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Abstract

In this paper, we develop easily implemented approximations of the prices of several interest rate derivatives. We study swaptions, constant maturity swaps ("CMS"), and CMS options. For swaption prices, we approximate swaption prices under one forward measure by using a Gram-Charlier expansion. This expansion is an orthogonal decomposition of a density function in additive form and involves bond moments in the coefficients. Hence, the swaptions price can be approximated easily and accurately. Higher-order approximations yield very accurate prices enough to price each transaction, and lower-order approximations are suitable for portfolio evaluation and risk management. In addition, we approximate CMS rates by using bond moments. We also approximate prices of CMS options by combining the two methods.

Keywords: Gram-Charlier expansion, bond moment, swaption, constant maturity swap, convexity adjustment

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Contents

I. Introduction

Many financial institutions hold large portfolios of interest rate derivatives transactions. In many cases, the maturity of these transactions is long and the number of contracts in a portfolio will not decrease in a short time. Therefore, an efficient calculation method is needed, not only for pricing specific transaction, but also for evaluation and risk management of the portfolio. An efficient calculation method has two features. (1) it is not computer-intensive for the valuation of a portfolio. (2) it can handle many types of products, including swaps, caps, and constant maturity swaps ("CMS"), within the same model to avoid inconsistent valuations among products. In this paper, we develop easily implemented approximations of the prices of several interest rate derivatives. We study swaptions, CMS and CMS options.

Jamshidian (1989) derived an exact analytical pricing formula for options on coupon bonds with a one-factor model. However, except for the forward swap measure approach of Jamshidian (1997), a closed formula for swaptions has not been obtained in multi-factor models due to the difficulty of identifying the exercise boundary with respect to the underlying factors. Brace (1997) proposed a rank-r approximation method for swaption prices in the LIBOR market model. Singleton and Umantsev (2002) linearized the exercise boundary (or the corresponding swap rate) with respect to the state variables under essentially affine term structure models. An innovative approximation method for swaptions was proposed by Collin-Dufresne and Goldstein (2002) (hereafter, "CDG"). They developed an approximation method under Gaussian and CIR (Cox-Ingersoll-Ross, [1985]) models by approximating the density function. For both models, their approach produces accurate approximation and fast calculation compared with Monte Carlo simulations.

For swaption prices, we simplify and complement the approach of CDG to obtain a swaption pricing formula that can be easily implemented for particular types of term structure model. We use a Gram–Charlier expansion of the density function of the underlying swap at the swaption expiry under the forward measure associated with the expiry. The coefficients of the series are expressed by the cumulants or, equivalently, by the moments. This approach works, as CDG explained, when the moments are calculated analytically, as in, e.g., affine term structure models and Gaussian Heath–Jarrow–Morton models (see, e.g., Musiela and Rutkowski [2005]). Our formula is closely related to several results based on the Malliavin calculus and the Fourier transform. This is because the Gram–Charlier expansion is obtained by using a Fourier inversion of the characteristic function. It is also a version of the Wiener Chaos expansion. Note that we approximate the density function of a swap value that takes both positive and negative values. Hence, it is not subject to the criticism that the distribution of a price that takes only positive values cannot be approximated by normal distributions.

Following CDG, we expand the density function of the underlying swap. However, whereas CDG carried out their calculations under many forward measures associated with the swaption expiry and cash-flow timing, we use only one forward measure associated with the swaption expiry. Thus, our formula is easier to implement and more accurate. Our numerical study of affine term structure models confirms that our method yields better approximations than that of CDG. We conclude that higher-order approximations are sufficiently accurate for pricing specific transactions, while lower-order approximations are speedily obtained and suitable for portfolio evaluation and risk management.

However, the error depends on the model parameters, such as the level of the yield curve or the mean reversion speed.

In addition, we approximate CMS rates by using moments of bonds ("bond moments"), and approximate the prices of CMS options by combining our two methods. Benhamou (2000) derived an approximation of the convexity adjustment of a CMS rate for lognormal zero coupon models by using a Wiener Chaos expansion. The literature on convexity adjustment is cited in Benhamou (2000). Calculating the convexity adjustment for a CMS rate is difficult because the swap value and duration, each of which is a linear combination of bond prices, are correlated. Hence, bond moments can be used to price the convexity adjustment. To approximate CMS option prices, we combine the two methods, the Gram–Charlier expansion and bond moments. Although there is a parallelism in the form of results between this paper and Benhamou (2000), our approach can be easily applied to wider classes of models.

The rest of this paper is organized as follows. In Section II, we develop an approximation method by using a Gram–Charlier expansion. We present formulae for swaptions, CMS, and option on CMS based on the cumulants of the underlying swap. In Section III, we describe affine term structure models and derive bond moments and Greeks. In Section IV, we perform numerical calculations for affine term structure models. Section V concludes the paper.

II. Approximating Interest Rate Derivative Price by using a Gram–Charlier Expansion

A. The valuation of a swaption

We denote by $P(t,T)$ the time-t price of a zero coupon bond with a maturity date of T. $(\Omega, \mathcal{F}, P)$ is a probability space with a J-dimensional standard Brownian motion W. We assume that tradable assets comprise zero coupon bonds and a money-market account, and that there is a risk-neutral measure, Q, with the Brownian motion W^Q.

We consider a swaption with the expiry T_0 and the fixed rate K (per payment and notional amount) during a period $[T_0, T_N]$. We fix the relevant dates, $T_0 < T_1 < \cdots < T_N$, which are set at regularly spaced time intervals, with $\delta = T_i - T_{i-1}$ for all i. The value $SV(t)$ of the underlying swap at time t is given by

$$
\begin{aligned}
SV(t) &= \begin{cases} -P(t,T_0) + \delta K \sum_{i=1}^{N} P(t,T_i) + P(t,T_N), & \text{for the receiver's swaption,} \\ P(t,T_0) - \delta K \sum_{i=1}^{N} P(t,T_i) - P(t,T_N), & \text{for the payer's swaption,} \end{cases} \\
&\equiv \sum_{i=0}^{N} a_i P(t,T_i), \qquad (1)
\end{aligned}
$$

where a_i is the amount of cash flow at time T_i. Following from the usual discussion about no-arbitrage and the change of measure, the swaption value, $SOV(t)$, at time t is first evaluated under the risk-neutral measure Q, and then converted to the expected value of the gain from exercising under the T_0-forward measure, P^{T_0}, as follows

$$\begin{aligned} SOV(t) &= E^Q\Big[e^{-\int_t^{T_0} r_s ds}\max(SV(T_0),0) \mid \mathcal{F}_t\Big] \\ &= P(t,T_0)E^{T_0}\Big[1_{\{SV(T_0)>0\}}SV(T_0) \mid \mathcal{F}_t\Big] \\ &= P(t,T_0)\int_0^\infty xf(x)dx, \end{aligned} \quad (2)$$

where f is the density function of the swap value SV at the expiry date T_0 under the T_0-forward measure conditioned by $\mathcal{F}_t$.

B. The Gram–Charlier expansion

Following Stuart and Ord (1987), we derive the Gram–Charlier expansion and show its relationship to the Edgeworth expansion. As shown below, the Edgeworth expansion is obtained by using the inverse Fourier transform of the characteristic function in a multiplicative form. The Gram–Charlier expansion is further expanded and reordered as an orthogonalized series of the Edgeworth expansion in additive form. The Gram–Charlier expansion is more useful for many practical purposes.

We define the Chebyshev–Hermite polynomial as $H_n(x) = (-1)^n\phi(x)^{-1}D^n\phi(x)$ with $H_0(x) = 1$, where $D = \frac{d}{dx}$ and $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$. [1] The Chebyshev–Hermite polynomials have the orthogonal property $\int_{-\infty}^{\infty} H_m(x)H_n(x)\phi(x)dx = \delta_{mn}n!$ with respect to the Gaussian measure, ν, which has a standard normal distribution, $N(0,1)$. As shown in equation (3) below, by using the properties of the Chebyshev–Hermite polynomials, the Gram–Charlier expansion is an orthogonal decomposition with $\{H_n\phi\}_n$ of a density function that has coefficients q_n, each of which depends on a given set of cumulants.

Proposition 1. *Assume that a random variable Y has the density function f and has cumulants c_k $(k \geq 1)$, all of which are finite and known. Then the followings hold.*
(i) f can be expanded as follows

$$f(x) = \sum_{n=0}^{\infty}\frac{q_n}{\sqrt{c_2}}H_n\Big(\frac{x-c_1}{\sqrt{c_2}}\Big)\phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big), \quad (3)$$

$$\begin{aligned} \textit{where}\quad q_n &= \frac{1}{n!}E\Big[H_n\Big(\frac{Y-c_1}{\sqrt{c_2}}\Big)\Big] \\ &= \begin{cases} 1, & \textit{if } n=0, \\ 0, & \textit{if } n=1,2, \\ \sum_{m=1}^{[n/3]}\sum^{**}\frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\Big(\frac{1}{\sqrt{c_2}}\Big)^n, & \textit{if } n\geq 3, \end{cases} \end{aligned} \quad (4)$$

$$\sum^{**} \quad \textit{means} \quad \sum_{k_1+\cdots+k_m=n,k_i\geq 3}.$$

[1] We use H_n for the *Chebyshev–Hermite* polynomial, which should not be confused with the *Hermite* polynomial, $\tilde{H}_n(x)$, defined by $\tilde{H}_n(x) = (-1)^n e^{x^2}D^n e^{-x^2} = 2^{n/2}H_n(\sqrt{2}x)$. By definition,

$$H_0(x)=1,\quad H_1(x)=x,\quad H_2(x)=x^2-1,\quad H_3(x)=x^3-3x,\quad H_4(x)=x^4-6x^2+3,$$
$$H_5(x)=x^5-10x^3+15x,\quad H_6(x)=x^6-15x^4+45x^2-15,\quad H_7(x)=x^7-21x^5+105x^3-105x.$$

(ii) for any $a \in \mathbb{R}$,

$$\begin{aligned} E[1_{\{Y>a\}}] &= N\big(\frac{c_1-a}{\sqrt{c_2}}\big)+\sum_{k=3}^{\infty}(-1)^{k-1}q_kH_{k-1}\big(\frac{c_1-a}{\sqrt{c_2}}\big)\phi\big(\frac{c_1-a}{\sqrt{c_2}}\big), \\ E[1_{\{Y>a\}}Y] &= c_1N\big(\frac{c_1-a}{\sqrt{c_2}}\big)+\sqrt{c_2}\phi\big(\frac{c_1-a}{\sqrt{c_2}}\big) \\ &\quad +\sum_{k=3}^{\infty}(-1)^kq_k\Big(-aH_{k-1}\big(\frac{c_1-a}{\sqrt{c_2}}\big)+\sqrt{c_2}H_{k-2}\big(\frac{c_1-a}{\sqrt{c_2}}\big)\Big)\phi\big(\frac{c_1-a}{\sqrt{c_2}}\big). \end{aligned}$$

Proof. The characteristic function G_Y of a random variable Y is defined by the Fourier transform of f as

$$G_Y(t)=\int_{-\infty}^{\infty}e^{itx}f(x)dx=e^{itc_1}\int_{-\infty}^{\infty}e^{i\sqrt{c_2}tx}\sqrt{c_2}f(c_1+\sqrt{c_2}x)dx. \tag{5}$$

On the other hand, by the definitions of the cumulants, this can be expressed as

$$G_Y(t)=\exp\Big[\sum_{k=1}^{\infty}\frac{c_k}{k!}(it)^k\Big]=e^{itc_1}\int_{-\infty}^{\infty}e^{i\sqrt{c_2}tx}\exp\Big[\sum_{k=3}^{\infty}\frac{(-1)^kc_k}{k!}\Big(\frac{D}{\sqrt{c_2}}\Big)^k\Big]\phi(x)dx. \tag{6}$$

This is because, for any sequence $\{a_n\}$, it follows that

$$\exp\Big(-\frac{c_2}{2}t^2+\sum_{n=1}^{\infty}a_n(-i\sqrt{c_2}t)^n\Big) = \int_{-\infty}^{\infty}e^{i\sqrt{c_2}tx}\exp\big(\sum_{n=1}^{\infty}a_nD^n\big)\phi(x)dx.$$

We further expand the integrand of equation (6) by using the Taylor expansion. We then reorder the terms as follows

$$\begin{aligned} &\exp\Big[\sum_{k=3}^{\infty}\frac{(-1)^kc_k}{k!}\Big(\frac{D}{\sqrt{c_2}}\Big)^k\Big]\phi(x) \\ =&\ \Big(1+\sum_{m=1}^{\infty}\frac{1}{m!}\Big[\sum_{k=3}^{\infty}\frac{(-1)^kc_k}{k!}\Big(\frac{D}{\sqrt{c_2}}\Big)^k\Big]^m\Big)\phi(x) \\ =&\ \Big(1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{k_1,\cdots,k_m\geq 3}\frac{(-1)^{k_1+\cdots+k_m}c_{k_1}\cdots c_{k_m}}{k_1!\cdots k_m!}\Big(\frac{D}{\sqrt{c_2}}\Big)^{k_1+\cdots+k_m}\Big)\phi(x) \\ =&\ \Big(1+\sum{}^{*}\frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\Big(\frac{1}{\sqrt{c_2}}\Big)^nH_n(x)\Big)\phi(x), \end{aligned}$$

where $\sum^*$means $\sum_{n=3}^{\infty}\sum_{m=1}^{[n/3]}\sum_{k_1+\cdots+k_m=n,k_i\geq 3}$. We use the relationship $H_n(x)\phi(x) = (-1)^nD^n\phi(x)$ in the last equality. Then, equation (6) can be written as

$$e^{itc_1}\int_{-\infty}^{\infty}e^{i\sqrt{c_2}tx}\phi(x)dx+e^{itc_1}\int_{-\infty}^{\infty}e^{i\sqrt{c_2}tx}\sum{}^{*}\frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\Big(\frac{1}{\sqrt{c_2}}\Big)^nH_n(x)\phi(x)dx. \tag{7}$$

By using the inverse Fourier transforms of both equations (5) and (7) and by changing the relevant variable, we obtain the following Gram–Charlier expansion around the mean

[2]:

$$f(x) = \frac{1}{\sqrt{c_2}}\phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big) + \frac{1}{\sqrt{c_2}}\sum{}^{*} \frac{c_{k_1}\cdots c_{k_m}}{m!k_1!\cdots k_m!}\Big(\frac{1}{\sqrt{c_2}}\Big)^n H_n\Big(\frac{x-c_1}{\sqrt{c_2}}\Big)\phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big).$$

The proof of (ii) is straightforward by using (i) and the properties of Chebyshev–Hermite polynomials. □

The Gram–Charlier expansion may be interpreted as the Wiener Chaos expansion of $\tilde{f}/\phi \in L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, where $\tilde{f}(x) = \sqrt{c_2}f(c_1 + \sqrt{c_2}x)$ is the density function of the standardized random variable $(Y - c_1)/\sqrt{c_2}$. The Wiener Chaos expansion states that the Chebyshev–Hermite polynomials form a complete orthonormal system in the Hilbert space, $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$ (see Nualart [1995], p.7). The advantages of the Gram–Charlier expansion are that it is written in additive form and the coefficients q_n are easily expressed by the given cumulants as follows [3]

$$q_0 = 1, \quad q_1 = q_2 = 0, \quad q_3 = \frac{c_3}{3!c_2^{3/2}}, \quad q_4 = \frac{c_4}{4!c_2^2},$$
$$q_5 = \frac{c_5}{5!c_2^{5/2}}, \quad q_6 = \frac{c_6 + 10c_3^2}{6!c_2^3}, \quad q_7 = \frac{c_7 + 35c_3c_4}{7!c_2^{7/2}}. \tag{8}$$

The cumulants, c_j, can be calculated from the moments, μ_j, around zero. [4]

In the proof of Proposition 1, the inverse Fourier transforms of both equations (5) and (6) yield the Edgeworth expansion

$$f(x) = \frac{1}{\sqrt{c_2}}\exp\Big[\sum_{k=3}^{\infty}\frac{(-1)^k c_k}{k!}D^k\Big]\phi\Big(\frac{x-c_1}{\sqrt{c_2}}\Big). \tag{9}$$

However, this multiplicative form is not useful for approximating option prices. Hence, we require an additive form. Both the Gram-Charlier and the Edgeworth expansions are equivalent (have the same value) when the summation is taken over infinite terms,

[2]For the density function of a standardized random variable, an expansion around zero $f(x) = \sum_{k=0}^{\infty} q_k H_k(x)\phi(x)$, where $q_k = \frac{1}{k!}E[H_k(Y)] = \sum_{l=0}^{[k/2]}\frac{(-1)^l}{l!(k-2l)!2^l}E[Y^{k-2l}]$, is known as a Gram–Charlier series of type A (Stuart and Ord [1987]).

[3]In this context, it is well known that $3!q_3$ represents skewness and $4!q_4$ represents the excess kurtosis.

[4]See Stuart and Ord (1987). For example,

$$\begin{aligned}
c_1 &= \mu_1, \quad c_2 = \mu_2 - \mu_1^2, \quad c_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3,\\
c_4 &= \mu_4 - 4\mu_1\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4,\\
c_5 &= \mu_5 - 5\mu_1\mu_4 - 10\mu_2\mu_3 + 20\mu_1^2\mu_3 + 30\mu_1\mu_2^2 - 60\mu_1^3\mu_2 + 24\mu_1^5,\\
c_6 &= \mu_6 - 6\mu_1\mu_5 - 15\mu_2\mu_4 + 30\mu_1^2\mu_4 - 10\mu_3^2 + 120\mu_1\mu_2\mu_3 - 120\mu_1^3\mu_3\\
&\quad + 30\mu_2^3 - 270\mu_1^2\mu_2^2 + 360\mu_1^4\mu_2 - 120\mu_1^6,\\
c_7 &= \mu_7 - 7\mu_1\mu_6 - 21\mu_2\mu_5 - 35\mu_3\mu_4 + 140\mu_1\mu_3^2 - 630\mu_1\mu_2^3 + 210\mu_1\mu_2\mu_4\\
&\quad - 1260\mu_1^2\mu_2\mu_3 + 42\mu_1^2\mu_5 + 2520\mu_1^3\mu_2^2 - 210\mu_1^3\mu_4 + 210\mu_2^2\mu_3 + 840\mu_1^4\mu_3\\
&\quad - 2520\mu_1^5\mu_2 + 720\mu_1^7.
\end{aligned}$$

but the truncated sum may lead to differences between them. However, in many practical applications, the finite sum is the same after a further approximation is made on the Edgeworth expansion. The approximation based on the Gram–Charlier expansion ignores the "orthogonalized moments", q_n, in the context of equation (8), whereas the approximation based on the Edgeworth expansion ignores the higher cumulants, c_n.

C. Swaption

Suppose that we know the j-th cumulant, c_j, of the underlying swap value at the expiry T_0 under the forward measure P^{T_0} that is associated with the option expiry with condition $\mathcal{F}_t$. We set the following: $C_j = c_j P(t,T_0)^j$ and $q_k = q_k(c_1,\ldots,c_k) = q_k(C_1,\ldots,C_k)$, which is calculated from equation (8). Then, the swaption value is obtained from equation (2) as

$$
\begin{aligned}
& SOV(t) \\
= \ & P(t,T_0)E^{T_0}\Big[1_{\{SV(T_0)>0\}}SV(T_0) \mid \mathcal{F}_t\Big] \\
= \ & P(t,T_0)\Big[c_1 N\big(\frac{c_1}{\sqrt{c_2}}\big) + \sqrt{c_2}\phi\big(\frac{c_1}{\sqrt{c_2}}\big) + \sqrt{c_2}\phi\big(\frac{c_1}{\sqrt{c_2}}\big)\sum_{k=3}^{\infty}(-1)^k q_k H_{k-2}\big(\frac{c_1}{\sqrt{c_2}}\big)\Big] \\
= \ & C_1 N\big(\frac{C_1}{\sqrt{C_2}}\big) + \sqrt{C_2}\phi\big(\frac{C_1}{\sqrt{C_2}}\big) + \sqrt{C_2}\phi\big(\frac{C_1}{\sqrt{C_2}}\big)\sum_{k=3}^{\infty}(-1)^k q_k H_{k-2}\big(\frac{C_1}{\sqrt{C_2}}\big). \qquad (10)
\end{aligned}
$$

In particular, the truncated sum of equation (10) yields an approximation of the swaption value.

Proposition 2. *The swaption value is approximated as*

$$
SOV(t) \approx C_1 N\big(\frac{C_1}{\sqrt{C_2}}\big) + \sqrt{C_2}\phi\big(\frac{C_1}{\sqrt{C_2}}\big) + \sqrt{C_2}\phi\big(\frac{C_1}{\sqrt{C_2}}\big)\sum_{k=3}^{L}(-1)^k q_k H_{k-2}\big(\frac{C_1}{\sqrt{C_2}}\big). \qquad (11)
$$

We refer to this expression as the L-th order approximated price, GCL.

Hence, the calculation of the swaption is reduced to the value of cumulants c_j of the underlying swap or the swap moments

$$
\begin{aligned}
M_m(t) &= E^{T_0}\Big[(SV(T_0))^m \mid \mathcal{F}_t\Big] \\
&= E^{T_0}\Big[\Big(\sum_{i=0}^{N} a_i P(T_0,T_i)\Big)^m \mid \mathcal{F}_t\Big] \\
&= \sum_{0\le i_1,\ldots,i_m\le N} a_{i_1}\cdots a_{i_m}\mu^{T_0}(t,T_0,\{T_{i_1},\ldots,T_{i_m}\}), \qquad (12)
\end{aligned}
$$

where T_0 is the expiry date of the swaption, $T_1,\ldots,T_m$ are the coupon payment dates and

$$
\mu^T(t,T_0,\{T_1,\ldots,T_m\}) \equiv E^T\Big[\prod_{i=1}^{m} P(T_0,T_i) \mid \mathcal{F}_t\Big] \qquad (13)
$$

is the bond moments under the T-forward measure with $T \geq T_0$. As shown subsequently, $\mu^{T_0}(t, T_0, \{T_1, \ldots, T_m\})$ can be calculated analytically for particular classes of interest rate models.

It is worth mentioning that other approaches based on the Edgeworth expansion can be used to reach the same conclusion as that implied by equation (11). For a given forward measure, the two expansions, the Gram–Charlier expansion and the Edgeworth expansion, are numerically equivalent and the truncated sums are numerically equivalent when an appropriate approximation is made.

The asymptotic expansion approach is developed by using Malliavin calculus (see, e.g., Kunitomo and Takahashi [2001]). Essentially, it used a third-order Edgeworth expansion and an approximation as

$$\begin{aligned} f(x) &\approx \frac{1}{\sqrt{c_2}} \exp\Big[\frac{-c_3}{3!} D^3\Big] \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \\ &\approx \frac{1}{\sqrt{c_2}} \Big[1 - \frac{c_3}{3!} D^3\Big] \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \\ &= \frac{1}{\sqrt{c_2}} \Big[1 + \frac{c_3}{3!(\sqrt{c_2})^3} H_3\Big(\frac{x - c_1}{\sqrt{c_2}}\Big)\Big] \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big). \end{aligned}$$

It seems that the numerical performance of Kunitomo and Takahashi (2001) is similar to ours. Kawai (2003) approximates swaptions by using an asymptotic expansion approach in the LIBOR market model. In existing studies, the swaption value is often decomposed into weighted cash-flow values based on the exercise probabilities under the forward measures associated with the cash-flow timing,

$$SOV(t) = \sum_{i=0}^{N} a_i P(t, T_i) E^{T_i}\Big[1_{\{SV(T_0) > 0\}} \mid \mathcal{F}_t\Big]. \tag{14}$$

When calculating the probability of ending up in-the-money under the forward measure, CDG used a seventh-order Edgeworth expansion. They ignored terms higher than D^7 in a further expanded series by a Taylor expansion of the exponential,

$$\begin{aligned} f(x) &\approx \frac{1}{\sqrt{c_2}} \exp\Big[\sum_{k=3}^{7} \frac{(-1)^k c_k}{k!} D^k\Big] \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \\ &\approx \frac{1}{\sqrt{c_2}} \Big(1 + \sum_{k=3}^{7} \frac{(-1)^k c_k}{k!} D^k + \frac{1}{2}\big((\frac{c_3}{3!} D^3)^2 - 2\frac{c_3}{3!}\frac{c_4}{4!} D^3 D^4\big)\Big) \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big) \\ &= \frac{1}{\sqrt{c_2}} \Big(1 + q_3 H_3 + q_4 H_4 + q_5 H_5 + q_6 H_6 + q_7 H_7\Big) \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big), \end{aligned}$$

where $H_k = H_k\Big(\frac{x - c_1}{\sqrt{c_2}}\Big)$. This approximation of the density is exactly the same as ours when $L = 7$. Furthermore, CDG found that c_6 and c_7 were negligible in equation (8) relative to c_3^2 and $c_3 c_4$, respectively. Thus, q_6 and q_7 are represented by low-degree cumulants, which reduces computational time.

The truncated Edgeworth expansion,

$$\frac{1}{\sqrt{c_2}} \exp\Big[\sum_{k=3}^{L} \frac{(-1)^k c_k}{k!} D^k\Big] \phi\Big(\frac{x - c_1}{\sqrt{c_2}}\Big)$$

makes full use of the properties of a finite set of cumulants, $c_1, \ldots, c_L$. CDG's recalculation of q_6 and q_7 with $c_6 = c_7 = 0$ may be regarded as a way of using as much information on c_3, c_4, and c_5 as possible. However, the truncated Gram–Charlier expansion,

$$\sum_{n=0}^{L} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x - c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x - c_1}{\sqrt{c_2}}\right),$$

does not fully reflect the properties of the cumulants $c_3, \ldots, c_L$ because of the truncation. This is a disadvantage of the orthogonal decomposition and may explain why a higher-order approximation does not necessarily yield a better approximation than a lower-order approximation.

Equations (2) and (14) are equivalent. However, in the applications, their computational efficiency and approximation errors differ. When the swaption value is expressed as equation (2), we work under the T_0-forward measure, P^{T_0}. On the other hand, CDG proposed taking the set of forward measures $\{P^{T_i}\}_i$ and working with equation (14). In this case, appropriate formulae must be used to calculate probability $E^{T_i}\left[1_{\{SV(T_0)>0\}} \mid \mathcal{F}_t\right]$ for each measure. Using equation (2) rather than equation (14) reduces the time taken to compute the moments because there are fewer underlying measures. In terms of approximation error, one would conjecture that equation (14) might accumulate the error of each expectation. The main difference between CDG's approach and ours is the choice of the measures. Thus, CDG's results may differ from ours. In the section on numerical examples, we compare the results of the two approaches.

D. Constant Maturity Swap (CMS)

In this subsection, we demonstrate the another usefulness of the bond moments in an approximation of the convexity adjustment of a CMS rate. A CMS is a swap contract between two parties to exchange a fixed rate and a floating rate, which has a reference rate that is a swap rate with a specified time to maturity. The fixed rate to be exchanged on a CMS is called the CMS rate. Similar products to CMS include 15-year floating coupon Japanese Government Bonds ("JGB") with a 10-year JGB coupon [5], and Long-Term Prime Rate swaps with a reference rate that is a coupon of a 5-year bank debenture.

We fix the relevant dates, $T_0 < T_1 < \ldots < T_n < \ldots < T_{n+m}$, which are set at regularly spaced time intervals with $\delta = T_i - T_{i-1}$ for all i. We consider a CMS to be traded at time $t < T_0$ for the exchange of a fixed rate $CMS(t)$ with the observed swap rates for a maturity of $\tau = m\delta$ in arrears during the period $[T_0, T_n]$. The fixing dates are T_i $(i = 0, \ldots, n-1)$ and the payment dates are T_{i+1}. By the usual discussion, the time-t value of the floating rate amount that is fixed at time T_i and settled at time T_{i+1} is written as

$$\delta P(t, T_{i+1}) E^{T_{i+1}}[S(T_i, T_i, \tau) \mid \mathcal{F}_t],$$

where $S(u, T_i, \tau)$ is the forward swap rate for the period $[T_i, T_i + \tau]$ at time u,

$$S(u, T_i, \tau) = \frac{P(u, T_i) - P(u, T_i + \tau)}{\delta \sum_{j=i+1}^{i+m} P(u, T_j)}.$$

[5] This kind of bond, whose coupon is linked to a yield or a coupon rate of Government bonds with a predetermined time to maturity, is called a constant maturity treasury ("CMT").

The fixed rate on the CMS is given by

$$CMS(t) = \frac{\sum_{i=1}^{n} P(t,T_{i+1}) E^{T_{i+1}}[S(T_i,T_i,\tau) \mid \mathcal{F}_t]}{\sum_{i=1}^{n} P(t,T_i)}. \tag{15}$$

(See Musiela and Rutkowski [2005].)

The expectation of the swap rate, $E^{T_{i+1}}[S(T_i,T_i,\tau) \mid \mathcal{F}_t]$, is close to the forward swap rate $S(t,T_i,\tau)$. They coincide if the swap rate is a martingale under the forward measure. Otherwise, a difference between them exists and it is called the convexity adjustment in a broad sense (bCA). [6] Hence, it is sufficient to consider the bCA of the single-period CMS rate; $n = 1$, and

$$\text{bCA} = E^{T_1}[S(T_0,T_0,\tau) \mid \mathcal{F}_t] - S(t,T_0,\tau).$$

Now let $t \le u \le T_0$. We consider a receiving swap with a coupon rate of $S(t,T_0,\tau)$ for a period of $[T_0,T_m]$. Recall that the time-u swap value is given by

$$SV(u) = \delta \sum_{j=1}^{m} P(u,T_j)(S(t,T_0,\tau) - S(u,T_0,\tau)).$$

We define the duration (or basis point value) of the swap $Dur(u)$ as follows

$$Dur(u) = \delta \sum_{j=1}^{m} P(u,T_j). \tag{16}$$

Given that

$$\begin{aligned} S(u,T_0,\tau) &= S(t,T_0,\tau) + \frac{\sum_{j=1}^{m} P(u,T_j)(S(u,T_0,\tau) - S(t,T_0,\tau))}{\sum_{j=1}^{m} P(u,T_j)} \\ &= S(t,T_0,\tau) - \frac{SV(u)}{Dur(u)}, \end{aligned} \tag{17}$$

by taking expectations of both sides, we have

$$\text{bCA} = -E^{T_1}[SV(T_0) Dur(T_0)^{-1} \mid \mathcal{F}_t]. \tag{18}$$

A major problem in evaluating the CMS is that no general analytical expression exists for the expectation of $SV(T_0)Dur(T_0)^{-1}$. We propose a simple approximation of this expectation based on bond moments. We make use of the fact that both the swap value and the duration are affine functions of bond prices. Let us denote by $D(t) = \delta \sum_{j=1}^{m} P(t,T_j)/P(t,T_0)$ the forward duration of the swap. This would be close to the conditional mean of a random variable, $Dur(T_0)$. We can approximate the stochastic duration by a first- or second-order deterministic duration as follows

$$\begin{aligned} Dur(T_0)^{-1} &= \frac{D(t)^{-1}}{1 + \frac{Dur(T_0) - D(t)}{D(t)}} \\ &\approx \begin{cases} \frac{1}{D(t)}\left(2 - \frac{Dur(T_0)}{D(t)}\right) & \text{(for the first-order approx.),} \\ \frac{1}{D(t)}\left(3 - 3\frac{Dur(T_0)}{D(t)} + \left(\frac{Dur(T_0)}{D(t)}\right)^2\right) & \text{(for the second-order approx.).} \end{cases} \end{aligned} \tag{19}$$

[6] bCA for the LIBOR is zero because a forward LIBOR of maturity date T_{i+1} is a martingale under the T_{i+1}-forward measure $S(t,T_i,\delta) = E^{T_{i+1}}[S(T_i,T_i,\delta) \mid \mathcal{F}_t]$.

Note that $(1+x)^{-1} \approx 1-x$ (the first-order approximation) and $(1+x)^{-1} \approx 1-x+x^2$ (the second-order approximation). Then, for the first-order approximation, we obtain the following result, which can be modified for the second-order approximation. Bond moments allow us to calculate the convexity adjustment easily.

Proposition 3. *The first-order approximation of the single-period CMS rate is given by*

$$E^{T_1}[S(T_0,T_0,\tau) \mid \mathcal{F}_t] \approx S(t,T_0,\tau) - \sum_{j=0}^{m} a_j \Big(\frac{2\mu^{T_1}(t,T_0,\{T_j\})}{D(t)} - \delta \sum_{k=1}^{m} \frac{\mu^{T_1}(t,T_0,\{T_j,T_k\})}{D(t)^2} \Big),$$

where [7]

$$a_j = \begin{cases} -1 & \text{if } j=0, \\ \delta S(t,T_0,\tau) & \text{if } 0<j<m, \\ 1+\delta S(t,T_0,\tau) & \text{if } j=m. \end{cases}$$

Note that the bCA represents convexity adjustment with different timings for the observation, T_0, and the payment, T_1. We can consider a convexity adjustment with the same timing and call it the convexity adjustment in a narrow sense (nCA). [8] By noting that $E^{T_0}[SV(T_0) \mid \mathcal{F}_t] = 0$, equation (18) can be decomposed into two terms, as follows

$$\begin{aligned} \text{bCA} &= -E^{T_1}[SV(T_0)Dur(T_0)^{-1} \mid \mathcal{F}_t] \\ &= -Cov^{T_0}[SV(T_0), Dur(T_0)^{-1} \mid \mathcal{F}_t] \\ &\quad + \Big(E^{T_0}[SV(T_0)Dur(T_0)^{-1} \mid \mathcal{F}_t] - E^{T_1}[SV(T_0)Dur(T_0)^{-1} \mid \mathcal{F}_t] \Big). \end{aligned}$$

The first term, $-Cov_t^{T_0}[SV(T_0), Dur(T_0)^{-1}](= -E^{T_0}[SV(T_0)Dur(T_0)^{-1} \mid \mathcal{F}_t])$, is the nCA, which represents adjustment based on the same timing for the observation and the payment. The remaining term in the bracket, $E^{T_0}[\cdots] - E^{T_1}[\cdots]$, represents timing adjustment (TA). This is because the observation, T_0, and the settlement, T_1, have different timing.

E. CMS Option

The methods we have discussed so far are useful. The approximated price of an option contract on a CMS swap can be obtained by combining the two methods to approximate a swaption price by the Gram–Charlier expansion (in Section II.C) and a convexity adjustments of a CMS rate with bond moments (in Section II.D). We present the approximated price of a floor of a single period CMS.

CMS options are often incorporated in structured products such as callable bond and capped floater. The 15-year JGB incorporates a floor for the CMT rate since the coupon is set as the maximum of either zero or the current 10-year JGB coupon minus some constant alpha.[9] Therefore, the valuation of these options is of great interest to practitioners.

[7] a_j $(j=0,\ldots,m)$ is the cash flow at time T_j, used to represent the swap with the fixed rate of $S(t,T_0,\tau)$.

[8] Convexity adjustment for bond yields is an example of nCA.

[9] In the 15-year JGB issued on 2005 July, alpha is set at 0.95 percent; i.e., the floor is struck at 0.95 percent.

A swap rate is observed on T_0. The observed swap is assumed to start on T_0 and is assumed to include the coupon exchanges on $T_1, \ldots, T_m$ with $\delta = T_i - T_{i-1}$ and $\tau = m\delta$. The strike rate of the floor is K and the payment of the floor is made on T_1. The value is then given by

$$\begin{aligned} CMSFloor(t) &= E^Q\Big[\exp\Big(-\int_t^{T_1} r_s ds\Big)\delta \max(K - S_{T_0}, 0) \mid \mathcal{F}_t\Big] \\ &= \delta P(t, T_1) E^{T_1}\big[\max(K - S(T_0, T_0, \tau), 0) \mid \mathcal{F}_t\big]. \end{aligned}$$

From equations (17) and (19), we can approximate the observed swap rate as

$$\begin{aligned} S(T_0, T_0, \tau) &= S(t, T_0, \tau) - \frac{SV(T_0)}{Dur(T_0)} \\ &\approx S(t, T_0, \tau) - SV(T_0)\Big(\frac{2}{D(t)} - \frac{1}{D(t)^2} Dur(T_0)\Big) \end{aligned}$$

Thus, we obtain an approximated price of the CMS floor as

$$\begin{aligned} & CMSFloor(t) \\ \approx\ & \delta P(t, T_1) E^{T_1}\Big[\max\Big(K - S(t, T_0, \tau) + \frac{2}{D(t)} SV(T_0) - \frac{1}{D(t)^2} SV(T_0) Dur(T_0), 0\Big) \mid \mathcal{F}_t\Big]. \end{aligned}$$

This can be approximated further by using the Gram–Charlier expansion and the bond moments. The positive part of the cash flow at the expiry consists of three parts: (i) a constant, $K - S(t, T_0, \tau)$; (ii) a linear combination of bond prices, $\frac{2}{D(t)} SV(T_0)$; and (iii) a quadratic combination of bond prices$-\frac{1}{D(t)^2} SV(T_0) Dur(T_0)$. Thus, while it is straightforward to calculate the swap moments, this requires higher order bond moments because of the quadratic terms.

III. Affine Term Sructure Models and the Greeks

Our methods for approximating swaptions and CMS prices are independent of any model if bond moments can be obtained analytically or numerically. If a specific model with state variables is applied to the underlying model, we may be able to calculate the Greeks of the swaptions with respect to the state variables based on the approximation. Examples include not only affine term structure models (ATSMs), but also Gaussian quadratic term structure models (see, e.g., Ahn *et al.* [2002]) and Gaussian Heath–Jarrow–Morton models. In this section, we introduce ATSMs, on which the numerical examples in Section IV are based. Then, we evaluate the deltas of a swaption price for ATSMs.

In an ATSM, the bond price is expressed in the form of an exponentially affine function,

$$P(t, T) = \exp(A(t, T) + B(t, T)^\top X(t)), \tag{20}$$

of a vector of factors (or state variables), $X = (X_1, \ldots, X_J)^\top$, which follows $dX(t) = \mu_X(X(t), t)dt + \sigma_X(X(t), t)dW^Q(t)$ under Q. Duffie and Kan (1996) characterized affine models. Under certain conditions, expressing the bond price in the above form is equivalent to assuming that the short rate r, the drift term μ_X, and $\sigma_X {\sigma_X}^\top$ are affine functions of X. It is sufficient to restrict our attention to the case in which r and X satisfy $r(t) = \delta_0 + {\delta_X}^\top X(t)$, and $dX(t) = K^Q(\theta^Q - X(t))dt + \Sigma D(X(t))dW^Q(t)$, respectively, where $\delta_0, \alpha \in \mathbb{R}$, $\delta_X, \theta^Q, \beta \in \mathbb{R}^J$, and $K^Q \in \mathbb{R}^{J\times J}$, $D(X(t))$ is the diagonal matrix $D(X(t)) = \mathrm{diag}\left[\sqrt{\alpha_1 + {\beta_1}^\top X(t)}, \ldots, \sqrt{\alpha_J + {\beta_J}^\top X(t)}\right]$, and $\Sigma \in \mathbb{R}^{J\times J}$ is a matrix such that $\Sigma\Sigma^\top$ is a covariance matrix.

The Feynman–Kac formula yields the following system of ordinary differential equations:

$$
\begin{aligned}
\frac{\partial}{\partial t}A(t,T) &= -(K^Q\theta^Q)^\top B(t,T) - \frac{1}{2}\sum_{j=1}^{J} {(\Sigma^\top B(t,T))_j}^2 \alpha_j + \delta_0, \quad A(T,T) = 0, \\
\frac{\partial}{\partial t}B(t,T) &= {K^Q}^\top B(t,T) - \frac{1}{2}\sum_{j=1}^{J} {(\Sigma^\top B(t,T))_j}^2 \beta_j + \delta_X, \quad B(T,T) = 0.
\end{aligned} \tag{21}
$$

This system can be solved in a closed form for special cases and can be solved numerically in many other cases. An important characteristic of ATSMs is that the zero rate, $R(t,T)$, and the instantaneous forward rate, $f(t,T)$, for a maturity date of T at time t, are affine functions of X. They are given by

$$
\begin{aligned}
R(t,T) &= -(T-t)^{-1}(A(t,T) + B(t,T)^\top X(t)), \\
f(t,T) &= -\frac{\partial A(t,T)}{\partial T} - \frac{\partial B(t,T)}{\partial T}X(t).
\end{aligned}
$$

As is the bond price, the moments are exponentially affine, and are of the form

$$
\mu^T(t, T_0, \{T_1, \ldots, T_m\}) = \frac{\exp(M(t) + N(t)^\top X(t))}{P(t,T)},
$$

where $M(t) = M(t, T, T_0, \{T_1, \ldots, T_m\})$ and $N(t) = N(t, T, T_0, \{T_1, \ldots, T_m\})$ satisfy the same system of ordinary differential equations

$$
\begin{aligned}
\frac{\partial}{\partial t}M(t) &= -(K^Q\theta^Q)^\top N(t) - \frac{1}{2}\sum_{j=1}^{J} {(\Sigma^\top N(t))_j}^2 \alpha_j + \delta_0, \\
\frac{\partial}{\partial t}N(t) &= {K^Q}^\top N(t) - \frac{1}{2}\sum_{j=1}^{J} {(\Sigma^\top N(t))_j}^2 \beta_j + \delta_X.
\end{aligned} \tag{22}
$$

The terminal conditions are

$$
M(T_0) = \sum_{i=1}^{m} A(T_0, T_i) + A(T_0, T), \quad N(T_0) = \sum_{i=1}^{m} B(T_0, T_i) + B(T_0, T).
$$

These follow from the Feynman–Kac formula. In Gaussian-type and CIR-type models, there are explicit solutions for A, B, M, and N. These solutions are given in the Appendix. Given these formulae, it is straightforward to apply the Gram–Charlier expansion for swaption prices.

Using these functional forms, it is easy to show that the swaption delta with respect to the initial value $X_i(0)$ is given by

$$\begin{aligned}
\frac{\partial SOV(0)}{\partial X_i(0)} &= \frac{1}{2C_2}\frac{\partial C_2}{\partial X_i(0)} SOV(0) \\
&+ \Big(\frac{\partial C_1}{\partial X_i(0)} - \frac{C_1}{2C_2}\frac{\partial C_2}{\partial X_i(0)}\Big)\Big(N(\frac{C_1}{\sqrt{C_2}}) + \sum_{k=3}^{\infty}(-1)^{k-1} q_k H_{k-1}(\frac{C_1}{\sqrt{C_2}})\phi(\frac{C_1}{\sqrt{C_2}})\Big) \\
&+ \sqrt{C_2}\sum_{k=3}^{\infty}(-1)^k \frac{\partial q_k}{\partial X_i(0)} H_{k-2}(\frac{C_1}{\sqrt{C_2}})\phi(\frac{C_1}{\sqrt{C_2}}). \qquad (23)
\end{aligned}$$

The Greeks with respect to other parameters can be obtained in a similar way.

IV. Numerical examples

In this section, we give numerical examples using two ATSMs (a three-factor Gaussian model and a two-factor CIR model), following CDG. We compute the (receiver's) swaption prices of various strikes by using a Gram–Charlier expansion and compare them with prices calculated by a Monte Carlo simulation with respect to accuracy and computational burden. In addition, approximations of CMS rates are investigated in each model.

A. Parameters

Parameters are selected as shown in Table 1, so that the induced rates approximately fit observed Japanese Yen data. One of the reasons for choosing the Japanese Yen data is its closeness to the non-positive area for interest rates, which will distinguish various numerical performances. Figure 1 shows three series of Japanese Yen swap rates, the observed yield curve in markets on 23 February 2005, and the yield curves implied by the two models. Table 2 shows the option data implied by the Gaussian model, which include the at-the-money-forward (ATMF) receivers' swaption prices in basis points (bp) and two types of volatilities in percents (pct), which we will explain shortly. The reader can grasp the level of swaption prices before we discuss the accuracy of the approximation.

As there is no standard pricing formula for interest rate derivatives, a swaption is traded by two parties after they agree on the premium (swaption price). Volatility is treated simply as a reference for their negotiations. Usually, swaption prices are quoted in units of basis points by market traders and brokers. There are two types of reference volatility. *Yield volatility* is used when using Black's formula to value an option. Yield volatility is also known as relative volatility because of its form. The other type of volatility, *absolute volatility*, is yield volatility multiplied by the ATMF rate. Yield volatilities are more widely used than absolute volatilities among practitioners in financial markets. However, a benefit of the absolute volatility is that its intuitive meaning roughly represents the annual standard deviation of a particular swap rate to be observed. Other advantages are that the surface of the absolute volatilities is relatively flat and there is little dependence on the level of yield curve. We consider a one-year into a 10-year swaption ("1 into 10") as a reference because our parameters imply a relatively low cumulative absolute volatility compared with other expiries and maturities.

First, we check the approximated distributions of 10-year swap rates (and values) one year later. Figure 2 shows the distribution of the swap rates. Figures 3 and 4 show the density function of the value of a 10-year receiving swap one year later, for the Gaussian and CIR models, respectively. The swap rate is set to the ATMF rate. In the case of the Gram–Charlier expansion, the density function of the swap value is obtained by using equation (3) up to a particular order. For the Monte Carlo method, the density is obtained by simulation with exact distributions at expiry for both the Gaussian and CIR models. Note that the Gram–Charlier expansion provides a good approximation for the Gaussian model. However, in the CIR model, it overweights the probability around the lower rates by one percent more than the ATMF rate and underweights it around the higher rates.

	Gaussian	CIR
δ_0	-0.0065	-0.02
δ	[1 1 1]	[1 1]
K	[0.05 0.1 1]	[0.05 0.5]
θ	[0.015 0.02 0.02]	[0.085 0.01]
Σ	diag[0.01 0.02 0.03]V	diag[0.08 0.05]V
$VV^\top$	$\begin{pmatrix} 1 & -0.8 & 0.7 \\ -0.8 & 1 & -0.9 \\ 0.7 & -0.9 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
X_0	[0.005 -0.02 0.02]	[0.01 0.01]

Table 1: Parameters

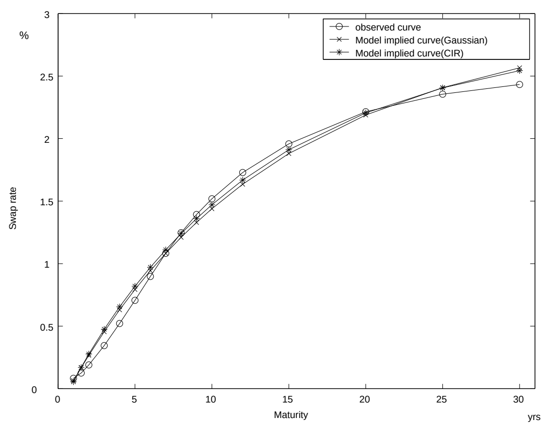

Figure 1: Japanese Yen swap rates and model-implied swap rates

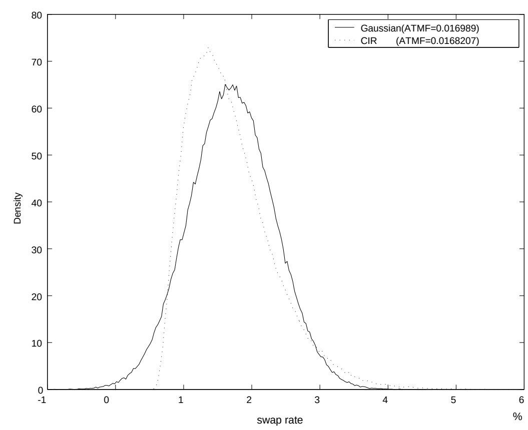

Figure 2: Distribution of swap rate (1 into 10)

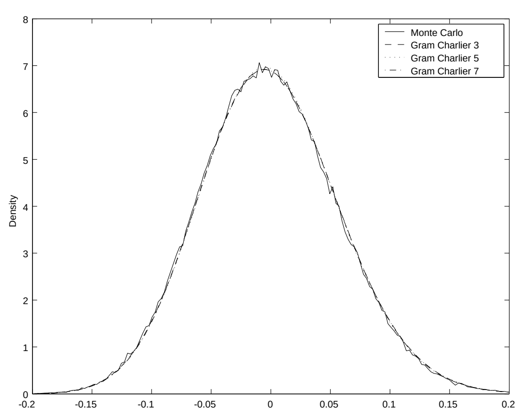

Figure 3: Distribution of swap value (Gaussian, 1 into 10)

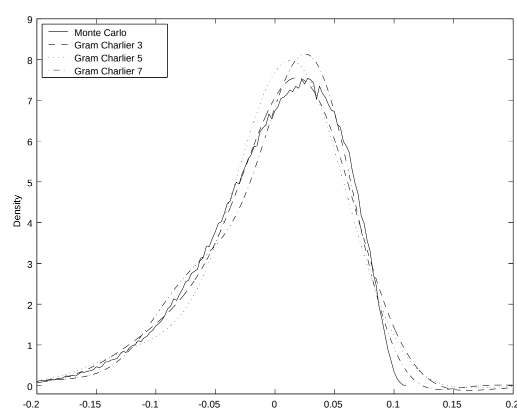

Figure 4: Distribution of swap value (CIR, 1 into 10)

Premium (bp)	Maturity					
Expiry	1	2	3	5	7	10
1	32	57	84	138	183	231
2	44	84	125	200	259	322
3	54	105	154	241	309	381
5	67	129	186	285	362	444
7	73	138	199	302	382	467
10	74	140	200	302	381	466

Yield vol. (pct)	Maturity					
Expiry	1	2	3	5	7	10
1	200	117	89.8	64.9	50.6	37.4
2	102	79.8	68.5	53.4	43.1	33.1
3	7.4	62.7	55.3	44.9	37.4	29.5
5	5.1	43.5	39.6	33.7	29.4	23.7
7	4.0	33.1	30.9	27.0	23.7	20.0
10	3.1	24.7	23.3	20.8	18.9	16.1

Absolute vol. (pct)	Maturity					
Expiry	1	2	3	5	7	10
1	0.94	0.76	0.74	0.73	0.70	0.64
2	0.86	0.80	0.79	0.76	0.71	0.64
3	0.86	0.82	0.80	0.76	0.71	0.63
5	0.88	0.80	0.77	0.72	0.67	0.60
7	0.87	0.75	0.73	0.67	0.62	0.56
10	0.82	0.68	0.65	0.61	0.56	0.51

Table 2: ATMF receiver's swaption prices based on the Gaussian model (top), yield volatilities (middle), and absolute volatilities (bottom) (expiries and maturities measured in years)

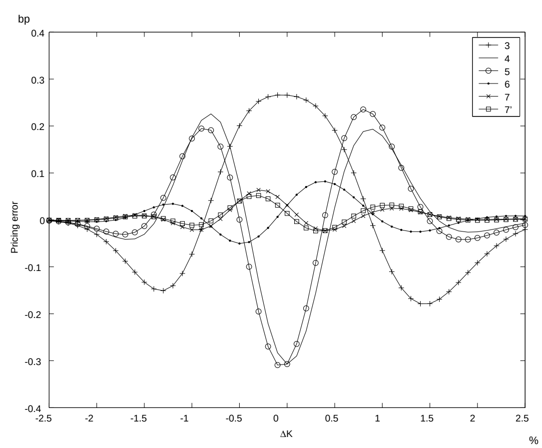

Figure 5: Pricing errors by strikes and orders of approximation
(1 into 10, Gaussian)

B. Swaption prices based on the Gaussian model

In this subsection, we analyze the pricing errors of the swaption prices in the Gaussian model. Figure 5 illustrates pricing errors for a one-year into a 10-year swaption with several strike rates (from ATMF−2.5% to ATMF+2.5%). The horizontal axis represents the difference, ΔK, between the strike rate and the ATMF rate. The pricing error is calculated as the approximated price equation (11) based on the Gram–Charlier expansion ("GC price") minus the price from the Monte Carlo simulation ("MC price"). GC7′ is obtained from GC7 with the sixth and seventh cumulants being set equal to 0 in equation (8). [10] The MC price is obtained by simulating 400 million times (20 million runs multiplied by 20 to calculate MC error) with the negative correlation technique, and using Gaussian distribution of state variables at the expiry to avoid the discretizing error. The standard error is of the order of 10^{-6} for a one-year into 10-year swaption.

All pricing errors are within 0.3 bp for GC3, GC4, and GC5, and within 0.1 bp for GC6, GC7′, and GC7. The results of GC7′ are similar to those of GC7. Note that the higher-order approximations (GC4 and GC5) do not necessarily produce more accurate prices than lower order approximations (GC3). The reason is that the Gram–Charlier expansion is an orthogonal expansion. Table 3 shows the GC prices and MC prices used in Figure 5. Note that for the ATMF swaption ($C_1 = 0$), there are no contributions from the odd-order term because $H_{2k+1}(0) = 0$. In addition, the contribution of each term of the odd (even) order behaves like an odd (even) function of ΔK because of the properties of the Chebyshev–Hermite polynomials.

Similar wave patterns of pricing errors are reported for other combinations of expiries and maturities in Figures 6 and 7. The magnitudes of the fluctuations depend on the expiry and maturity. Nevertheless, the errors for various strikes in five-year into 10-year swaptions based on GC3 are, at most, 2 bp. Due to the higher absolute volatilities of shorter maturities (one-year into 5-year swaptions) compared with one-year into 10-year swaptions, the option delta is higher for the same distance from the ATMF rate, so that the shape of Figure 6 is a zoomed-in picture of a certain part of Figure 5. A similar explanation applies to Figure 7. This is because the standard deviation of a swap rate at the expiry T grows at the order $\sqrt{T}$ when the absolute volatility is constant. Figure 8 illustrates pricing errors for ATMF swaptions based on a seventh-order approximation. This figure shows that most pricing errors for ATMFs are within 1 bp.

Table 4 shows the calculation time for each method based on using Visual C in a 2.4 GHz Pentium 4 CPU. The time for the Gram–Charlier expansion increases substantially with the order of the approximation and the maturity of the underlying swaps. This is due to the associated increase in the number of terms in the summation.

By considering the accuracy and the computational time, we conclude that a higher order approximation GC7′ yields very accurate prices enough to price a specific transaction, and that a lower order one GC3 attains good approximation in a very short time so that it is suitable for the portfolio evaluation and the risk management. However, the level of the accuracy depends on the model parameters. One should note that we compare the pricing errors of several approximation orders by using the same model parameters. It is obvious that the error depends on the model parameters such as the level of the yield curve or the mean reversion speed.

[10] CDG use the same method as GC7′ to calculate the probabilities of ending in in-the-money under several relevant forward measures.

ΔK	-1%	-0.5%	0%	0.5%	1%
3rd	12.600	68.438	230.926	535.646	945.868
4th	12.849	68.311	230.353	535.482	946.112
5th	12.847	68.237	230.353	535.558	946.130
6th	12.692	68.187	230.691	535.532	945.930
$7'$th	12.662	68.277	230.674	535.440	945.964
7th	12.652	68.278	230.691	535.435	945.955
MC	12.673	68.237	230.660	535.455	945.933

Table 3: GC prices and MC prices (bp)

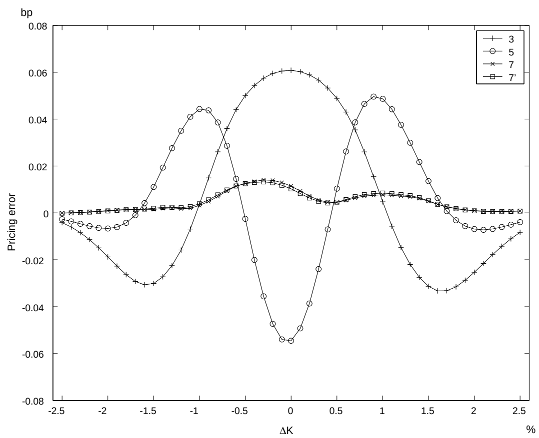

Figure 6: 1 into 5

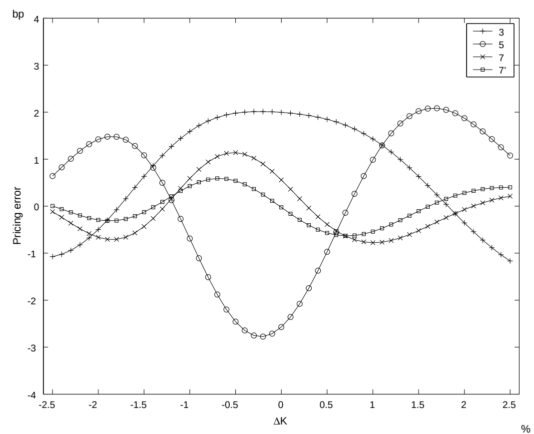

Figure 7: 5 into 10

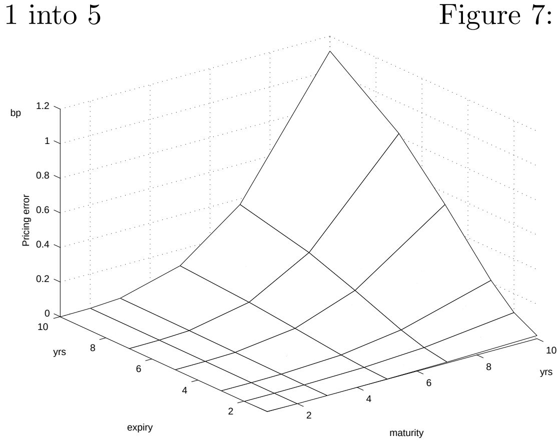

Figure 8: Pricing errors for ATMF (Gaussian, GC7)

	1 into 5 ATMF	1 into 7 ATMF	1 into 10 ATMF
GC3	<0.000	<0.000	<0.000
GC5	<0.000	0.031	0.156
GC7′	<0.000	0.031	0.156
GC7	0.063	0.438	3.453
MC(20million)	78.266	90.766	109.531

Table 4: Calculation times (sec., Gaussian)

C. Swaption prices based on the CIR model

It is important for practitioners to recognize the pattern and the level of the pricing errors before implementing our approach. As expected, the CIR model has larger standard errors from the Monte Carlo simulation and poorer approximations than those of the Gaussian model, by about one digit. However, the main features are similar to those of the Gaussian model.

Figure 9 shows the pricing errors for one-year into 10-year swaptions from the CIR model. The corresponding figure for the Gaussian model is Figure 5. For the CIR model, the MC price is obtained by simulating 100 million times (5 million runs multiplied by 20) by using noncentral chi-squared distributions of the state variables at the expiry to avoid the discretizing error. The basic features of the pricing errors of the CIR model look similar to those of the Gaussian. The pricing errors are no more than 4 bp based on GC3 and are about 2 bp based on GC7. Figure 10 shows the pricing errors from the ATMF based on a seventh-order approximation. It seems that these errors for the CIR model are not at critical levels for the practitioners' purposes in their daily activities, such as pricing and risk management. Calculation times are similar for the CIR and Gaussian models. Therefore, the conclusion from the Gaussian model also applies to the CIR model.

As mentioned in footnote 10, CDG use the same method used for GC7$'$ to calculate the probabilities of being in-the-money under several relevant forward measures. Figure 11 compares the performance of the CDG approach to our approach for coupon-bond option prices (a two-year option on 12-year bond) by using the same parameters used by CDG. This figure indicates that our approach is better than that of CDG. Especially, difference between GC7$'$ and CDG shows accumulated pricing error due to the number of forward measures. Again, accuracy depends on the model parameters. The order of the errors are quite different between Figures 9 and 11 since the pricing error for the CIR model will be smaller if the underlying yield is higher.

D. Convexity adjustments for CMS rates

In this subsection, we investigate how accurately our approximation methods with bond moments calculate the convexity adjustments of CMS rates. The convexity adjustments (bCA, nCA, and TA) of one-period CMS rates under the Gaussian model are calculated by the Monte Carlo method, as shown in Table 5, to grasp the magnitudes. The longer the time to the observation or the longer the maturity of the observed swaps, the bigger the adjustments are. The pricing errors are reported in Tables 6, 7, and 8 for the Gaussian with the first-order approximation, the CIR with the first-order approximation, and the CIR with the second-order approximation, respectively.

The Gaussian first-order approximation performs well. The pricing errors are, at most, 0.29 bp for the CA in a broad sense, 0.24 bp for the CA in a narrow sense, and 0.05 bp for the timing adjustment (Table 6). The CIR first-order approximation does not perform well, as Table 7 shows. The second-order approximation reduces the errors from Table 7 by roughly half, so that they are relatively small (see Table 8). Our methods are practical.

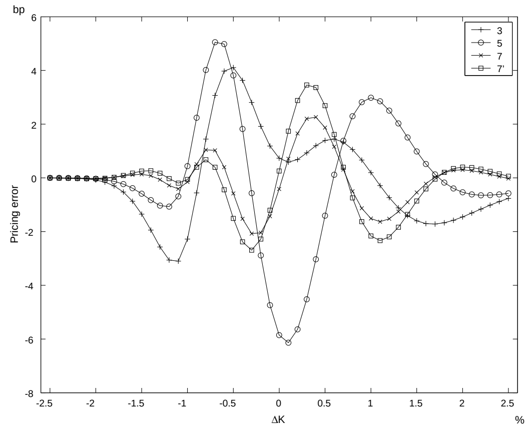

Figure 9: Pricing errors by strikes and orders of approximation (1 into 10, CIR)

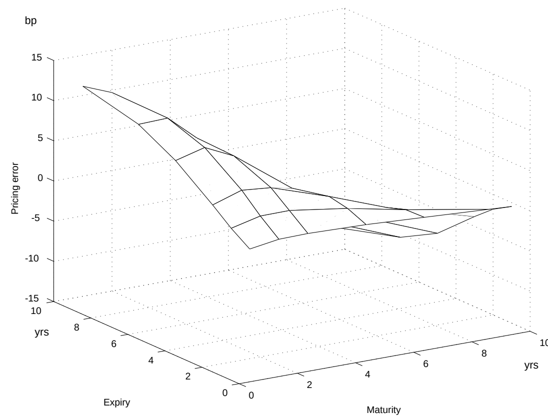

Figure 10: Pricing error for ATMF (CIR, GC7)

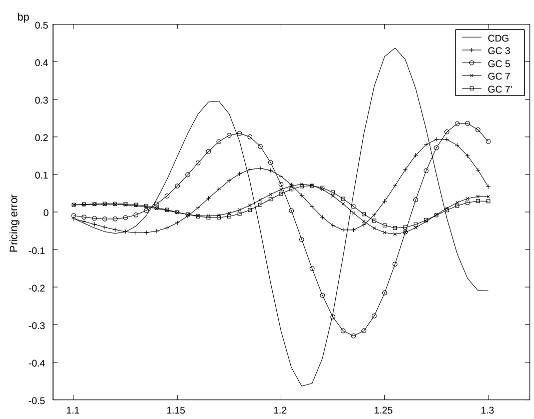

Figure 11: Comparison based on CDG parameters

(bCA)	1	3	5	7	10	20
1	0.14	0.65	1.18	1.60	2.00	2.30
3	0.46	2.24	3.74	4.83	5.81	6.45
5	0.76	3.49	5.65	7.17	8.54	9.51
10	1.14	5.08	8.08	10.21	12.19	13.99
(nCA)	1	3	5	7	10	20
1	0.51	0.85	1.32	1.72	2.10	2.38
3	1.47	3.05	4.45	5.47	6.37	6.85
5	2.38	4.86	6.86	8.26	9.50	10.18
10	3.56	7.18	9.96	11.90	13.66	15.03
(TA)	1	3	5	7	10	20
1	-0.37	-0.20	-0.14	-0.12	-0.10	-0.08
3	-1.01	-0.82	-0.71	-0.64	-0.56	-0.40
5	-1.62	-1.37	-1.22	-1.10	-0.95	-0.67
10	-2.43	-2.10	-1.87	-1.69	-1.48	-1.04

Table 5: Adjustments for single-period CMS swap rates (bp) for the Gaussian model (payment due six months later than the observation): Convexity adjustments in the broad sense (bCA, top); Convexity adjustments in the narrow sense (nCA, middle); and Timing adjustments (TA, bottom). Rows 1 to 10 express observation years, and columns 1 to 20 are maturities (in years) of observed swap rates.

(bCA)	1	3	5	7	10	20
1	0.00	0.00	0.00	0.00	0.00	0.01
3	0.00	0.00	0.01	0.02	0.04	0.06
5	0.00	0.01	0.03	0.05	0.08	0.14
10	0.00	0.02	0.05	0.09	0.16	0.29
(nCA)	1	3	5	7	10	20
1	0.00	0.00	0.00	0.00	0.00	0.01
3	0.00	0.00	0.01	0.02	0.03	0.05
5	0.00	0.01	0.02	0.03	0.06	0.11
10	0.00	0.01	0.04	0.07	0.12	0.24
(TA)	1	3	5	7	10	20
1	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.01
5	0.00	0.00	0.01	0.01	0.02	0.02
10	0.00	0.01	0.02	0.02	0.04	0.05

Table 6: Pricing errors of adjustments for single-period CMS swap rates (bp) for the Gaussian model: price approximated by bond moments $-$ MC price

(bCA)	1	3	5	7	10	20
1	-0.00	-0.01	-0.03	-0.05	-0.07	-0.11
3	-0.03	-0.13	-0.26	-0.40	-0.58	-0.91
5	-0.08	-0.33	-0.66	-0.99	-1.42	-2.19
10	-0.24	-0.98	-1.88	-2.77	-3.90	-5.94
(nCA)	1	3	5	7	10	20
1	-0.00	-0.02	-0.03	-0.05	-0.07	-0.12
3	-0.03	-0.13	-0.27	-0.41	-0.60	-0.94
5	-0.08	-0.35	-0.69	-1.03	-1.48	-2.28
10	-0.25	-1.04	-2.00	-2.94	-4.15	-6.30
(TA)	1	3	5	7	10	20
1	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.01	0.01	0.02	0.03
5	0.00	0.01	0.03	0.04	0.06	0.09
10	0.02	0.06	0.12	0.18	0.24	0.36

Table 7: Pricing errors of adjustments for single-period CMS swap rates (bp) for the CIR model (first-order approximation)

(bCA)	1	3	5	7	10	20
1	0.00	-0.00	-0.00	-0.01	-0.01	-0.02
3	-0.00	-0.01	-0.04	-0.08	-0.14	-0.30
5	-0.01	-0.05	-0.12	-0.24	-0.42	-0.89
10	-0.03	-0.19	-0.50	-0.89	-1.53	-3.09
(nCA)	1	3	5	7	10	20
1	0.00	-0.00	-0.00	-0.01	-0.01	-0.02
3	-0.00	-0.01	-0.04	-0.08	-0.14	-0.31
5	-0.01	-0.05	-0.13	-0.25	-0.44	-0.93
10	-0.03	-0.20	-0.53	-0.94	-1.61	-3.25
(TA)	1	3	5	7	10	20
1	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.00	0.00	0.00	0.00	0.01
5	0.00	0.00	0.00	0.01	0.02	0.03
10	0.00	0.01	0.03	0.05	0.08	0.16

Table 8: Pricing errors of adjustments for single-period CMS swap rates (bp) for the CIR model (second-order approximation)

V. Conclusion

We have developed easy-to-use approximation methods for pricing several interest rate derivatives by using the Gram–Charlier expansion and by using bond moments. Approximation accuracy depends on the underlying model, the detailed characteristics of the products, and the model parameters. From numerical studies for swaption prices, our method yields smaller pricing errors than the method used by Collin-Dufresne and Goldstein (2002). We conclude that a higher order approximation GC7′ yields very accurate prices enough to price a specific transaction, and that a lower order one GC3 attains good approximation in a very short time so that it is suitable for the portfolio evaluation and the risk management. Using bond moments to calculate the convexity adjustments of constant maturity swap rates is novel and the approximation method performs well. We also derive an approximated option price on CMS rates by combining the two methods. These approximations can be applied to CMT products such as 15-year floating coupon JGB.

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A. Affine term structure models

A. $\mathbb{A}_0(J)$ Gaussian Model

The coefficients of a J-factor Gaussian model, $\mathbb{A}_0(J)$, are given by

$$\begin{aligned}&\delta_X = \mathbf{1}_J, \quad K^Q = \mathrm{diag}[K_1, \dots, K_J] \quad (0 < K_1 < \cdots < K_J),\\ &\Sigma = \mathrm{diag}[\sigma_1, \dots, \sigma_J]V, \quad \text{where } VV^\top = (\rho_{ij})_{ij}, \quad D(X(t)) = I_J.\end{aligned}$$

Bond prices and bond moments can be obtained from

$$\begin{aligned}
A(t,T) &= -(T-t)\Big(\delta_0 + \sum_{i=1}^{J}(1 - D(K_i(T-t)))\theta_i - \frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J}\frac{\rho_{ij}\sigma_i\sigma_j}{K_iK_j}\\
&\quad \times \big(1 - D(K_i(T-t)) - D(K_j(T-t)) + D((K_i+K_j)(T-t))\big)\Big),\\
B_j(t,T) &= -\tau D(K_j(T-t)),\\
M(t) &= A(t,T_0) + F_0 + \tau\sum_{j=1}^{J}K_j\theta_jF_jD(K_j\tau) + \frac{\tau}{2}\sum_{i,j=1}^{J}\rho_{ij}\sigma_i\sigma_j\Big(F_iF_jD((K_i+K_j)\tau)\\
&\quad + F_i\frac{D((K_i+K_j)\tau) - D(K_i\tau)}{K_j} + F_j\frac{D((K_i+K_j)\tau) - D(K_j\tau)}{K_i}\Big),\\
N_j(t) &= B_j(t,T_0) + F_j\exp(-K_j(T_0-t)),
\end{aligned}$$

where $D(x) = \frac{1-e^{-x}}{x}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^m A(T_0,T_i) + A(T_0,T)$ and $F_j = \sum_{i=1}^m B_j(T_0,T_i) + B_j(T_0,T)$.

B. $\mathbb{A}_J(J)$ CIR Model

The coefficients of a J-factor CIR model, $\mathbb{A}_J(J)$, are given by

$$\begin{aligned}&\delta_X = 1_J, \quad K^Q = \mathrm{diag}[K_1, \dots, K_J], \quad \theta = (\theta_1, \dots, \theta_J)^\top, \quad (\theta_j > 0),\\ &\Sigma = \mathrm{diag}[\sigma_1, \dots, \sigma_J], \quad D(X(t)) = \mathrm{diag}[\sqrt{X_1(t)}, \dots, \sqrt{X_J(t)}].\end{aligned}$$

Bond prices and bond moments can be obtained from

$$\begin{aligned}
A(t,T) &= -\delta_0(T-t) - \sum_{j=1}^{J}K_j\theta_j\Big[\frac{2}{\sigma_j^2}\ln\frac{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)+2\gamma_j}{2\gamma_j} + \frac{2}{K_j-\gamma_j}(T-t)\Big],\\
B_j(t,T) &= \frac{-2(e^{\gamma_j(T-t)}-1)}{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)+2\gamma_j},\\
M(t) &= F_0 - \delta_0\tau - \sum_{j=1}^{J}K_j\theta_j\Big[\frac{2}{\sigma_j^2}\ln\frac{(K_j+\gamma_j-\sigma_j^2F_j)(e^{\gamma_j\tau}-1)+2\gamma_j}{2\gamma_j}\\
&\quad + \frac{(K_j+\gamma_j)F_j+2}{K_j-\gamma_j-\sigma_j^2F_j}\tau\Big],\\
N_j(t) &= \frac{-((K_j-\gamma_j)F_j+2)(e^{\gamma_j\tau}-1)+2\gamma_jF_j}{(K_j+\gamma_j-\sigma_j^2F_j)(e^{\gamma_j\tau}-1)+2\gamma_j},
\end{aligned}$$

where $\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^m A(T_0,T_i) + A(T_0,T)$ and $F_j = \sum_{i=1}^m B_j(T_0,T_i) + B_j(T_0,T)$.