}}$$

for time $s(> t)$.

This lemma is easy to prove. This is shown by merely taking the partial derivatives of (2):

$$-\frac{\partial}{\partial t_i} Q[\tau_1 > t_1, \ldots, \tau_I > t_I | \mathcal{G}_T \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^I]|_{t_1=s,\ldots,t_I=s} .$$

A model with three defaultable firms is constructed in the latter half of this section to evaluate the contingent claims with counterparty credit risks. In general, it is necessary to refer to counterparty credit risks to evaluate credit default swaps. The first entity in this model is the reference entity, referred to as firm 1. The second and third entities (firms 2 and 3) in this model are the buyer and the seller of credit default swaps. There are two kinds of bonds in this market, default-free bonds and defaultable bonds issued by firm 1, the former paying the notational principal fixed at 1 at the maturity date, denoted as $p(t, u)$ with maturity u ($u \leq T$) at time t. Even if the issuer of defaultable bonds defaults, defaultable bonds do not necessarily become worthless. This is because any value of the firm remaining is distributed to the bondholders. In this article, the following rule governing the pay-off ratio for defaultable bonds is imposed for defaultable zero coupon bonds.

Assumption 2 *(pay-off ratio) 1) A defaultable zero coupon bond pays the notational principal fixed at* \$1 *at the maturity date, if the issuer is solvent. 2) A defaultable zero coupon bond pays* \$$\delta$ *at the maturity date, if default has occurred before the maturity date* ($0 \leq \delta < 1$).

The following artificial defaultable bonds are introduced for technical reasons.

(i)Zero recovery bonds: This bond has a cash flow at the maturity date if the reference entity (firm 1) is not in default, and has no pay-off if default has occurred before the maturity date. The price of zero-recovery bonds with maturity date, u, at time, t, is denoted by $w_1(t, u)$.
(ii)Zero recovery bonds of the basket type: This bond has a cash flow at the maturity date if none of the three firms defined above are in default before the maturity date. No cash flow occurs if any of these firms are in default before the maturity date. The price of these bonds with maturity date, u, at time, t, is denoted by $w(t, u)$.

The prices of the default-free bonds and zero-recovery bonds issued by firm 1 are given by

$$\begin{aligned}
p(t,T) &= E^Q[\exp(-\int_t^T r(s)ds)|\mathcal{G}_t] \\
w_1(t,T) &= E^Q[\exp(-\int_t^T r(s)ds)1_{\{\tau_1>T\}}|\mathcal{F}_t] \\
&= E^Q[\exp(-\int_t^T r(s)ds)E^Q[1_{\{\tau_1>T\}}|\mathcal{G}_T \vee \mathcal{F}_t]|\mathcal{F}_t] \\
&= E^Q[\exp(-\int_t^T r(s)+h_1(s)ds)1_{\{\tau_1>t\}}|\mathcal{F}_t] \\
&= E^Q[\exp(-\int_t^T r(s)+h_1(s)ds)|\mathcal{G}_t]1_{\{\tau_1>t\}} .
\end{aligned}$$

The price of zero-recovery bonds of the basket type is also given by

$$\begin{aligned} w(t,T) &= E^Q[\exp(-\int_t^T r(s)ds)1_{\{\tau_1,\tau_2,\tau_3>T\}}|\mathcal{F}_t] \\ &= E^Q[\exp(-\int_t^T r(s)+h_1(s)+h_2(s)+h_3(s)ds)|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}} . \end{aligned}$$

Functions w_1 and w have a form that the expectation operator multiplied by the indicator function. The symbol "tilde" means that the indicator function $1_{\{\tau_1,\tau_2,\tau_3>t\}}$ is removed from these functions, i.e., functions $\tilde{w}_1$ and $\tilde{w}$ stand for

$$\begin{aligned} \tilde{w}_1(t,T) &= E^Q[\exp(-\int_t^T r(s)+h_1(s)ds)|\mathcal{G}_t] \\ \tilde{w}(t,T) &= E^Q[\exp(-\int_t^T r(s)+h_1(s)+h_2(s)+h_3(s)ds)|\mathcal{G}_t] . \end{aligned}$$

The arbitrage-free price of defaultable bonds issued by firm 1 is given by

$$v_1(t,T) = \delta p(t,T) + (1-\delta)w_1(t,T) .$$

This is obtained by the arbitrage-free argument. The valuation of forward-start credit default swaps of the Bermudan type is studied in this article. These securities are introduced by Chen and Filipović (2003), and the dates of cash flows for these securities are determined in advance. By equipping these conditions, the pricing formulas for swaptions on these securities become significantly simple. The swap contract is made at time $T_0(>0)$ and the cash flows occur at time $\{T_1,\ldots,T_n\}$. The assumption that these time intervals are equal is made, i.e.,

$$\Delta = T_m - T_{m-1} \quad (m=1,\ldots,n) .$$

In this article, credit default swaps are securities with the following properties.

(a) The default of firm i ($i=1,2,3$) does not occur until time T_m, i.e., $T_m < \tau_1,\tau_2,\tau_3$, the buyer of the credit default swap pays the fixed rate, c, to the seller of the credit default swap at time T_m.
(b) If the default of firm 1 occurs during the time interval $(T_{m-1},T_m]$, and the default times of firms 2 and 3 satisfy the conditions $T_m < \tau_3$ and $T_{m-1} < \tau_2$, then the seller of the credit default swap must pay the amount

$$1 - v_1(T_m,T) = 1 - \delta p(T_m,T)$$

to the buyer of the credit default swap.
(c) There is no payment and the contract terminates in all other cases.

The expected discounted income for the seller of the credit default swap at time $t(\le T_0)$ is denoted by cB_t. B_t is given by

$$B_t = E^Q[\sum_{m=1}^n \exp(-\int_t^{T_m} r(s)ds)\Delta 1_{\{\tau_1,\tau_2,\tau_3>T_m\}}|\mathcal{F}_t] = \Delta\sum_{m=1}^n w(t,T_m)1_{\{\tau_1,\tau_2,\tau_3>t\}} .$$

The expected discounted income for the seller of the credit default swap at time $t(\leq T_0)$ is denoted by S_t. It is given by

$$\begin{aligned} S_t &= E^Q[\sum_{m=1}^{n} e^{-\int_t^{T_m} r(s)ds}(1-\delta p(T_m,T))1_{\{T_{m-1}<\tau_1<T_m\}}1_{\{T_{m-1}<\tau_2\}}1_{\{T_m\leq\tau_3\}}|\mathcal{F}_t] \\ &= \sum_{m=1}^{n}(S_t^{1m}-\delta S_t^{2m}-S_t^{3m}+\delta S_t^{4m}) \end{aligned}$$

where $S_t^{1m}, S_t^{2m}, S_t^{3m}, S_t^{4m}$ are given by

$$\begin{aligned} S_t^{1m} &= E^Q[e^{-\int_t^{T_m} r(s)ds}1_{\{\tau_1,\tau_2>T_{m-1},\tau_3>T_m\}}|\mathcal{F}_t] \\ S_t^{2m} &= E^Q[e^{-\int_t^{T_m} r(s)ds}p(T_m,T)1_{\{\tau_1,\tau_2>T_{m-1},\tau_3>T_m\}}|\mathcal{F}_t] \\ S_t^{3m} &= E^Q[e^{-\int_t^{T_m} r(s)ds}1_{\{\tau_1>T_m,\tau_2>T_{m-1},\tau_3>T_m\}}|\mathcal{F}_t] \\ S_t^{4m} &= E^Q[e^{-\int_t^{T_m} r(s)ds}p(T_m,T)1_{\{\tau_1>T_m,\tau_2>T_{m-1},\tau_3>T_m\}}|\mathcal{F}_t]\ . \end{aligned}$$

Tedious calculations lead to alternative expressions of $S_t^{1m}, S_t^{2m}, S_t^{3m}$ and S_t^{4m}. They are given by

$$\begin{aligned} S_t^{1m} &= E^Q[e^{-\int_t^{T_m}(r(s)+h_3(s))ds-\int_t^{T_{m-1}}(h_1(s)+h_2(s))ds}|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}} \\ S_t^{2m} &= E^Q[e^{-\int_t^{T} r(s)ds-\int_t^{T_{m-1}} h_1(s)+h_2(s)ds-\int_t^{T_m} h_3(s)ds}|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}} \\ S_t^{3m} &= E^Q[e^{-\int_t^{T_m}(r(s)+h_1(s)+h_3(s))ds-\int_t^{T_{m-1}} h_2(s)ds}|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}} \\ S_t^{4m} &= E^Q[e^{-\int_t^{T_m}(r(s)+h_1(s)+h_3(s))ds-\int_t^{T_{m-1}} h_2(s)ds}p(T_m,T)|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}} \\ &= E^Q[e^{-\int_t^{T} r(s)ds-\int_t^{T_m}(h_1(s)+h_3(s))ds-\int_t^{T_{m-1}} h_2(s)ds}|\mathcal{G}_t]1_{\{\tau_1,\tau_2,\tau_3>t\}}\ . \end{aligned}$$

These conditional expectations are calculated by the partial differential equation (PDE) approach, if the four-dimensional stochastic process $(r(t),h_1(t),h_2(t),h_3(t))$ has the affine structure. This is discussed by Chen and Filipović (2003).

The premium for a forward-start credit default swap at the initial time, 0, is represented using the conditional expectations. If it is denoted by CDS, the relation

$$CDS \times B_0 = S_0$$

must be satisfied. CDS is represented by

$$CDS = \frac{S_0}{B_0} = \frac{\sum_{m=1}^{n}(\tilde{S}_0^{1m}-\delta\tilde{S}_0^{2m}-\tilde{S}_0^{3m}+\delta\tilde{S}_0^{4m})}{\Delta\sum_{m=1}^{n}\tilde{w}(0,T_m)}\ .$$

3 Asymptotic Expansion Methods

In this section, the pricing of credit default swaps and swaptions on credit default swaps are considered using the asymptotic expansion approach. This method was first introduced by Watanabe (1987) and has been applied to mathematical statistics by Yoshida

(1992). The asymptotic expansion approach is also a powerful tool for mathematical finance as shown by recent applications by Kunitomo and Takahashi (2001, 2003). The asymptotic approximation formulas for credit derivatives have been investigated by Muroi (2005). The validity of this method is discussed in the Appendix.

3.1 Pricing Credit Default Swaps

The risk-free rates and the default hazard rates for each firm i ($i = 1, 2, 3$) are denoted by $r^\epsilon(t)$ and $h_i^\epsilon(t)$ ($i = 1, 2, 3$), respectively. They satisfy the following stochastic differential equation (SDE) with a small parameter, $\epsilon \in (0, 1]$,

$$r^\epsilon(t) = x_0 + \int_0^t \mu_0(\bar{x}_0 - r^\epsilon(s))ds + \sum_{j=0}^{J} \epsilon \int_0^t \sigma_{0j}(r^\epsilon(s), h_1^\epsilon(s), h_2^\epsilon(s), h_3^\epsilon(s))dW_s^j \quad (3)$$

$$h_i^\epsilon(t) = x_i + \int_0^t \mu_i(\bar{x}_i - h_i^\epsilon(s))ds + \sum_{j=0}^{J} \epsilon \int_0^t \sigma_{ij}(r^\epsilon(s), h_1^\epsilon(s), h_2^\epsilon(s), h_3^\epsilon(s))dW_s^j, \quad (4)$$

where $(W_t^0, \ldots, W_t^J)$ is a $J + 1$ dimensional standard Brownian motion. This model contains many specific and popular examples, such as the Gaussian models and the (multifactor) CIR models (Cox, Ingersoll and Ross models).

The zero-th order terms of the stochastic expansions for $r^\epsilon(t)$ and $h_i^\epsilon(t)$ ($i = 1, 2, 3$) are denoted by $X_0(t)$ and $X_i(t)$ ($i = 1, 2, 3$), i.e.

$$X_0(t) = r^\epsilon(t)|_{\epsilon=0}, \quad X_i(t) = h_i^\epsilon(t)|_{\epsilon=0} \quad (i = 1, 2, 3) .$$

These quantities are solutions of the integral equations

$$X_i(t) = x_i + \int_0^t \mu_i(\bar{x}_i - X_i(s))ds$$

and they are given by

$$X_i(t) = \bar{x}_i + (x_i - \bar{x}_i)e^{-\mu_i t} .$$

The first-order terms of stochastic expansions for $r^\epsilon(t)$ and $h_i^\epsilon(t)$ ($i = 1, 2, 3$) are represented by $A_0(t)$ and $A_i(t)$ ($i = 1, 2, 3$). They are given by

$$A_0(t) = \frac{\partial r^\epsilon(t)}{\partial \epsilon}|_{\epsilon=0}, \quad A_i(t) = \frac{\partial h_i^\epsilon(t)}{\partial \epsilon}|_{\epsilon=0} \quad (i = 1, 2, 3) .$$

These are the solutions of the integral equations

$$A_i(t) = -\int_0^t \mu_i A_i(s)ds + \sum_{j=0}^{J} \int_0^t \sigma_{ij}(X_0(s), \ldots, X_3(s))dW_s^j$$

and they are given by

$$A_i(t) = \sum_{j=0}^{J} \int_0^t \sigma_{ij}^A(t, s)dW_s^j, \quad \sigma_{ij}^A(t, s) = Y_i(t)(Y_i(s))^{-1}\sigma_{ij}(X_0(s), \ldots, X_3(s)) ,$$

where $Y_i(t)$ is a solution of the following ordinary differential equation (ODE),

$$\frac{dY_i(t)}{dt} = -\mu_i Y_i(t), \quad Y_i(0) = 1, \ (i = 0, \dots, 3) .$$

This equation is solved easily and the solution is $Y_i(t) = e^{-\mu_i t}$. The second-order terms of the stochastic expansions for the stochastic processes $r^\epsilon(t)$ and $h_i^\epsilon(t)$ $(i = 1, 2, 3)$ are denoted by $B_0(t)$ and $B_i(t)$ $(i = 1, 2, 3)$. They are given by

$$B_0(t) = \frac{1}{2}\frac{\partial^2 r^\epsilon(t)}{\partial \epsilon^2}|_{\epsilon=0}, \quad B_i(t) = \frac{1}{2}\frac{\partial^2 h_i^\epsilon(t)}{\partial \epsilon^2}|_{\epsilon=0} \quad (i = 1, 2, 3) .$$

These are the solutions of the integral equations,

$$B_i(t) = -\int_0^t \mu_i B_i(s) ds + \sum_{j=0}^{J}\sum_{l=0}^{3} \int_0^t \partial_l \sigma_{ij}(X_0(s), \dots, X_3(s)) A_l(s) dW_s^j .$$

The solutions are

$$B_i(t) = \sum_{j,k=0}^{J}\sum_{l=0}^{3} \int_0^t \sigma_{ijl}^B(t,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j$$

where

$$\sigma_{ijk}^B(t,s) = Y_i(t)(Y_i(s))^{-1} \partial_k \sigma_{ij}(X_0(s), \dots, X_3(s)) .$$

The Taylor expansion with the small parameter ϵ yields the expansion formulas for $r^\epsilon(t)$ and $h_i^\epsilon(t)$ $(i = 1, 2, 3)$. They are given by

$$r^\epsilon(t) = X_0(t) + \epsilon A_0(t) + \epsilon^2 B_0(t) + o_Q(\epsilon^2) , \quad h_i^\epsilon(t) = X_i(t) + \epsilon A_i(t) + \epsilon^2 B_i(t) + o_Q(\epsilon^2) .$$

The new random variables,

$$R_1^{ij}(t,T) = \int_0^t r_{ij}^A(T,t,s) dW_s^j, \quad \tilde{R}_1^{ij}(t,T) = \int_t^T r_{ij}^A(T,s,s) dW_s^j$$

and

$$\begin{aligned} R_2^{ijk}(t,T) &= \sum_{l=0}^{3} \int_0^t r_{ijl}^B(T,t,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j \\ \tilde{R}_2^{ijk}(t,T) &= \sum_{l=0}^{3} \int_t^T r_{ijl}^B(T,s,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j , \end{aligned}$$

are introduced to calculate the integrals $\int_t^T A_i(s) ds$ and $\int_t^T B_i(s) ds$ in the next steps. The functions $r_{ij}^A(T,t,s)$ and $r_{ijk}^B(T,t,s)$ are given by

$$\begin{aligned} r_{ij}^A(T,t,s) &= (\int_t^T Y_i(u) du)(Y_i(s))^{-1} \sigma_{ij}(X_0(s), \dots, X_3(s)) \\ r_{ijk}^B(T,t,s) &= (\int_t^T Y_i(u) du)(Y_i(s))^{-1} \partial_k \sigma_{ij}(X_0(s), \dots, X_3(s)) . \end{aligned}$$

These integrals are represented by

$$\int_t^T A_i(s)ds = \sum_{j=0}^{J}(R_1^{ij}(t,T)+\tilde{R}_1^{ij}(t,T)) \ and \ \int_t^T B_i(s)ds = \sum_{j,k=0}^{J}(R_2^{ijk}(t,T)+\tilde{R}_2^{ijk}(t,T)) \ .$$

These representations lead to the price of discount bond with the maturity date T at time $t \in [0,T]$, which is given by

$$p(t,T) = e^{-\int_t^T X_0(s)ds-\epsilon\sum_{j=0}^{J} R_1^{0,j}(t,T)-\epsilon^2\sum_{j=0}^{J} R_2^{0,j}(t,T)}(1+\frac{\epsilon^2}{2}\sum_{j=0}^{J}\int_t^T r_{0j}^A(T,u,u)^2du)+o_Q(\epsilon^2) \ .$$

If the issuer of bonds is not in default at time $t \in [0,T]$, the price of zero-recovery bonds with maturity date T at time t is given by

$$\begin{aligned}\tilde{w}_1(t,T) &= e^{-\int_t^T X_0(s)+X_1(s)ds-\epsilon\sum_{j=0}^{J}(R_1^{0,j}(t,T)+R_1^{1,j}(t,T))-\epsilon^2\sum_{j=0}^{J}(R_2^{0,j}(t,T)+R_2^{1,j}(t,T))} \\ &\times(1+\frac{\epsilon^2}{2}\sum_{j=0}^{J}\int_t^T(r_{0j}^A(T,u,u)+r_{1j}^A(T,u,u))^2du)+o_Q(\epsilon^2) \ .\end{aligned}$$

The price of defaultable bonds of the basket type is given by

$$\begin{aligned}\tilde{w}(t,T) &= e^{-\sum_{i=0}^{3}\int_t^T X_i(s)ds-\epsilon\sum_{i=0}^{3}\sum_{j=0}^{J} R_1^{i,j}(t,T)-\epsilon^2\sum_{i=0}^{3}\sum_{j=0}^{J} R_2^{i,j}(t,T)} \\ &\times(1+\frac{\epsilon^2}{2}\sum_{j=0}^{J}\int_t^T(\sum_{i=0}^{3} r_{ij}^A(T,u,u))^2du)+o_Q(\epsilon^2) \ ,\end{aligned}$$

if the issuer of defaultable bonds is not in default at time t. Four kinds of conditional expectations introduced in the previous section, $\tilde{S}_t^{1m}, \tilde{S}_t^{2m}, \tilde{S}_t^{3m}, \tilde{S}_t^{4m}$, are also calculated further to evaluate the price of the credit default swap. The quantity $\tilde{S}_t^{1m}$ is given by

$$\begin{aligned}\tilde{S}_t^{1m} &= \alpha_1^m(t)e^{-\epsilon\sum_{j=0}^{J} R_1^{0j}(t,T_m)+R_1^{1j}(t,T_{m-1})+R_1^{2j}(t,T_{m-1})+R_1^{3j}(t,T_m)}\times \\ &\times e^{-\epsilon^2\sum_{j,k=0}^{J} R_2^{0jk}(t,T_m)+R_2^{1jk}(t,T_{m-1})+R_2^{2jk}(t,T_{m-1})+R_2^{3jk}(t,T_m)}+o_Q(\epsilon^2)\end{aligned}$$

where $\alpha_1^m(t)$ and $q_{1j}^A(T_m,t,u)$ are

$$\begin{aligned}\alpha_1^m(t) &= e^{-\int_t^{T_m}(X_0(s)+X_3(s))ds-\int_t^{T_{m-1}}(X_1(s)+X_2(s))ds} \\ &\times\{1+\frac{\epsilon^2}{2}\sum_{j=0}^{J}\int_t^{T_m}(q_{1j}^A(T_m,u,u))^2du\} \\ q_{1j}^A(T_m,t,u) &= (r_{0j}^A(T_m,t,u)+r_{3j}^A(T_m,t,u)) \\ &+(r_{1j}^A(T_{m-1},t,u)+r_{2j}^A(T_{m-1},t,u))1_{\{u\le T_{m-1}\}} \ .\end{aligned}$$

$\tilde{S}_t^{2m}, \tilde{S}_t^{3m}$ and $\tilde{S}_t^{4m}$ are given by

$$\begin{aligned}\tilde{S}_t^{2m} &= \alpha_2^m(t)e^{-\epsilon\sum_{j=0}^{J} R_1^{0j}(t,T)+R_1^{1j}(t,T_{m-1})+R_1^{2j}(t,T_{m-1})+R_1^{3j}(t,T_m)}\times \\ &\times e^{-\epsilon^2\sum_{j,k=0}^{J} R_2^{0jk}(t,T)+R_2^{1jk}(t,T_{m-1})+R_2^{2jk}(t,T_{m-1})+R_2^{3jk}(t,T_m)}+o_Q(\epsilon^2)\end{aligned}$$

$$
\begin{aligned}
\tilde{S}_t^{3m} &= \alpha_3^m(t) e^{-\epsilon \sum_{j=0}^J R_1^{0j}(t,T_m)+R_1^{1j}(t,T_m)+R_1^{2j}(t,T_{m-1})+R_1^{3j}(t,T_m)} \times \\
&\quad \times e^{-\epsilon^2 \sum_{j,k=0}^J R_2^{0jk}(t,T_m)+R_2^{1jk}(t,T_m)+R_2^{2jk}(t,T_{m-1})+R_2^{3jk}(t,T_m)} + o_Q(\epsilon^2) \\
\tilde{S}_t^{4m} &= \alpha_4^m(t) e^{-\epsilon \sum_{j=0}^J R_1^{0j}(t,T)+R_1^{1j}(t,T_m)+R_1^{2j}(t,T_{m-1})+R_1^{3j}(t,T_m)} \times \\
&\quad \times e^{-\epsilon^2 \sum_{j,k=0}^J R_2^{0jk}(t,T)+R_2^{1jk}(t,T_m)+R_2^{2jk}(t,T_{m-1})+R_2^{3jk}(t,T_m)} + o_Q(\epsilon^2) \, ,
\end{aligned}
$$

where $\alpha_i^m(t)$ and $q_{ij}^A(T_m, t, u)$ $(i = 2, 3, 4)$ are given by

$$
\begin{aligned}
\alpha_2^m(t) &= e^{-\int_t^T X_0(s)ds - \int_t^{T_{m-1}}(X_1(s)+X_2(s))ds - \int_t^{T_m} X_3(s)ds} \\
&\quad \times \{1 + \frac{\epsilon^2}{2} \sum_{j=0}^J \int_t^T (q_{2j}^A(T_m, u, u))^2 du\} \\
\alpha_3^m(t) &= e^{-\int_t^{T_m}(X_0(s)+X_1(s)+X_3(s))ds - \int_t^{T_{m-1}} X_2(s)ds} \\
&\quad \times \{1 + \frac{\epsilon^2}{2} \sum_{j=0}^J \int_t^{T_m} (q_{3j}^A(T_m, u, u))^2 du\} \\
\alpha_4^m(t) &= e^{-\int_t^T X_0(s)ds - \int_t^{T_m}(X_1(s)+X_3(s))ds - \int_t^{T_{m-1}} X_2(s)ds} \\
&\quad \times \{1 + \frac{\epsilon^2}{2} \sum_{j=0}^J \int_t^T (q_{4j}^A(T_m, u, u))^2 du\} \\
q_{2j}^A(T_m, t, u) &= r_{0j}^A(T, t, u) + (r_{1j}^A(T_{m-1}, t, u) + r_{2j}^A(T_{m-1}, t, u)) 1_{\{u \le T_{m-1}\}} \\
&\quad + r_{3j}^A(T_m, t, u) 1_{\{u \le T_m\}} \\
q_{3j}^A(T_m, t, u) &= r_{0j}^A(T_m, t, u) + r_{1j}^A(T_m, t, u) + r_{2j}^A(T_{m-1}, t, u) 1_{\{u \le T_{m-1}\}} \\
&\quad + r_{3j}^A(T_m, t, u) \\
q_{4j}^A(T_m, t, u) &= r_{0j}^A(T, t, u) + (r_{1j}^A(T_m, t, u) + r_{3j}^A(T_m, t, u)) 1_{\{u \le T_m\}} \\
&\quad + r_{2j}^A(T_{m-1}, t, u) 1_{\{u \le T_{m-1}\}} \, .
\end{aligned}
$$

These results lead to the premium of forward credit default swaps at time $t \le T_0$.

Theorem 1 *The premium of the forward credit default swap at the initial time* 0 *is given by*

$$
\begin{aligned}
CDS(0) &= \frac{\sum_{m=1}^n (\tilde{S}_0^{1m} - \delta \tilde{S}_0^{2m} - \tilde{S}_0^{3m} + \delta \tilde{S}_0^{4m})}{\Delta \sum_{m=1}^n \tilde{w}(0, T_m)} 1_{\{\tau_1, \tau_2, \tau_3 > 0\}} \\
&= \frac{\sum_{m=1}^n (\alpha_1^m(0) - \delta \alpha_2^m(0) - \alpha_3^m(0) + \delta \alpha_4^m(0)) + o_Q(\epsilon^2)}{\Delta \sum_{m=1}^n \tilde{w}(0, T_m) + o_Q(\epsilon^2)} 1_{\{\tau_1, \tau_2, \tau_3 > 0\}} \, .
\end{aligned}
$$

3.2 Pricing Credit Default Swaptions

This subsection presents the asymptotic approximation formula for the price of a credit default swaption. Consider a credit default swaption on a credit default swap with a strike price K and expiry date T_0. The payoff of this contract at the maturity date T_0 is given by $(S_{T_0}-KB_{T_0})^+$. The swaption price at the initial time 0 is given by

$$
\begin{aligned}
&E[e^{-\int_0^{T_0} r^\epsilon(s)+h_1^\epsilon(s)+h_2^\epsilon(s)+h_3^\epsilon(s)ds}\times \\
&\quad \times\{\sum_{m=1}^{n}(\tilde{S}_{T_0}^{1m}-\delta\tilde{S}_{T_0}^{2m}-\tilde{S}_{T_0}^{3m}+\delta\tilde{S}_{T_0}^{4m}-K\Delta\tilde{w}(T_0,T_m))\}^+]1_{\{\tau_1,\tau_2,\tau_3>0\}} \qquad (5)
\end{aligned}
$$

where the random variable C is given by $C=S_{T_0}/B_{T_0}$. A random variable g is defined by

$$
g=e^{-\int_0^{T_0} r^\epsilon(s)+h_1^\epsilon(s)+h_2^\epsilon(s)+h_3^\epsilon(s)ds}\{\sum_{m=1}^{n}(\tilde{S}_{T_0}^{1m}-\delta\tilde{S}_{T_0}^{2m}-\tilde{S}_{T_0}^{3m}+\delta\tilde{S}_{T_0}^{4m}-K\Delta\tilde{w}(T_0,T_m))\}\ .
$$

The asymptotic expansion of the random variable $e^{-\int_0^{T_0} r^\epsilon(s)+h_1^\epsilon(s)+h_2^\epsilon(s)+h_3^\epsilon(s)ds}\tilde{S}_{T_0}^{im}$ is calculated and is given by

$$
e^{-\int_0^{T_0} r^\epsilon(s)+h_1^\epsilon(s)+h_2^\epsilon(s)+h_3^\epsilon(s)ds}\tilde{S}_{T_0}^{im}=g_{im}^0+\epsilon g_{im}^1+\epsilon^2 g_{im}^2+o_Q(\epsilon^2)
$$

where g_{im}^0, g_{im}^1 and g_{im}^2 $(i=1,2,3,4)$ are given by

$$
\begin{aligned}
g_{im}^0 &= e^{-\int_0^{T_0} X_0(s)+\cdots+X_3(s)ds}\alpha_i^m(T_0) \\
g_{im}^1 &= -g_{im}^0\sum_{j=0}^{J}\int_0^{T_0} q_{ij}^A(T_m,u,u)dW_u^j \\
g_{im}^2 &= -g_{im}^0\sum_{j,k=0}^{J}\sum_{l=0}^{3}\int_0^{T_0} q_{ijl}^B(T_m,s,s)\int_0^s \sigma_{lk}^A(s,u)dW_u^k dW_s^j \\
&\quad +\frac{g_{im}^0}{2}\{\sum_{j=0}^{J}\int_0^{T_0} q_{ij}^A(T_m,u,u)dW_u^j\}^2 \quad (i=1,2,3,4)\ .
\end{aligned}
$$

A function $q_{ijk}^B(T_m,t,u)$ $(i=1,2,3,4; j,k=0,\ldots,J)$ is defined by

$$
\begin{aligned}
q_{1jk}^B(T_m,t,u) &= (r_{0jk}^B(T_m,t,u)+r_{3jk}^B(T_m,t,u)) \\
&\quad +(r_{1jk}^B(T_{m-1},t,u)+r_{2jk}^B(T_{m-1},t,u))1_{\{u\le T_{m-1}\}} \\
q_{2jk}^B(T_m,t,u) &= r_{0jk}^B(T,t,u)+(r_{1jk}^B(T_{m-1},t,u)+r_{2jk}^B(T_{m-1},t,u))1_{\{u\le T_{m-1}\}} \\
&\quad +r_{3jk}^B(T_m,t,u)1_{\{u\le T_m\}} \\
q_{3jk}^B(T_m,t,u) &= r_{0jk}^B(T_m,t,u)+r_{1jk}^B(T_m,t,u)+r_{2jk}^B(T_{m-1},t,u)1_{\{u\le T_{m-1}\}} \\
&\quad +r_{3jk}^B(T_m,t,u) \\
q_{4jk}^B(T_m,t,u) &= r_{0jk}^B(T,t,u)+(r_{1jk}^B(T_m,t,u)+r_{3jk}^B(T_m,t,u))1_{\{u\le T_m\}} \\
&\quad +r_{2jk}^B(T_{m-1},t,u)1_{\{u\le T_{m-1}\}}\ .
\end{aligned}
$$

Similar calculations lead to

$$
e^{-\int_0^{T_0} r^\epsilon(s)+h_1^\epsilon(s)+h_2^\epsilon(s)+h_3^\epsilon(s)ds}\tilde{w}(T_0,T_m)=g_{5m}^0+\epsilon g_{5m}^1+\epsilon^2 g_{5m}^2+o_Q(\epsilon^2)
$$

where g^0_{5m}, g^1_{5m} and g^2_{5m} $(i = 1, 2, 3, 4)$ are given by

$$
\begin{aligned}
g^0_{5m} &= e^{-\int_0^{T_0} X_0(s)+\cdots+X_3(s)ds}(1 + \frac{\epsilon^2}{2}\sum_{j=0}^{J}\int_{T_0}^{T_m}(\sum_{i=0}^{3} r^A_{ij}(T_m, u, u))^2 du) \\
g^1_{5m} &= -g^0_{5m}\sum_{j=0}^{J}\int_0^{T_0}\sum_{i=0}^{3} r^A_{ij}(T_m, u, u)dW^j_u \\
g^2_{5m} &= -g^0_{5m}\sum_{j,k=0}^{J}\sum_{l=0}^{3}\int_0^{T_0}\sum_{i=0}^{3} r^B_{ijl}(T_m, s, s)\int_0^u \sigma^A_{lk}(s, u)dW^k_u dW^j_s \\
&\quad + \frac{g^0_{5m}}{2}\{\sum_{j=0}^{J}\int_0^{T_0}\sum_{i=0}^{3} r^A_{ij}(T_m, u, u)dW^j_u\}^2 \quad (i = 1, 2, 3, 4) .
\end{aligned}
$$

Random variables g^0, g^1 and g^2 are newly defined by

$$
g^i = \sum_{m=1}^{n}(g^i_{1m} - \delta g^i_{2m} - g^i_{3m} + \delta g^i_{4m} - K\Delta g^i_{5m}) \quad (i = 0, 1, 2) .
$$

Although the zero-th order term is order 1, i.e., $g^0_{jm} = O(1)$, g^0 is very small if the swaption is nearly at the money (ATM) and the condition $g^0 = \epsilon y = O(\epsilon)$ is imposed. If the random variable g^1 is denoted by

$$
g^1 = \int_0^{T_0}\sigma^j_g(t)dW^j_t ,
$$

then $\sigma^j_g(t)$ is represented by

$$
\begin{aligned}
\sigma^j_g(t) &= \sum_{m=1}^{n}\{-g^0_{1m}q^A_{1j}(T_m, t, t) + \delta g^0_{2m}q^A_{2j}(T_m, t, t) \\
&\quad + g^0_{3m}q^A_{3j}(T_m, t, t) - \delta g^0_{4m}q^A_{4j}(T_m, t, t) + K\Delta g^0_{5m}\sum_{i=0}^{3} r^A_{ij}(T_m, t, t)\} .
\end{aligned}
$$

The price of swaption P is given by

$$
P = E[g1_{\{g\geq 0\}}] .
$$

If the density function for the random variable

$$
X^\epsilon_{T_0} = \frac{1}{\epsilon}(g - g^0) = g^1 + \epsilon g^2 + \cdots
$$

is identified, the price of the swaption given in (5) is derived. The characteristic function of the random variable $X^\epsilon_{T_0}$ is denoted by $\phi_X(t)$ and is represented by

$$
\phi_X(t) = E[e^{itX^\epsilon_{T_0}}] = E[e^{it(g^1+\epsilon g^2+\cdots)}] = E[e^{itg^1}(1 + \epsilon it E[g^2|g^1])] + o_Q(\epsilon) .
$$

The following condition

$$
\Sigma = \int_0^{T_0}\sum_{j=0}^{J}\sigma^j_g(t)^2 dt > 0
$$

is imposed. This condition assures nondegeneracy of the Malliavin covariance matrix defined in Section 6.

Lemma 4 *The density function for the random variable $X^\epsilon_{T_0}$ is denoted by $f^\epsilon_X(x)$ and is given by*

$$f^\epsilon_X(x) = n[x;0,\Sigma] + \epsilon[\frac{c}{\Sigma}x^3 + (\frac{f}{\Sigma} - 2c)x]n[x;0,\Sigma] + o(\epsilon^2)\ ,$$

where the function $n[x;0,\Sigma]$ represents the density function for the normal random variable with mean 0 and variance Σ. The constants c and f are given by

$$E[g_2|g_1 = x] = cx^2 + f\ .$$

This conditional expectation is calculated by using the following lemma.

Lemma 5 *$\{\mathbf{w}_t\}$ is an n-dimensional Wiener process and the vector $\mathbf{x}$ is a k-dimensional vector. There exists a nonstochastic function $\mathbf{q}_1(t) : \mathbf{R}^1 \to \mathbf{R}^{k\times n}$ and a positive semi-definite matrix $\mathbf{\Sigma} = \int_0^T \mathbf{q}_1(t)\mathbf{q}_1(t)'dt$.*
(1) *There exists a nonstochastic function $\mathbf{q}_i(t) : \mathbf{R}^1 \to \mathbf{R}^{m\times n}$ $(i = 2,3)$. The following identity must be satisfied for $0 \le s \le t \le T$:*

$$E[\int_0^t[\int_0^s \mathbf{q}_2(u)d\mathbf{w}_u]'\mathbf{q}_3(s)d\mathbf{w}_s|\int_0^T \mathbf{q}_1(u)d\mathbf{w}_u = \mathbf{x}]$$
$$= \mathbf{tr}\int_0^t[\int_0^s \mathbf{q}_2(u)\mathbf{q}_1(u)'du]\mathbf{q}_1(s)\mathbf{q}_3(s)'ds\mathbf{\Sigma}^{-1}[\mathbf{xx}' - \mathbf{\Sigma}]\mathbf{\Sigma}^{-1}\ .$$

(2) *There exists a nonstochastic function, $\mathbf{q}_i(t) : \mathbf{R}^1 \to \mathbf{R}^n$ $(i = 2,3)$. The following identity must be satisfied for $0 \le s \le t \le T$:*

$$E[\int_0^s \mathbf{q}_2(u)d\mathbf{w}_u][\int_0^t \mathbf{q}_3(v)d\mathbf{w}_v]|\int_0^T \mathbf{q}_1(u)d\mathbf{w}_u = \mathbf{x}]$$
$$= \int_0^s \mathbf{q}_2(u)\mathbf{q}_3(u)'du + [\int_0^s \mathbf{q}_2(u)\mathbf{q}_1(u)'du]\mathbf{\Sigma}^{-1}[\mathbf{xx}' - \mathbf{\Sigma}]\mathbf{\Sigma}^{-1}[\int_0^t \mathbf{q}_1(u)\mathbf{q}_3(u)'du]\ .$$

The constant c is represented by using new constants

$$\begin{aligned}
c_{im} &= -g^0_{im}\sum_{j,k=0}^{J}\sum_{l=0}^{3}\int_0^{T_0} q^B_{ijl}(T_m,s,s)\sigma^j_g(s)\int_0^s \sigma^A_{lk}(s,u)\sigma^k_g(u)duds \\
&\quad + \frac{g^0_{im}}{2}\sum_{j=0}^{3}\{\int_0^{T_0} q^A_{ij}(T_m,s,s)\sigma^j_g(s)ds\}^2 \quad (i = 1,\ldots,4) \\
c_{5m} &= -g^0_{5m}\sum_{j,k=0}^{J}\sum_{l=0}^{3}\int_0^{T_0}\sum_{i=0}^{3} r^B_{ijl}(T_m,s,s)\sigma^j_g(s)\int_0^s \sigma^A_{lk}(s,u)\sigma^k_g(u)duds \\
&\quad + \frac{g^0_{5m}}{2}\sum_{j=0}^{3}\{\int_0^{T_0}\sum_{i=0}^{3} r^A_{ij}(T_m,s,s)\sigma^j_g(s)ds\}^2
\end{aligned}$$

and $c = \sum_{m=1}^n (c_{1m} - \delta c_{2m} - c_{3m} + \delta c_{4m} - K\Delta c_{5m})/\Sigma^2$. The constant, f, is also defined by exploiting new constants

$$b_{im} = \frac{g^0_{im}}{2}\int_0^{T_0}\sum_{j=0}^{J} q^A_{ij}(T_m,s,s)^2 ds \quad (i = 1,\ldots,4)$$

$$\begin{aligned} b_{5m} &= \frac{g^0_{5m}}{2} \int_0^{T_0} \sum_{j=0}^{J} (\sum_{i=0}^{3} r^A_{ij}(T_m, s, s))^2 ds \\ b &= \sum_{m=1}^{n} (b_{1m} - \delta b_{2m} - b_{3m} + \delta b_{4m} - K\Delta b_{5m}) \end{aligned}$$

and is given by $f = b - c\Sigma$. The constants c and f lead to the swaption price,

$$\begin{aligned} P &= E^Q[g_0 + \epsilon X^\epsilon_i]^+ \\ &= \int_{g_0+\epsilon x \geq 0} (g_0 + \epsilon x) f^\epsilon_X(x) dx \\ &= \int_{g_0+\epsilon x \geq 0} (g_0 + \epsilon x) n[x; 0, \Sigma]\{1 + \epsilon[\frac{c}{\Sigma} x^3 + (\frac{f}{\Sigma} - 2c)x] + \epsilon^2 h(x)\} dx + o(\epsilon^2) \ , \end{aligned}$$

where $h(x)$ is a polynomial function. Further calculations lead to

$$P = g_0 \int_{-y}^{\infty} n[x; 0, \Sigma] dx + \epsilon \int_{-y}^{\infty} x n[x; 0, \Sigma] dx + \epsilon^2 \int_{-y}^{\infty} (cx^2 + f) n[x; 0, \Sigma] dx + o(\epsilon^2)$$

where y is given by $y = g_0/\epsilon$ $(= O(1))$. The integrals are calculated by using the following formulas

$$\begin{aligned} \int_{-y}^{\infty} x n[x; 0, \Sigma] dx &= \Sigma n[y; 0, \Sigma] \ , \\ \int_{-y}^{\infty} x^2 n[x; 0, \Sigma] dx &= \Sigma N(\frac{y}{\Sigma^{1/2}}) - y\Sigma n[y; 0, \Sigma] \ , \end{aligned}$$

where $N(\cdot)$ is the distribution function for a standard normal random variable. The numerical results using this pricing formula are provided in Section 5.

4 Pricing Credit Default Swaps and Swaptions of the Basket Type

The pricing of credit default swaps and credit default swaptions with counterparty credit risks have been discussed in earlier sections. In order to evaluate credit derivatives within these models, the default hazard rates for multiple firms are considered. The pricing of credit default swaps of the basket type is included as another example where a model with multiple defaultable entities should be considered. Credit default swaps of the basket type, such as first-to-default swaps, are contingent claims that are frequently traded on an OTC (on-the-counter) basis and have been analyzed by Kijima (2000) and Kijima and Muromachi (2000). Credit default swaps of the basket type are default swaps where the basket is underlain by more than one entity. There are several kinds of credit default swaps of this type, including first-to-default swaps and nth-to-default swaps. The pricing of first-to-default swaps and their swaptions are considered in the current article. Because there are I-defaultable entities and interest rate risk, this model is an $I + 1$ factor model. Forward credit default swaps of the basket type are securities that start the contract at the future date T_0 and the payment is made only at the predetermined time points

$\{T_1, \ldots, T_n\}$. These time increments are assumed to be equal in this article, i.e., $\Delta = T_m - T_{m-1}$ $(m = 1, \ldots, n)$. A credit default swap of the basket type in this article is defined:

(a) If no default has occurred at time T_m then the buyer of the swap pays a premium, c, at time T_m.
(b) If the firm i in the basket is in default in the period $(T_{m-1}, T_m]$ and this is the first default among the I firms in the basket, the seller of the swap pays the amount γ_i to the buyer and the contract terminates.
(c) If the default of any firm has not occurred up to the maturity date T_n, the seller of the swap pays γ_0 at the maturity date.

The discounted expected income to the seller of the credit default swap is given by

$$\begin{aligned}
S'_t &= \sum_{m=1}^{n}\sum_{i=1}^{I}\{E^Q[\gamma_i \exp(-\int_t^{T_m} r(s)ds)1_{\{T_{m-1}<\tau_i\le T_m\}}1_{\{\tau_i\le\tau_1,\ldots,\tau_i\le\tau_I\}}|\mathcal{F}_t] \\
&\quad + E^Q[\gamma_0 \exp(-\int_t^{T} r(s)ds)1_{\{T<\tau_1,\ldots,T<\tau_I\}}|\mathcal{F}_t]\} \\
&= \sum_{m=1}^{n}\sum_{i=1}^{I}\{E^Q[\gamma_i \exp(-\int_t^{T_m} r(s)ds)\int_t^{T_m} h_i(s)\exp(-\sum_{j=1}^{I}\int_t^{s} h_j(u)du)ds|\mathcal{G}_t] \\
&\quad - E^Q[\gamma_i \exp(-\int_t^{T_m} r(s)ds)\int_t^{T_{m-1}} h_i(s)\exp(-\sum_{j=1}^{I}\int_t^{s} h_j(u)du)ds|\mathcal{G}_t]\}1_{\{\tau_1,\ldots,\tau_I>t\}} \\
&\quad + \gamma_0 E^Q[\exp(-\int_t^{T}(r(s)+\sum_{i=1}^{I}h_i(s))ds)|\mathcal{G}_t]1_{\{\tau_1,\ldots,\tau_I>t\}} \ .
\end{aligned}$$

Note that Lemma 3 is used to derive this formula. On the other hand, the discounted expected pay off for the buyer of the swap is given by

$$c'B'_t = c'\Delta\sum_{m=1}^{n} w(t, T_m) \ .$$

Because the discounted pay off for the buyer and seller of a basket-type credit default swap must be the same, the premium is given by

$$BCDS(t) = \frac{S'_t}{B'_t} \ .$$

As a special case, consider the situation where $\gamma_1, \ldots, \gamma_I$ take the same quantity, γ, i.e., $\gamma_1, \ldots, \gamma_I \equiv \gamma$. The pricing formulas for the basket-type credit default swap premium become simpler. The new variable,

$$S_t^{5m} = E[\exp(-\int_t^{T_m} r(s) - \sum_{i=1}^{I}\int_t^{T_{m-1}} h_i(s)ds)|\mathcal{G}_t]1_{\{\tau_1,\ldots,\tau_I>t\}} \ ,$$

is introduced to evaluate a credit default swap of the basket type in this case. As defined by the symbol "tilde" in Section 2, the $\tilde{S}_t^5$ means that the indicator function is removed from S_t^5.

The expected discounted value of the cash flow received by the seller of a credit default swap of the basket type is given by

$$S'_t = \{\gamma_0 \tilde{w}(t,T) + \gamma \sum_{m=1}^{n} (\tilde{S}_t^{5m} - \tilde{w}(t,T_m))\} 1_{\{\tau_1,\ldots,\tau_I > t\}} \ .$$

It is possible to evaluate this value by using the PDE approach, if the stochastic process $(r(t), h_1(t), \ldots, h_I(t))$ has the affine structure. This quantity can also be evaluated using the asymptotic expansion approach. The rational value of a forward credit default swap premium at time t $(0 \le t \le T_0)$ is given by

$$BCDS(t) = \frac{S'_t}{B'_t} = \frac{\gamma_0 \tilde{w}(t,T) + \gamma \sum_{m=1}^{n} (\tilde{S}_t^{5m} - \tilde{w}(t,T_m))}{\Delta \sum_{m=1}^{n} \tilde{w}(t,T_m)} 1_{\{\tau_1,\ldots,\tau_I > t\}} \ .$$

The valuation methods of swaption on a credit default swap of the basket type with maturity date T_0 and strike price K are also discussed in this section. The random variable C' is introduced by

$$C' = \frac{\tilde{S}'_{T_0}}{\tilde{B}'_{T_0}} = \frac{\gamma_0 \tilde{w}(T_0,T) + \gamma \sum_{m=1}^{n} (\tilde{S}_{T_0}^{5m} - \tilde{w}(T_0,T_m))}{\Delta \sum_{m=1}^{n} \tilde{w}(T_0,T_m)} \ .$$

The pay-off function for swaptions at the expiration date T_0 is given by

$$(\gamma_0 w(t,T) + \gamma \sum_{m=1}^{n} (S_t^{5m} - w(t,T_m)) - K\Delta \sum_{m=1}^{n} w(t,T_m)) 1_{\{C' > K\}} \ .$$

The price of swaptions is evaluated using the asymptotic expansion approach employed in this article, and the numerical method is discussed in the next section. The method is almost the same as the pricing of swaptions on credit default swaps with counterparty credit risks. The numerical results are provided in the next section. It is also possible to derive the price of a swaption on the second-to-default swaps introduced by Kijima and Muromachi (2000). However, the pricing formulas may be more complicated and this article does not consider this sort of problem.

5 Numerical Results

In previous sections, the pricing formulas of credit derivatives, such as credit default swaps with counterparty credit risks, have been obtained. These prices have been evaluated with the asymptotic expansion method proposed in this article. The numerical results in this section are calculated using four kinds of models, the Gaussian model, the CIR model, the CIR interest rate model and the CEV (Constant Elasticity of Variance) interest rate model. These models are special cases of (3) and (4). The prices of the credit default swap and swaption are obtained with the asymptotic expansion method. Monte Carlo methods are also examined for purpose of comparison.

(i) Gaussian Model :

The short interest rate model is represented by

$$dr^{\epsilon}(t) = \alpha(\bar{r} - r^{\epsilon}(t))dt + \epsilon\sigma dW_t^0,\ r^{\epsilon}(0) = r$$

where $\alpha = 0.2, \bar{r} = 0.05, \sigma = 1.5$ and $r = 0.05$. The small parameter ϵ is given by $\epsilon = 0.01$. The default time for firm i $(i = 1, 2, 3)$ is denoted by τ_i. The hazard rate process for this stopping time is given by

$$dh_i^{\epsilon}(t) = \beta_i(\bar{h}_i - h_i^{\epsilon}(t))dt + \epsilon\sum_{j=0}^{3}\sigma_{ij}dW_t^j,\ h_i^{\epsilon}(0) = h_i$$

where a stochastic process (W^0, W^1, W^2, W^3) is a four-dimensional standard Wiener process and the parameters in these SDEs are given by $\beta_1 = 0.1, \beta_2 = 0.1, \beta_3 = 0.2, \bar{h}_1 = 0.04, \bar{h}_2 = 0.04, \bar{h}_3 = 0.03, h_1 = 0.04, h_2 = 0.03$ and $h_3 = 0.04$. The parameters in diffusion terms are $\sigma_{10} = -0.4, \sigma_{20} = -0.6, \sigma_{30} = -0.5, \sigma_{11} = 1.0, \sigma_{21} = 0.5, \sigma_{31} = -0.4, \sigma_{32} = 0.4, \sigma_{22} = 1.5$ and $\sigma_{33} = 1.0$. Other parameters are fixed at 0.

(ii) CIR model:

The spot interest rate and the hazard rate processes are given by the CIR model. In this model, the spot interest rate process and the hazard rate processes are assumed to be independent of each other. The spot interest rate is given by

$$dr^{\epsilon}(t) = \alpha(\bar{r} - r^{\epsilon}(t))dt + \epsilon\sigma\sqrt{r^{\epsilon}(t)}dW_t^0,\ r^{\epsilon}(0) = r$$

where $\alpha = 0.2, \bar{r} = 0.05, \sigma = 7.5, r = 0.05$.

The default hazard rate process for firm i $(i = 1, 2, 3)$ is governed by the SDE

$$dh_i^{\epsilon}(t) = \beta_i(\bar{h}_i - h_i^{\epsilon}(t))dt + \epsilon\sigma_{ii}\sqrt{h_i^{\epsilon}(t)}dW_t^i,\ h_i^{\epsilon}(0) = h_i$$

where $\beta_1 = 0.1, \beta_2 = 0.1, \beta_3 = 0.2, \bar{h}_1 = 0.04, \bar{h}_2 = 0.04, \bar{h}_3 = 0.03$. The hazard rates at the initial time are given by $h_1 = 0.04, h_2 = 0.03$ and $h_3 = 0.04$. The parameters in the diffusion terms are $\sigma_{11} = 5.0, \sigma_{22} = 7.5$ and $\sigma_{33} = 5.0$.

(iii) CIR interest rate model:

The spot interest rate is given by

$$dr^{\epsilon}(t) = \alpha(\bar{r} - r^{\epsilon}(t))dt + \epsilon\sigma\sqrt{r^{\epsilon}(t)}dW_t^0,\ r^{\epsilon}(0) = r$$

where the parameters are the same as those of the CIR model. The hazard rate process for the default time of firm i $(i = 1, 2, 3)$ is

$$dh_i^{\epsilon}(t) = \beta_i(\bar{h}_i - h_i^{\epsilon}(t))dt + \epsilon\sigma_{i0}\sqrt{r^{\epsilon}(t)}dW_t^0 + \epsilon\sum_{j=1}^{3}\sigma_{ij}dW_t^j,\ h_i^{\epsilon}(0) = h_i\ .$$

The parameters for hazard rate processes are given by $\sigma = 7.5, \sigma_{10} = -2.0, \sigma_{20} = -3.0$ and $\sigma_{30} = -2.5$. Other parameters are the same as the Gaussian model.

(iv) CEV interest rate models:

The spot interest rate is given by

$$dr^{\epsilon}(t) = \alpha(\bar{r} - r^{\epsilon}(t))dt + \epsilon\sigma(r^{\epsilon}(t))^{\xi}dW_t^0,\ r^{\epsilon}(0) = r$$

and the hazard rate process for firm i ($i = 1, 2, 3$) is

$$dh_i^{\epsilon}(t) = \beta_i(\bar{h}_i - h_i^{\epsilon}(t))dt + \epsilon\sigma_{i0}(r^{\epsilon}(t))^{\xi}dW_t^0 + \epsilon\sum_{j=1}^{3}\sigma_{ij}dW_t^j,\ h_i^{\epsilon}(0) = h_i\ .$$

The parameters in this model are the same as the CIR interest rate model, except that ξ is assumed to be 0.7. It should be noted that a special feature of this model is that the closed form analytic formulas for the prices of bonds and swaps are not available, because they are not included in the affine model.

The models for the pricing of forward credit default swaps are constructed. The recovery rate for defaultable bonds is fixed at $\delta = 0.5$. The start time of the forward credit default swap and the maturity date of swaption are $T_0 = 1$ (year). The swap buyer must pay premiums every half year, in other words $\Delta = 0.5$. The time to maturity of the swaption is 3 years or 5 years at the expiry date, i.e., the maturity date of the credit default swap is $T_6 = 4$ (years) or $T_{10} = 6$ (years). Accordingly, the swap buyer must pay the premium six times and ten times, respectively. The maturity date of the defaultable bonds is fixed at $T = T_m$. In other words, the maturity date of defaultable bonds is either four years or six years.

Table 1 presents the prices of credit default swaps with counterparty credit risks, and Table 3 presents the price of credit default swaps of the basket type. The prices of credit default swaps in the CEV interest rate models under the heading "Analytic" in Tables 1 and 3 are not calculated, because the CEV model is not included in the affine models and the pricing formula for credit default swaps in this model is not easy to obtain. These tables reveal that the asymptotic expansion approach enables us to evaluate credit default swaps very accurately. The figures that appear under the heading "Analytic" in Tables 1 and 3 with the Gaussian model and the CIR interest rate model are calculated by solving the PDE method. Because these models are affine models, pricing formulas are obtained easily and accurately by solving the Riccati equations with the Runge–Kutta method. Three firm models are also constructed to evaluate basket-credit default swaps. The spot interest rate process and the hazard rate processes for the three firms are the same as those of the models for credit default swaps with counterparty credit risks. The recovery rates of credit default swaps of the basket type are given by $\gamma_0 = 0$ and $\gamma = 0.5$, respectively.

The prices of credit default swaptions are also calculated numerically using the asymptotic expansion approach. For comparative purposes, the prices of these securities are obtained by Monte Carlo methods with the number of simulations $N = 1,000,000$ and the number of time steps $M = 5,000$ for one year. In order to evaluate swaptions using Monte Carlo methods, the pay-off function must be represented as a function of the spot interest rate $r^{\epsilon}(t)$ and the hazard rate for i-th firm (i=1,2,3), $h_i^{\epsilon}(t)$. The pay-off function

Table 1. Premium for CDS

Start after 1 year, maturity 4 years

	Analytic	Asymptotic
Gaussian	0.021685	0.021685
CIR model	0.021711	0.021710
CIR interest	0.021675	0.021675
CEV interest		0.021711

Start after 1 year, maturity 6 years

	Analytic	Asymptotic
Gaussian	0.022615	0.022616
CIR model	0.022680	0.022677
CIR interest	0.022593	0.022592
CEV interest		0.022678

"Asymptotic" stands for the asymptotic expansion methods.
"CIR interest" stands for the CIR interest rate model.
"CEV interest" stands for the CIR interest rate model.

Table 2. Price of swaptions

Maturity 1 year, CDS maturity 4 years

	Asymptotic	M.C.
Gaussian	0.003563	0.003572
CIR model	0.003339	0.003331
CIR interest	0.003606	0.003619

Maturity 1 year, CDS maturity 6 years

	Asymptotic	M.C.
Gaussian	0.006369	0.006394
CIR	0.006092	0.006070
CIR interest	0.0064	0.006441

"M.C." stands for the Monte Carlo methods.

Table 3. Premium for Basket CDS

Start after 1 year, maturity 4 years

	Analytic	Asymptotic
Gaussian	0.055017	0.055018
CIR	0.055252	0.055248
CIR interest	0.055024	0.055025
CEV interest		0.054999

Start after 1 year, maturity 6 years

	Analytic	Asymptotic
Gaussian	0.054432	0.054438
CIR	0.054853	0.054843
CIR interest	0.054436	0.054443
CEV interest		0.054426

Table 4. Price of swaptions on Basket CDS

Maturity 1 year, CDS maturity 4 years

	Asymptotic	M.C.
Gaussian	0.009360	0.009382
CIR	0.006533	0.006508
CIR interest	0.009683	0.009687

Maturity 1 year, CDS maturity 6 years

	Asymptotic	M.C.
Gaussian	0.011665	0.011694
CIR	0.008183	0.008096
CIR interest	0.012078	0.012080

Table 5. Price of swaptions with different maturity (CIR interest rate)

Maturity (years)	1	2	3	4	5
Asymptotic	0.0064	0.006897	0.006624	0.006058	0.005393
M.C.	0.006435	0.006953	0.006708	0.006168	0.005529

Figure 1: Price of CDS as a function of hazard rates for firms 2 and 3 (CIR model).

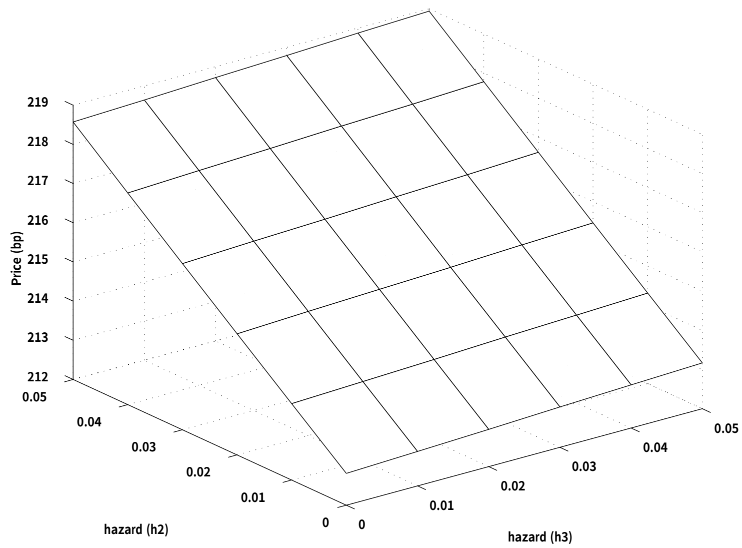


Figure 2: Price of swaptions as a function of hazard rates for firms 2 and 3 (CIR model).

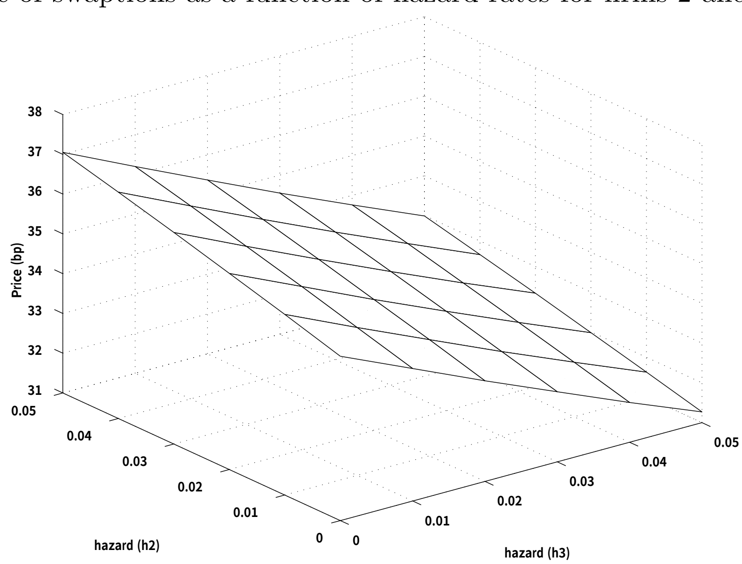
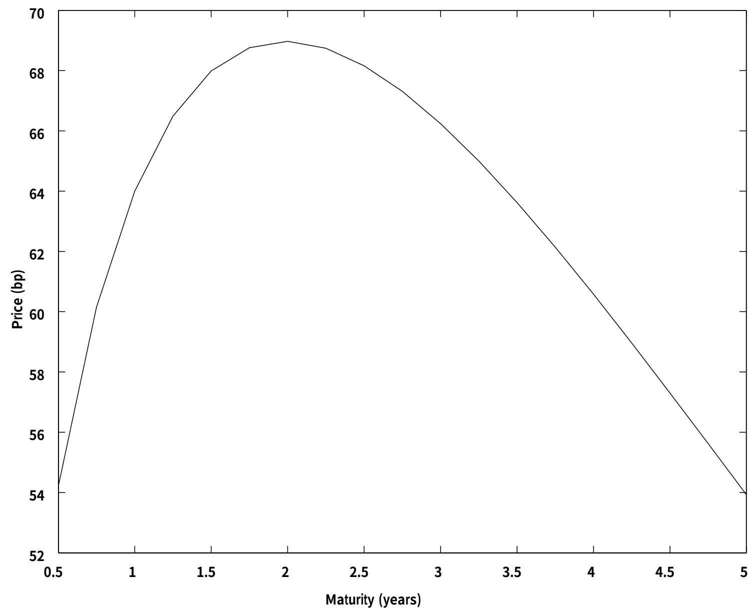


Figure 3: Price of swaptions as a function of the maturity date (CIR interest rate model).



for swaptions is obtained by solving PDEs. These are solved easily and accurately if the model has the affine structure. The strike price of credit default swaptions is fixed at 0.022 and the numerical results are shown in Table 2. The numerical results for credit default swaptions of the basket type are shown in Table 4. The strike price for basket swaptions is given by 0.055. Table 5 demonstrates the prices of swaptions with the different maturities in the "(iii) CIR interest rate model" with the time to maturity as five years, i.e., the maturity date of swaps is given by $T_{10} = T_0 + 5$. The number of time steps for one year for Monte Carlo studies is changed to $M = 2,000$ to reduce the computational time. This table reveals that the asymptotic expansion method is very accurate, if the time to maturity of swaption is not long. It appears that the pricing error becomes larger as the time to the maturity date of the swaption becomes longer.

Figures 1 and 2 present the prices of the credit default swap and the credit default swaption as a function of the hazard rates of firms 2 and 3, respectively. The model used in these figures is the CIR model. The parameters, however, are the same as the "(ii) CIR model" except h_i and $\bar{h}_i$ $(i = 2, 3)$. The parameters h_i and $\bar{h}_i$ are fixed at the same level, i.e., $h_i = \bar{h}_i$ $(i = 2, 3)$, and the initial default hazard rates h_2 and h_3 are moving from 0.0 to 0.05, respectively. Figure 1 shows that the swap premium is sensitive to the hazard rate for firm 2, but firm 3 . This is because the premiums are paid at the last second in each time interval, $[T_i, T_{i+1}]$. Figure 3 presents the prices of the swaption as a function of the maturity date.

The correlation effect between the price of credit default swaps of the basket type and their swaptions is also discussed. It is known that the correlation effect for the theoretical price of a basket swap is not strong in models based on conditional independence: see the numerical example in Kijima and Muromachi (2000). However, it is not known whether the correlation effect of basket swaptions is significant. In order to study the correlation effect, two models, both Gaussian, are considered. The parameters used in each model are

Table 6. Correlation effect of basket derivatives

	Basket Swaps	Basket Swaptions
Correlated	0.055567	0.001352
Uncorrelated	0.054351	0.008137

the same as those used in the "(i) Gaussian model" except for the volatility coefficients. In the model without the correlation effect, the volatility coefficients are given by $\sigma_{00} = 0.015$, $\sigma_{11} = 0.01$, $\sigma_{22} = 0.02$, $\sigma_{33} = 0.01$. Other parameters are fixed at 0. In the model with the correlation effect, the volatility coefficient for the interest rate process is given by $\sigma_{00} = 0.015$. The volatility coefficients for the hazard rates for the three firms are given by $\sigma_{10} = (-2/7) * 0.01$, $\sigma_{11} = (5/7) * 0.01$, $\sigma_{12} = (2/7) * 0.02$, $\sigma_{13} = (4/7) * 0.01$. $\sigma_{20} = (-2/7)*0.02$, $\sigma_{11} = (4/7)*0.02$, $\sigma_{22} = (5/7)*0.02$, $\sigma_{13} = (2/7)*0.02$, $\sigma_{10} = (-2/7)*0.01$, $\sigma_{11} = (2/7) * 0.01$, $\sigma_{12} = (4/7) * 0.02$, $\sigma_{13} = (5/7) * 0.01$. The size of the volatility for the hazard rate of each firm is equal in the two models. The maturity date of the swaptions is $T_0 = 1$ (year) and the expiry dates of the swaps are $T_6 = 4$ (years). The numerical results are shown in Table 6. Table 6 reveals that the correlation effect of the price of the swaption on the basket swap is significant, although the correlation effect of the value of the basket swap is not very strong.

6 Concluding Remarks

This paper studies the pricing of credit derivatives with multiple default entities, such as credit default swaps and swaptions, in a model with counterparty credit risks. The model employed is based on the conditional independence framework. The prices of these credit derivatives are often intractable, and this article exploits the asymptotic expansion approach introduced by Kunitomo and Takahashi (2001). This method enables us to evaluate credit derivatives such as these more easily and systematically. The validity of the asymptotic expansion method is shown in the appendix. The pricing problems of credit default swaps of the basket type and their swaptions have also been considered.

As shown in this article, the asymptotic expansion approach is applicable for evaluating the prices of many kinds of credit derivatives. Accordingly, I believe that the asymptotic expansion approach will become one of the primary tools for evaluating credit derivatives.

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Appendix

The asymptotic expansion methods used in this article were first applied in the financial literature by Kunitomo and Takahashi (2001, 2003). Originally proposed by Watanabe (1987), they have also been applied to the area of mathematical statistics by Yoshida (1992). Asymptotic expansion methods are applicable to many kinds of models in mathematical finance, including the Black–Scholes option pricing models, Heath, Jarrow and Morton's interest rate model, the pricing of equity derivatives with stochastic interest rates, and volatility models. The approach is rigorously justified by Malliavin calculus. See Ikeda and Watanabe (1989), Nualart (1995), Øksendal (1997) and Shigekawa (1997) for a standard textbook treatment of Malliavin calculus. The validity of the asymptotic expansion approach is discussed in this section.

Consider a Brownian motion defined on the canonical space $\Omega = C_0([0,T];\mathcal{R}^d)$ that consists of all continuous functions with a compact support defined on $[0,T]$. This space is a Banach space with the uniform norm, $||x||_\infty = \sup_{0\le t\le T}|x(t)|$. The subspace H^1 of $C_0([0,T];\mathcal{R}^d)$ that consists of all absolute continuous functions with a squared integrable density, i.e.,

$$x(t) = \int_0^t \frac{d}{ds}x(s)ds, \quad \frac{d}{dt}x(t) \in H = L^2([0,T]:\mathcal{R}^d)$$

is introduced. This space is often called "Cameron-Martin space". By equipping the inner product,

$$\langle x,y\rangle_{H^1} = \langle \frac{d}{dt}x, \frac{d}{dt}y\rangle_H = \sum_{i=1}^d \int_0^T \frac{d}{dt}x(t)\frac{d}{dt}y(t)dt \ ,$$

to the space H^1, the structure of Hilbert space H is transported to the space H^1. The relation,

$$||x||_\infty \le \int_0^T |\frac{d}{dt}x(t)|dt \le ||\frac{d}{dt}x(t)||_H \le ||x||_{H^1} \ ,$$

leads to the fact that the injection of H^1 into $C_0([0,T];\mathcal{R}^d)$ is continuous. Let $h_1,\ldots,h_n$ and w be elements of H^1 and $C_0([0,T];\mathcal{R}^d)$, respectively. A random variable F is called a polynomial function if there exists a polynomial function f such that

$$F = f(W(h_1),\ldots,W(h_n)) \ ,$$

where $W(h)$ represents the stochastic integral $W(h) = \int_0^T h(t)dw_t$.

The function $k(t)$ is supposed to be an element of H^1. There exists $\gamma(t)$, which is an element of H with a form $k(t) = \int_0^t \gamma(s)ds$. The directional derivatives for F with a direction $k \in H^1$ is defined by

$$\langle DF,\gamma\rangle_H = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[f(W(h_1)+\epsilon\langle h_1,\gamma\rangle_H,\ldots,W(h_n)+\epsilon\langle h_n,\gamma\rangle_H) - f(W(h_1),\ldots W(h_n))] \ .$$

If a function $\psi(t,\omega)$ exists and the relation

$$\langle DF,\gamma\rangle_H = \int_0^t \psi(s,\omega)\gamma(s)ds$$

is satisfied, the random variable $\psi(t,\omega)$ is termed the differential of F and is denoted by D_tF. D_tF is also represented by

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i \ .$$

Regard $\{D_tF\}$ as an element of stochastic processes, and this element is denoted by DF, i.e., DF is defined by $\{\psi(\cdot, w)\}$.

Let F be an $\mathcal{F}_T$ measurable random variable such that $||F||^2_{L^2(\Omega)} = E^P[F^2] < \infty$. There exists some square-integrable deterministic symmetric function $\hat{f}_n$ defined on $[0,T]^n$, such that

$$F = \sum_{n=0}^{\infty} I_n(\hat{f}_n) \ ,$$

where $I_n(\cdot)$ stands for the n-fold stochastic integral,

$$I_n(f) = \int_0^T \cdots \int_0^{t_n} \int_0^{t_2} f(t_1, \ldots, t_n) dW_{t_1} \cdots dW_{t_n} \ .$$

This is shown in Proposition 1.2.1 in Nualart (1995) and this expansion is called the Wiener chaos expansion. The functional, $J_n(\cdot)$, is the orthogonal projection on the nth Wiener chaos, which means that there is a relation $J_n(F) = I_n(f_n)$ for each square integrable random variable, $F = \sum_{n=0}^{\infty} I_n(f_n)$. A norm $||F||_{p,s}$ is defined by

$$||F||_{p,s} = || \sum_{k=0}^{\infty} (1+k)^{s/2} J_k F||_p \ .$$

The functional space D_p^s represents the completion of the family of polynomial random variable $\mathcal{P}$ with the norm $|| \cdot ||_{p,s}$. The functional space D_q^{-s} is the dual of D_p^s, if the relation $1/p + 1/q = 1$ is satisfied. The family of smooth random variables D^{∞} and its dual $\tilde{D}^{-\infty}$ are defined by

$$D^{\infty} = \cap_{s>0} \cap_{1<p<+\infty} D_p^s \text{ and } \tilde{D}^{-\infty} = \cup_{s>0} \cap_{1<p<+\infty} D_p^{-s} \ .$$

Suppose that $F = (F^1, \ldots, F^m)$ is a random variable whose component belongs to the space D_p^1. The Malliavin covariance matrix $\sigma_{MC}(F)$ for the random variable is defined by

$$\sigma_{MC}(F) = \{\langle DF^i, DF^j \rangle_H\}_{i,j} = \{\int_0^T \psi_i(t,\cdot)\psi_j(t,\cdot)dt\}_{i,j} \ .$$

The Malliavin's covariance matrix is symmetric nonnegative definite. Note that the Malliavin's covariance matrix is a random variable, because $\psi(t,\cdot)$ is a random variable.

Let the random variable $X^{\epsilon}(\omega)$ stand for the Wiener functional with a small parameter, ϵ. If the relation

$$\limsup_{\epsilon \to 0} \frac{||X^{\epsilon}||_{p,s}}{\epsilon^k} < +\infty$$

is satisfied, $X^{\epsilon}(\omega)$ is denoted by $X^{\epsilon}(\omega) = O(\epsilon^k)$ in D_p^s.

The random variable $X^\epsilon(\omega)$ has an asymptotic expansion, if for all $p > 1, s > 0$ and every $k = 1, 2, \ldots$,

$$X^\epsilon(\omega) - (g_1 + \epsilon g_2 + \cdots + \epsilon^{k-1} g_k) = O(\epsilon^k)$$

is included in D_p^s as $\epsilon \to 0$. Then the random variable

$$X^\epsilon(\omega) \sim g_1 + \epsilon g_2 + \cdots$$

is included in D^∞ where $g_1, g_2, \ldots \in D^\infty$. For each $k = 1, 2, \ldots$, there exists $s > 0$ such that for all $p > 1$, the random variable,

$$X^\epsilon(\omega) - (g_1 + \epsilon g_2 + \cdots + \epsilon^{k-1} g_k) = O(\epsilon^k) \ ,$$

is included in D_p^{-s}. Where $g_1, g_2, \ldots \in D_p^{-s}$ as $\epsilon \to 0$, the random variable $X^\epsilon(\omega) \in \tilde{D}^{-\infty}$ is said to have an asymptotic expansion in $\tilde{D}^{-\infty}$ and is written as

$$X^\epsilon(\omega) \sim g_1 + \epsilon g_2 + \cdots$$

and $g_1, g_2, \ldots \in \tilde{D}^{-\infty}$.

A new function $\psi(y)$, which is used in Theorem 2, is introduced. This is a smooth function and the relation $0 \leq \psi(y) \leq 1$ is satisfied for any real variable, y. The function $\psi(y)$ satisfies $\psi(y) = 0$ for $y > 1$, $\psi(y) = 1$ for $y < 1/2$. $\phi^\epsilon(y)$ is another smooth function and the random variable η^ϵ is supposed to be included in D^∞. The condition that the composite functional $\psi(\eta^\epsilon)\phi^\epsilon(X^\epsilon)I_{\mathcal{B}}(X^\epsilon) \in \tilde{D}^{-\infty}$ has an asymptotic expansion in $\tilde{D}^{-\infty}$ is given in Yoshida (1992). These are summarized in Theorem 2.

Theorem 2 *Let the following six conditions be satisfied:*
(i)$\{X^\epsilon(\omega); \epsilon \in (0,1]\} \in D^\infty$.
(ii)*The random variable $X^\epsilon(\omega)$ has the asymptotic expansion $X^\epsilon(\omega) \sim g_1 + \epsilon g_2 + \cdots$ in D^∞ with $g_1, g_2, \ldots \in D^\infty$ as $\epsilon \to 0$.*
(iii) *The random variable $\{\eta^\epsilon(\omega)\}$ is included in D^∞ and it is $O(1)$ in D^∞ as $\epsilon \to 0$.*
(iv)*For any $p > 1$,*

$$\sup_{\epsilon \in (0,1]} E[1_{\{|\eta^\epsilon| \leq 1\}}(\det[\sigma_{MC}(X^\epsilon)])^{-p}] < \infty \ . \tag{6}$$

(v)*For any $k \geq 1$,*

$$\lim_{\epsilon \to 0} \epsilon^{-k} P[|\eta^\epsilon| > \frac{1}{2}] = 0 \ .$$

(vi) *The function $\phi^\epsilon(x)$ in (x, ϵ) on $\mathcal{R}^n \times (0,1]$ is a smooth function. The derivative of this function is polynomial growth order in x uniformly in ϵ.*

Under these six conditions, the random variable $\psi(\eta^\epsilon)\phi^\epsilon(X^\epsilon)I_{\mathcal{B}}(X^\epsilon)$ has an asymptotic expansion $\psi(\eta^\epsilon)\phi^\epsilon(X^\epsilon)I_{\mathcal{B}}(X^\epsilon) \sim \Phi_0 + \epsilon\Phi_1 + \cdots$ in $\tilde{D}^{-\infty}$ as $\epsilon \to 0$. The formal Taylor expansion leads the coefficients of an asymptotic expansion, $\Phi_0, \Phi_1, \ldots$.

As shown in Theorem 2 in Yoshida (1992), the integration-by-parts formula is exploited to prove the validity of the asymptotic expansion approach. The condition to use the integration-by-parts formula is given by (6). In order to use the integration-by-parts formula, the inverse matrix of the Malliavin's covariance matrix must be derived. This is discussed in Section 5 in Ikeda and Watanabe (1989). (6) is a condition to ensure the existence of the inverse matrix of Malliavin's covariance matrix. Under the six conditions introduced in Theorem 2, the asymptotic expansion,

$$\limsup_{\epsilon\to 0} |\frac{1}{\epsilon^k} E[\psi(\eta^\epsilon)\phi^\epsilon(X^\epsilon)I_{\mathcal{B}}(X^\epsilon) - (\Phi_0 + \epsilon\Phi_1 + \cdots + \epsilon^k\Phi_k)]| < \infty ,$$

is satisfied.

Let the stochastic interest rate and the hazard rate processes be governed by a four-dimensional SDE,

$$z^\epsilon(t) = x_0 + \int_0^t \mu(z^\epsilon(s), s)ds + \sum_{j=0}^{J} \epsilon \int_0^t \sigma(z^\epsilon(s), s)dW_s \tag{7}$$

where $z^\epsilon(t) = {}^t(z_0^\epsilon(t), \cdots, z_3^\epsilon(t)) = {}^t(r^\epsilon(t), h_1^\epsilon(t), h_2^\epsilon(t), h_3^\epsilon(t))$. The drift term and the diffusion term, μ and σ , are given by $\mu : \mathcal{R}^4 \to \mathcal{R}^4$ and $\sigma : \mathcal{R}^4 \to \mathcal{R}^4 \otimes \mathcal{R}^4$ and the stochastic process $W_t = (w_t^0, w_t^1, w_t^2, w_t^3)$ is a four-dimensional standard Brownian motion. These functions satisfy the conditions of boundedness,

$$\sup_{z\in R^4} |\frac{\partial^k \mu_i(z,s)}{\partial z_0^{k_0}\cdots\partial z_3^{k_3}}| < M_1(k), \quad \sup_{z\in R^4} |\frac{\partial^k \sigma_{ij}(z,s)}{\partial z_0^{k_0}\cdots\partial z_3^{k_3}}| < M_2(k) \tag{8}$$

and

$$\sup_{0\le s\le T} [|\mu(0,s)| + |\sigma(0,s)|] < M_3 \tag{9}$$

where $k = k_0 + \cdots + k_3$. Malliavin calculus is a theory of the differential of the functional defined on the space of the paths of the Brownian motions and it is necessary to possess a strong solution for the original SDE to apply this theory to the solution of the SDE. Under conditions (8) and (9), SDE (7) has a strong solution and the next theorem follows.

Theorem 3 *Under assumptions (8) and (9), the random variable $z^\epsilon(T) \in D^\infty$ has an asymptotic expansion*

$$z^\epsilon(T) \sim z^0(T) + \epsilon g_1(T) + \epsilon^2 g_2(T) + \cdots$$

as $\epsilon \to 0$ where $g_1(T), g_2(T), \ldots \in D^\infty(\mathcal{R}^d)$.

This theorem is given, for example, in Theorem 3.1 in Kunitomo and Takahashi (2003).

The Malliavin's covariance matrix has to be calculated to ensure condition (iv) in Theorem 2. The Malliavin's covariance matrix for SDE (7) is given by

$$\sigma_{MC}^{ij}((z^\epsilon(T))) = \{\sum_{k=0}^{3} \int_0^T [Y^\epsilon(T)Y^\epsilon(s)^{-1}\sigma(z^\epsilon(s),s)]^{ik}[Y^\epsilon(T)Y^\epsilon(s)^{-1}\sigma(z^\epsilon(s),s)]^{jk}ds\}_{i,j} \tag{10}$$

where the stochastic process $Y^\epsilon(t)$ is a solution of the SDE

$$dY_{il}^\epsilon(t) = \sum_{k=0}^{3} \partial_k\mu_i(z^\epsilon(t),t)Y_{kl}^\epsilon(t)dt + \epsilon\sum_{j=0}^{3}\sum_{k=0}^{3} \partial_k\sigma_{ij}(z^\epsilon(t),t)Y_{kl}^\epsilon(t)dw_t^j .$$

The next condition is imposed to ensure the assumptions in Theorem 2.

Condition 1 *For any $T > 0$, an $n \times n$ matrix Σ_T is defined by*

$$\Sigma_T = \sum_{k=0}^{3} \int_0^T [Y(T)Y(s)^{-1}\sigma(z^0(s), s)]^{ik}[Y(T)Y(s)^{-1}\sigma(z^0(s), s)]^{jk} ds \ .$$

where $Y(t) = Y^0(t)$. This matrix is assumed to be nondegenerate.

For an arbitrary positive real number c, the random variable η_c^ϵ is defined by

$$\eta_c^\epsilon = c \int_0^T |Y^\epsilon(T)(Y^\epsilon(s))^{-1}\sigma(z^\epsilon(s), s) - Y(T)(Y(s))^{-1}\sigma(z^0(s), s)|^2 ds \ .$$

These preparations enable us to state the next results about the validity of the asymptotic expansion approach.

Theorem 4 *Assume that Condition 1, (8) and (9) are satisfied. The conditions $\eta^\epsilon \in D^\infty$, the smoothness of $\psi(y)$ with $0 \leq \psi(y) \leq 1$, $\psi(y) = 0$ ($|y| > 1$) and $\psi(y) = 1$ ($|y| < 1/2$) are satisfied. Let the function $\phi^\epsilon(y)$ be a smooth function for a smooth function in (x, ϵ) with all derivatives of polynomial growth order in x uniformly in ϵ. The function $\psi(\eta_c^\epsilon)\phi(z^\epsilon(T))I_{\mathcal{B}}(z^\epsilon(T))$ has an asymptotic expansion*

$$\psi(\eta_c^\epsilon)\phi(z^\epsilon(T))I_{\mathcal{B}}(z^\epsilon(T)) \sim \Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots$$

in $D^{-\infty}$ as $\epsilon \to 0$, where $\mathcal{B}$ is a Borel set. The coefficients, $\Phi_0, \Phi_1, \Phi_2, \ldots$ are given by a formal Taylor expansion of X_T^ϵ.

This theorem leads to the asymptotic expansion,

$$\begin{aligned} E[\phi(z^\epsilon(T))1_{\mathcal{B}}(z^\epsilon(T))] &\sim E[\psi(\eta_c^\epsilon)\phi(z^\epsilon(T))I_{\mathcal{B}}(z^\epsilon(T))] \\ &\sim E[\Phi_0] + \epsilon E[\Phi_1] + \epsilon^2 E[\Phi_2] + \cdots \ . \end{aligned}$$