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Abstract

This paper characterizes the optimal monetary policy reaction function in the presence of a zero lower bound on the nominal interest rate. We analytically prove and numerically show that the function is highly nonlinear, more expansionary, and more aggressive than the Taylor rule. We then test its empirical validity taking the case of Japan in the 1990s. Qualitatively, we find some evidence of nonlinear monetary policy. Quantitatively, we find the actual monetary policy to be too contractionary during the first half of the decade, while the low interest policy during the latter half turns out to be fairly consistent with the simulated path.

Key words: Deflation, Liquidity trap, Pre-emptive monetary policy, Zero bound

JEL classification: E52, E58, C63

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1 Introduction

A zero lower bound on the nominal interest rate is becoming a serious concern. Many central banks, especially in industrialized countries, have been successful in reducing average inflation rates to a range of 0-3% in recent decades. In this kind of low inflation era, especially when a central bank is faced by a severe recession, a zero lower bound on the short-term nominal interest rate – a policy instrument for most of the central banks – could be a serious constraint for the implementation of monetary policy. In the extreme case, when the nominal interest rate actually binds at zero, a central bank will no longer be able to stimulate the economy via the nominal interest rate channel – a phenomenon also known as a *liquidity trap*. In such circumstances, standard monetary policy, by controlling the short-term nominal interest, will become totally ineffective and the economy will have to bear the cost of increased volatility. Blinder (2000), being keenly aware of this predicament, succinctly warns, "*Don't go there.*" The recent trend of low inflation accompanied by the issues stemming from a zero lower bound is the basic reason why it is becoming a realistic and serious concern for many central banks.

Reflecting the practical importance of a zero lower bound on the nominal interest rate, much research has been made regarding the conduct of monetary policy in the presence of such a constraint. The pioneering work is due to Fuhrer and Madigan (1997). They conducted an impulse response analysis taking into account of the zero lower bound and showed that stabilization policy is costly in the sense that it takes longer for the output and the inflation rate to return to a steady state than the case where there is no constraint. Orphanides and Wieland (1998), in their stochastic simulation study, showed that the probability of the economic state entering a liquidity trap will be lower when the inflation target is set higher and concluded that the social welfare loss can be reduced by setting a positive inflation target. Reifschneider and Williams (2000) provided an insightful study which shows that a variant of the Taylor rule is superior to a standard Taylor rule in stabilization capability by comparing the efficient policy frontier of stochastic simulation results. This implies that when the zero lower bound on the nominal interest rate is incorporated, the optimal policy is neither a linear function of state variables nor a standard Taylor rule. Orphanides and Wieland (2000) demonstrated that the optimal policy under the non-negativity constraint is a nonlinear function of the inflation rate using a numerical method. Their numerical evidence suggests a central bank to adopt an "aggressive" monetary policy as the nominal interest rate approaches the zero lower bound. Watanabe (2000) and Jung *et al.* (2001) investigated the optimal conditions for the termination of a "zero interest rate policy" based on the forward-looking economy model following Woodford (1999). Using a simulation technique, they show that the optimal path of the nominal interest rate depends on historical policy conduct as well as a commitment for future policy conduct.[1] Hunt and Laxton (2003) investigated the role of an inflation target in the presence of the zero lower bound. Using the MULTIMOD simulation model, they show that targeting too low an inflation rate will induce a central bank to be susceptible to a deflationary spiral and suggest that the inflation rate should be targeted higher than 2% in the long run.

Among multiple aspects of the zero bound problem, our focus is on the explicit form of the optimal monetary policy reaction function.[2] The past studies are commonly aware of the risk of the zero lower bound (or the liquidity trap), which can seriously affect the stabilization function of the central bank. In fact, most of the past studies, such as Blinder (2000), Goodfriend (2000), Reifschneider and Williams (2000), Orphanides and Wieland (2000), and Hunt and Laxton (2003) point out the possibility that the optimal monetary policy is affected by the zero lower bound before the constraint actually binds. For example, Goodfriend (2000) claims that monetary policy must be "pre-emptive" to prevent the constraint from binding. Unfortunately, however, the past studies have mainly relied on conjectures or numerical/simulation methods in showing this "pre-emptiveness" feature of monetary policy.

The main contribution of this paper is that we provide a mathematical foundation to this "pre-emptiveness" feature of monetary policy in the presence of the zero lower bound. In particular, we investigate how the optimal monetary policy reaction function is affected when the zero lower bound of the nominal interest rate is explicitly incorporated to a Svensson (1997a) and Ball (1999) type model.[3] Based on the styl-

[1] For more recent studies in the context of forward-looking private agents, see Benhabib et al. (2002) and Eggertsson and Woodford (2003).

[2] Another important issue stemming from the zero lower bound constraint is the so-called "buffer role" of the inflation target. As was first pointed out by Summers (1991), a central bank can pre-empt the risk of being caught in a liquidity trap by targeting a small but positive inflation rate in the long-run. However, this pre-emption strategy assumes that the inflation target (i.e., π^*) can be controlled by a central bank, which, in reality, may not be a readily available policy option in the short-run. In this sense, the pre-emption strategy of setting a higher inflation target can be said to be a policy option only available in the long-run. This paper discusses the optimal monetary policy reaction based on the assumption of a fixed inflation target, and analyzes the short-run pre-emption strategy in a situation where the nominal interest rate is the sole control variable of a central bank. A reader further interested in the long-run pre-emption strategy in which an inflation target is variable should refer to Orhphanides and Wieland (1998), Hunt and Laxton (2003) and Nishiyama (2003).

[3] In the absence of a zero lower bound on the nominal interest rate, it is well known that the Taylor rule, which was first introduced in a seminal work by Taylor (1993), can be rationalized as the optimal policy reaction function of a certain class of a central bank's dynamic optimization problem as shown in Svensson (1997a) and Ball (1999). When a central bank's loss function is quadratic and the economy can be captured by a first order difference system of which state variables are the current output and the inflation rate, the solution of the dynamic programming problem is a linear function of the state variables, that is, the current output and the inflation rate. This is the well known property of the so called Linear-Quadratic (LQ) regulator problem when the control variable is unbounded, which has been intensively examined both in the economics and engineering fields of study.

ized framework of a central bank's dynamic optimization problem following Svensson (1997a) and Ball (1999), we derive the analytical expression of the optimal monetary policy reaction function in the presence of the zero lower bound and prove that it is (i) highly nonlinear, (ii) more expansionary, and (iii) more aggressive than the Taylor rule, which is otherwise the optimal policy in the absence of the zero bound. These features of the optimal policy reaction function are basically consistent with the policy implications shown by past studies, providing a solid analytical foundation to "pre-emptive" monetary policy conduct in the presence of the zero lower bound.

In the real world, perhaps the most notable episode where the conduct of monetary policy has been severely affected by the zero lower bound constraint was during the period of stagnation of the Japanese economy in the 1990s. There has been substantial debate regarding the Bank of Japan's (BOJ) monetary policy conduct during the 1990s. For instance, Ahearne *et al.* (2002) point out that the BOJ's monetary policy conduct was too slow in cutting the call rate during the early 1990s and conclude that this slow response has been one of the factors that caused Japan's prolonged stagnation during the entire decade.[4] Bernanke and Gertler (1999) also report similar simulation evidence based on the dynamic general equilibrium model incorporating a financial accelerator. On the other hand, McCallum (1999), Kamada and Muto (2000), and Yamaguchi (2002) show empirical evidence that the estimated Taylor rule can explain the BOJ's monetary policy conduct until the mid-1990s fairly well, while exhibiting a wide discrepancy between the actual call rate and the predicted call rate implied by the estimated Taylor rule during the latter half of the decade. This implies that the BOJ's low interest rate policy during the latter half of the 1990s was too expansionary, if we take the estimated Taylor rule for granted.

In this paper, taking the case of Japanese monetary policy in the 1990s, we empirically test the validity of the qualitative and quantitative features implied in our model. Taking into account the left-censoring of the call rate due to the presence of the zero lower bound, we adopt the Tobit model in estimating the BOJ's policy reaction function during the 1990s. The Likelihood Ratio type test based on the Tobit estimation results (weakly) rejects the linearity of the BOJ's policy reaction function with respect to the inflation rate and the output gap, revealing some evidence for the concavity of the policy reaction function. This empirical evidence turns out to be qualitatively consistent with the "aggressive" monetary policy conduct implied in our model. However, since the qualitative evidence from the Tobit estimation is unindicative about the desirable *level* of the call rate during the 1990s, we also conduct quantitative analysis based on the numerically approximated optimal policy reaction function. By comparing the actual path of the call rate and the simulated path of the call rate during the 1990s, it

[4] For a counter-argument in defense of the BOJ's monetary policy conduct in the early 1990s, see Yamaguchi (2002).

turns out that the BOJ's monetary policy conduct in the first half of the decade was too contractionary and too slow in responding to the declining output and disinflation, while the low interest rate policy by the BOJ in the latter half of the decade turns out to be fairly consistent with the simulated path implied by the optimal policy reaction function.

The remainder of this paper is organized as follows. Section 2 describes the set up of the model and derives the analytical expression of the optimal policy reaction function. Section 3 discusses the numerical strategy in approximating the optimal policy reaction function and demonstrates the results. Section 4 addresses the empirical issues taking the case of Japan in the 1990s and tests the qualitative and quantitative implications of the optimal policy reaction function. Section 5 offers some concluding remarks and discusses the future extension of the model.

2 The Model

2.1 Setup of the Model and First Order Conditions

This section describes the standard problem setting to discuss optimal monetary policy and its modification when the zero lower bound on the nominal interest rate is incorporated. Since we assume the welfare loss function to be quadratic, which is the standard specification in the monetary policy literature, the problem can be regarded as a well defined stochastic linear-quadratic (LQ) regulator problem. It is well known that the problem leads to a quadratic value function and a linear policy function,[5] which depend on only the current state variables if there is no constraint on the control variable. In this sense, the Taylor rule can be rationalized as a class of optimal reaction functions as proved in Svensson (1997a) and Ball (1999).[6] In this paper, the specification of a central bank's loss function and the economy follows those of the Svensson-Ball type model, and then we incorporate the zero lower bound constraint on the nominal interest rate.

[5] When the policy reaction function is *optimal*, in other words, when it is a solution to the dynamic optimization problem of the central bank, it is equivalent to an "*optimal* policy function" or an "*optimal* feedback rule." Note that these two are purely technical terms of control theory, while "*optimal* policy reaction function" is usually used in economic literature. To avoid confusion, we will use *policy reaction function* throughout this paper, unless there is a special need to distinguish these concepts.

[6] Note that the model adopted here can be completely solved by standard dynamic programming methods, while the model in Woodford (1999) cannot, since in the Woodford model, the state equation has non-trivial rational expectations behavior. Backus and Driffil (1986) and Woodford (1999) state that with non-trivial rational expectation behavior, it is impossible to use standard dynamic programming techniques to obtain the optimal policy reaction function.

First, we assume that a central bank's loss function is given as

$$L_t = \frac{1}{2}\left\{y_t^2 + \lambda\left(\pi_t - \pi^*\right)^2\right\}, \tag{1}$$

where π and y denote the inflation rate and the output gap, respectively, and π^* is the target inflation rate of the central bank. λ is a positive weight which represents the preference of the central bank. The economy is described by the following conventional IS and AS type formulation,

$$y_{t+1} = \rho y_t - \delta\left(i_t - E_t\pi_{t+1}\right) + \upsilon_{t+1} \tag{2}$$

$$\pi_{t+1} = \pi_t + \alpha y_t + \varepsilon_{t+1}. \tag{3}$$

Eqn (2) and eqn (3) stand for the IS equation and AS equation, respectively, where υ and ε are assumed to be *i.i.d.* random disturbances. Although eqn (2) contains a forward-looking variable, this can be substituted by the following relationship between the expected inflation rate and the current inflation rate and the output gap,

$$E_t\pi_{t+1} = \pi_t + \alpha y_t. \tag{4}$$

Then, a central bank's problem is formulated as an intertemporal minimization problem with the objective function,

$$\min_{i_t} : E_t\sum_{i=0}^{\infty}\beta^i L_{t+i} \tag{5}$$

subject to the IS equation $y_{t+1} = (\rho + \alpha\delta)y_t - \delta\left(i_t - \pi_t\right) + \upsilon_{t+1}$ with the zero lower bound constraint on the nominal interest rate,

$$i_t \geq 0$$

and the AS equation (3). We follow the treatment of the non-negativity constraint as in Watanabe (2000) and apply Kuhn-Tucker conditions in this dynamic optimization problem. Since this problem can be interpreted as a conventional optimal bounded control problem with a linear system, we can set up a Bellman equation[7] with three Lagrange multipliers as follows,

$$\begin{aligned} V\left(y_t, \pi_t\right) = \min_{i_t}[&\frac{1}{2}\left\{y_t^2 + \lambda\left(\pi_t - \pi^*\right)^2\right\} - E_t\phi_{t+1}\{(\rho + \alpha\delta)y_t - \delta i_t + \delta\pi_t - y_{t+1}\} \\ &- E_t\mu_{t+1}(\pi_t + \alpha y_t - \pi_{t+1}) \\ &- \psi_t i_t \\ &+ \beta E_t V\left(y_{t+1}, \pi_{t+1}\right)]. \end{aligned} \tag{6}$$

[7] Here we assume a central bank's discount factor β to be sufficiently small for the Contraction Mapping Theorem to hold. For the lists of regularity conditions regarding the Contraction Mapping Theorem, see Stokey and Lucas (1989). Throughout this paper, we simply assume that regularity conditions are satisfied.

We must be very careful in writing the signs of Lagrange multipliers. As this is a minimization problem with a non-negativity constraint, each sign in front of the multiplier must be set so that the multiplier has a positive value when the constraint is binding. The first order conditions (FOCs) of this problem are as follows,

$$E_t\phi_{t+1}\delta = \psi_t \tag{7}$$

$$-\frac{1}{\beta}E_t\mu_{t+1} = \lambda E_t(\pi_{t+1}-\pi^*) - \delta E_t\phi_{t+2} - E_t\mu_{t+2} \tag{8}$$

$$-\frac{1}{\beta}E_t\phi_{t+1} = E_t y_{t+1} - (\rho+\alpha\delta)E_t\phi_{t+2} - \alpha E_t\mu_{t+2} \tag{9}$$

where eqn (7), eqn (8), and eqn (9) represent the FOCs with respect to the nominal interest rate, i, the inflation rate, π, and the output gap, y, respectively.

To gain some intuition regarding these FOCs, suppose there is no zero lower bound on the interest rate, so that FOC (7) becomes $E_t\phi_{t+1} = \psi_t = 0$ for any period t. Then the only constraint which substantially matters is μ_t – the Lagrange multiplier associated with the AS eqn (3), which can be interpreted as the shadow cost of AS shock – in this case. In other words, in the absence of the zero lower bound constraint, a central bank can completely neutralize the ex-ante cost stemming from the IS shock, while facing a trade-off between inflation and output stabilization in responding to AS shocks. In the context of our model discussed in this paper, under optimal monetary policy conduct, $E_t\phi_{t+1}$ (which represents the shadow cost of IS shocks) will be equal to zero, while $E_t\mu_{t+1}$ (which represents the shadow cost of AS shocks) remains strictly positive. Thus, aggregate demand disturbances cannot be a restraint on the central bank's monetary policy conduct in the absence of the zero lower bound.

In contrast, suppose that there exists a zero lower bound constraint and it is binding in the current period. Then we have $E_t\phi_{t+1} > 0$ which implies that a shadow cost is emerging from the IS shocks. In this case, even if the policy is conducted optimally, a central bank cannot completely offset IS shocks. Thus, in the presence of a zero lower bound constraint, a central bank is confronted with the difficult task of balancing the opportunity costs arising from both IS and AS shocks. Note that in our model, monetary policy has a one period lag before it affects the economy, and therefore the zero lower bound matters for the ability of a central bank to stabilize the economy in the following period.

Next, multiplying α on both sides of eqn (8) and subtracting eqn (9), we obtain the following equation,

$$\alpha E_t\mu_{t+1} = \beta E_t y_{t+1} - \beta\alpha\lambda E_t(\pi_{t+1}-\pi^*) - \beta\rho E_t\phi_{t+2} + E_t\phi_{t+1}. \tag{10}$$

By updating eqn (10) one period and substituting back to eqn (9) for $E_t\mu_{t+2}$, we obtain the following relationship between $E_t y_{t+1}$ and $E_t y_{t+2}$.

$$\beta E_t y_{t+2} - \alpha\beta\lambda E_t(\pi_{t+2} - \pi^*) - E_t y_{t+1} = \rho\beta E_t \phi_{t+3} - (1+\rho+\alpha\delta) E_t \phi_{t+2} + \frac{1}{\beta} E_t \phi_{t+1} \tag{11}$$

Alternatively, by using the FOC for the nominal interest rate – i.e., eqn (7) –, the above relationship can also be expressed as a function of the Lagrange multipliers, ψ, associated with the Kuhn-Tucker condition,

$$\beta E_t y_{t+2} - \alpha\beta\lambda E_t(\pi_{t+2} - \pi^*) - E_t y_{t+1} = \frac{\beta\rho}{\delta} E_t \psi_{t+2} - \frac{(1+\rho+\alpha\delta)}{\delta} E_t \psi_{t+1} + \frac{1}{\beta\delta}\psi_t. \tag{12}$$

Since the choice of $E_t y_{t+1}$ and i_t is technically equivalent,[8] eqn (12) can be considered to be the Euler equation of the control variable. Although it is technically possible to express the Euler eqn (12) in terms of i_t, for the convenience of algebraic manipulation, we will take $E_t y_{t+1}$ as the proxy of the control variable for a while. Now it should be noted that the Euler equation (12) contains expected values of $\{\psi_{t+i}\}_{i=0}^{2}$. This implies that even if a zero lower bound constraint is not binding in the current period (therefore $\psi_t = E_t \phi_{t+1} = 0$ and $i_t > 0$ at period t), the Euler equation is affected by $E_t \psi_{t+1}$ which is in general positive due to the non-zero probability of the event that the constraint is binding in the following period. Orphanides and Wieland (2000) presents numerical evidence to claim that even if the non-negativity constraint did not bind in any past period, optimal monetary policy is different from the case where there is no such constraint. Eqn (12) derived here provides the analytical expression which is consistent with their claim.

2.2 Optimal Monetary Policy Reaction Function

Our goal here is to derive the analytical expression of the optimal monetary policy reaction function when the constraint is not binding in the current period. Since $E_t y_{t+1} = \frac{1}{\alpha}(E_t \pi_{t+2} - E_t \pi_{t+1})$ and $E_t y_{t+2} = \frac{1}{\alpha}(E_t \pi_{t+3} - E_t \pi_{t+2})$ from the AS eqn (3), substituting these into the Euler eqn (12) yields the second-order difference equation in terms of $E_t \pi_{t+1}$ as follows,

$$\beta E_t(\pi_{t+3} - \pi^*) - \left(1+\beta+\alpha^2\beta\lambda\right) E_t(\pi_{t+2} - \pi^*) + E_t(\pi_{t+1} - \pi^*) = \alpha E_t \Psi_t \tag{13}$$

where

$$\Psi_t \equiv \frac{1}{\delta}\left(\rho\beta\psi_{t+2} - (1+\rho+\alpha\delta)\psi_{t+1} + \beta^{-1}\psi_t\right).$$

[8] In other words, $E_t y_{t+1}$ and i_t have a one-to-one relationship given the state (π_t, y_t). Note that by applying the conditional expectation operator E_t on both sides of the IS equation, it follows that $E_t y_{t+1} = (\rho + \alpha\delta) y_t + \delta\pi_t - \delta i_t$.

Let $\theta_{1,2}$ be the roots of the characteristic equation, $z^2 - \left(1+\beta+\alpha^2\beta\lambda\right) z + \beta = 0$. The roots of the characteristic equation can be expressed as,

$$\theta_{1,2} = \frac{1+\beta+\alpha^2\beta\lambda \pm \sqrt{(1+\beta+\alpha^2\beta\lambda)^2 - 4\beta}}{2}.$$

Now, since $(1+\beta+\alpha^2\beta\lambda)^2 > 4\beta$, this implies that both roots are real. Further, by noticing that $\alpha^2\beta\lambda > 0$, we can infer that one root is greater than one and the other is less than one – i.e., $\theta_1 > 1$ and $\theta_2 < 1$.[9] Then we can derive the unique solution to the second-order difference equation as follows,

$$\theta_1 E_t(\pi_{t+2} - \pi^*) = E_t(\pi_{t+1} - \pi^*) - \alpha \sum_{i=0}^{\infty} \theta_2^i E_t \Psi_{t+i}. \tag{14}$$

Recalling that $E_t\pi_{t+2} = E_t\pi_{t+1} + \alpha E_t y_{t+1}$ from the AS equation, the solution to the second-order difference equation can be transformed as

$$E_t y_{t+1} = \left(\frac{1-\theta_1}{\alpha\theta_1}\right)(E_t\pi_{t+1} - \pi^*) - \frac{1}{\theta_1}\sum_{i=0}^{\infty} \theta_2^i E_t \Psi_{t+i}, \tag{15}$$

which can be regarded as the optimal reaction function of $E_t y_{t+1}$ in response to $E_t\pi_{t+1}$ (the expected inflation rate at period t) and the streams of expected values of $\{\psi_{t+i}\}_{i=1}^{\infty}$ at period t.

Since our goal here is to derive the analytical expression of the optimal monetary policy reaction function in terms of the nominal interest rate, we substitute $E_t y_{t+1}$ and $E_t\pi_{t+1}$ using the IS equation and AS equation, respectively. This yields the following optimal monetary policy reaction function,

$$i^*(\pi_t, y_t) = \pi_t + \left(\alpha + \frac{\rho\theta_1 + \theta_1 - 1}{\delta\theta_1}\right) y_t + \left(\frac{\theta_1 - 1}{\alpha\delta\theta_1}\right)(\pi_t - \pi^*) + \underbrace{\left(\frac{1}{\delta\theta_1}\right)\sum_{i=0}^{\infty} \theta_2^i E_t \Psi_{t+i}}_{\Theta} \tag{16}$$

where

$$\theta_1 = \frac{\alpha^2\beta\lambda + \beta + 1 + \sqrt{(\alpha^2\beta\lambda + \beta + 1)^2 - 4\beta}}{2}, \quad \theta_1 > 1$$

$$\theta_2 = \frac{\alpha^2\beta\lambda + \beta + 1 - \sqrt{(\alpha^2\beta\lambda + \beta + 1)^2 - 4\beta}}{2}, \quad 0 < \theta_2 < 1. \tag{17}$$

[9] This can be easily seen by evaluating the polynomial function, $f(z) = z^2 - (1+\beta+\alpha^2\beta\lambda)z + \beta$, at $z = 1$. Since $f(1) = -\alpha^2\beta\lambda < 0$ and $f''(z) > 0$, it should be the case that one of the characteristic roots be greater than one and the other be less than one.

The above optimal monetary policy reaction function (16) is valid for the state (π_t, y_t) such that $i_t^* > 0$. For any other state where the zero lower bound constraint is binding, the optimal monetary policy is trivially $i_t^* = 0$.

Some remarks are in order regarding the fourth term of eqn (16), Θ. First of all, it should be noted that Θ can be expressed as a function of the current state π_t and y_t. This result indirectly follows from the assumption that the Contraction Mapping Theorem holds for the value function in eqn (6). Since a central bank's value function is a function of the current state (π_t, y_t) and the functional shape is time-invariant, as a consequence, the optimal feedback rule is also a function of the current state (π_t, y_t) and is time-invariant as well – i.e., the optimal monetary policy reaction function can be expressed as $i^*(\pi_t, y_t)$ for any period t. Now, since the first term through the third term in eqn (16) is obviously a linear (and time-invariant) function of (π_t, y_t), this automatically requires the term Θ to be a time-invariant function of the current state π_t and y_t. In order to see this result in a more intuitive way, notice that the term Θ can be expressed in terms of the sequence of $\{E_t\psi_{t+i}\}_{i=0}^{\infty}$. Further, notice that by differentiating eqn (6) with respect to y_{t+1} and invoking the envelope theorem, it follows that

$$E_t V_y(\pi_{t+1}, y_{t+1}) = -\frac{1}{\beta} E_t \phi_{t+1}.$$

By exploiting FOC(7) and using the law of iterated expectation, we can link $E_t\psi_{t+i}$ and $E_t V_y(\pi_{t+i}, y_{t+i})$ as follows

$$E_t V_y(\pi_{t+i}, y_{t+i}) = -\frac{1}{\beta\delta} E_t \psi_{t+i} \quad \text{for } \forall i > 0. \tag{18}$$

Now, using backward-substitution, π_{t+i} and y_{t+i} could be expressed in terms of the current state (π_t, y_t) and streams of random shocks $\{\upsilon_{t+j}\}_{j=1}^{i}$ and $\{\varepsilon_{t+j}\}_{j=1}^{i}$ under the optimal policy conduct. Thus, the expected value of $V_y(\pi_{t+i}, y_{t+i})$ conditioned upon the information set available at period t can be, indeed, shown to be a function of the current state π_t and y_t – i.e., $E_t V_y(\pi_{t+i}, y_{t+i}) = g_i(\pi_t, y_t)$. In this manner, at least in principle, it is possible to express $E_t\psi_{t+i}$ for any i as a function of the current state π_t and y_t, which in turn implies that the term Θ is a function of the current state as well. However, due to the nature of the LQ stochastic problem with a bounded control variable, the explicit functional form of Θ in terms of π_t and y_t does not exist in general. This is one of the reasons why we rely on the numerical method in Section 3. As a matter of fact, we numerically demonstrate Θ to be a function of the current state in Section 3.

Second remark regarding the term Θ is the non-linearity with respect to the current state variables π_t and y_t. It should be noted that the optimal monetary policy reaction function (16) is linear in state variables π_t and y_t except for the fourth term, Θ. Suppose there is no zero lower bound constraint, so that $\Psi_{t+i} = 0$ for any i. Then the optimal reaction function is linear and is exactly the same as the familiar result shown in Svensson (1997a), which can be regarded as one type of the Taylor rule. However, once the zero lower bound is introduced, even if the constraint is not binding at the current state, there exists a non-zero probability that such a constraint may bind in the future, which alters the optimal monetary policy conduct from the case where there is no zero lower bound. In other words, $E_t\psi_{t+i}$'s (which can be expressed as a function of the current state π_t and y_t for any i) will, in general, take positive values that the optimal reaction function will deviate from the standard Taylor rule. Indeed, this very deviation from the Taylor rule is captured by the term Θ in eqn (16). As such, by recalling that the term Θ is a function of the sequence of $\{E_t\psi_{t+i}\}_{i=0}^{\infty}$ and also considering that $E_t\psi_{t+i}$'s will likely take higher (lower) values when the nominal interest rate is close to (apart from) the zero lower bound constraint, there is no reason to believe that the term Θ is linear in the current state variables. Analyzing the term Θ further, the following two propositions can be made with regard to the qualitative properties of the optimal monetary policy reaction function in the presence of the zero lower bound constraint.

Proposition 1 (Expansionary Policy) *Let $i^{Taylor}(\pi_t, y_t)$ be the optimal monetary policy reaction function when there is no zero bound on the nominal interest rate. Let $i^*(\pi_t, y_t)$ be the optimal monetary policy reaction function in the presence of a zero bound on the nominal interest rate. Then for any state (π_t, y_t) where i^* is strictly greater than zero, the monetary policy i^* will be at least as expansionary as the monetary policy i^{Taylor}, i.e., $i^* \leq i^{Taylor}$.*

Proof. See Appendix. ■

Proposition 2 (Aggressive Policy) *For any state (π_t, y_t) where i^* is strictly greater than zero, the monetary policy i^* will be at least as aggressive as the monetary policy i^{Taylor}, i.e., $\partial i^*/\partial \pi_t \geq \partial i^{Taylor}/\partial \pi_t$ and $\partial i^*/\partial y_t \geq \partial i^{Taylor}/\partial y_t$.*

Proof. See Appendix. ■

2.3 Discussion

Proposition 1 implies the conventional wisdom described as "pre-emptive" monetary policy in terms of interest rate control implementation. But since the pre-emptiveness seems to have much broader meanings,[10] we suggest that one could acquire much clear insight about the idea if we regard this as an analogy of the well-known *precautionary savings*. As we will discuss later, since the value function is not quadratic any more, the certainty equivalence property does not hold in spite of the quadratic period-by-period

[10] For instance, see Blinder (2000).

loss function. Carroll and Kimball(1996) proved that when the third derivative of the utility function is strictly positive, the consumer increases savings when the income risk increases. What essentially matters here is the positive third derivative of the period-by-period utility function which makes the value function non-quadratic. In comparison with our case, the zero lower bound makes the value function non-quadratic, which induces the central bank to increase its *savings* (interpreted as a *lower* nominal interest rate) in the face of future risk. Thus the mechanism behind this kind of pre-emptive monetary policy is quite similar to that of precautionary savings at least from the mathematical point of view.

Proposition 2, which refers to the aggressiveness of optimal monetary policy, is closely related to the concavity of the optimal policy reaction function. Given that the optimal policy reaction function eqn (16) converges to the Taylor rule as the inflation rate and the output gap approach positive infinity, Proposition 2 seems to imply the concavity of the optimal policy reaction function. However, these two concepts are slightly different. Proposition 2 merely states that the slope of the policy reaction function (defined as aggressiveness) is never less than that of the Taylor rule, while concavity of the policy reaction function implies that the slope itself is monotone and non-increasing in the output gap and the inflation rate. In other words, concavity means that central banks have to respond to changes in the state of the economy more substantially in recessions than in booms. We do not provide the analytical proof of this kind of concavity, but rather we numerically show it in the following section.

Before we move on to the next section, it is useful to shed some light on the monetary policy debate between those proponents suggesting to "preserve the ammunition" and those suggesting to "take aggressive action" in face of the zero lower bound constraint. Proponents[11] of "preserving the ammunition" suggest cutting the interest rate parsimoniously as the interest rate approaches the zero bound constraint. Their reasoning is as follows. By cutting the interest rate parsimoniously, a central bank can preserve room for maneuvering. In this way, a central bank can preserve power to stimulate the economy which can be potentially used to salvage the economy, especially when the economy has entered a deflationary spiral. On the other hand, proponents[12] of "taking aggressive action" suggest cutting the interest rate aggressively as the interest rate approaches the zero bound constraint. Their reasoning is basically in line with our argument in this paper. In other words, in the presence of the zero bound constraint, it is in the interest of a central bank to lean toward more expansionary monetary policy so that they can prevent the economy from falling into a deflationary spiral. Thus, in this regard, this

[11] This view is occasionally seen in some non-technical journals or newspapers intended for the general public. However, to the best of our knowledge, we have never encountered this kind of view in academic journals.

[12] This is the view shared by Reifschneider and Williams (2000), Orphanides and Wieland (2000), Ahearne *et al.* (2002), and Hunt and Laxton (2003) among others.

paper can be thought to be another paper that supports the view of "taking aggressive action" in the face of the zero bound constraint. Now, in the context of the model in this paper, what will go wrong when a central bank conducts monetary policy so as to preserve the ammunition? A major flaw is that such policy is putting a *contractionary* pressure (relative to the optimal monetary policy) on the economy when the economy is on the verge of a deflationary spiral. Namely, by not providing a sufficient stimulus to the economy, a central bank is inadvertently aiding, rather than pre-empting, the deflationary pressure to grow, raising the risk of falling into a deflationary spiral. Although such a policy might preserve some room for a rate-cut even when the economy actually falls into a deflationary spiral, such a rate-cut will be "too-little, too-late" in extricating the economy from a state of deflationary spiral. This is the reason why the policy conduct of "preserving the ammunition" is deemed sub-optimal, moreover destructive, at least in the model considered in this paper.

3 Numerical Analysis

Although the analytical expression has been derived in eqn (16), it is in general not possible to obtain a closed form expression of the optimal policy reaction function. In order to further analyze the nature of the optimal policy reaction function, we take a numerical approach in this section.[13] In particular, we adopt the numerical approximation method known as the collocation method (see Judd, 1998, Ch.11 and 12 and Miranda and Fackler, 2002, Ch.8 and 9) in this paper. As can be seen from the Bellman equation posed in eqn (6), the dynamic programming problem requires us to solve the mathematical problem known as the functional fixed-point problem. In other words, our goal is to pin down the function $V(\pi, y)$ such that the Bellman equation (6) is met. Now, due to the infinite-dimensional nature of a functional space, one is inherently faced with an infinite-dimensional fixed-point problem. The idea of the collocation method is to reduce this infinite-dimensional fixed-point problem into a finite-dimensional problem by approximating the function $V(\pi, y)$ by a finite number of basis functions[14] – a strategy also known as discretization of the functional space. Thus, in the sense that the collocation method is no more than an approximation, the numerical analysis in this section should not be regarded as the proof of the qualitative features of the optimal policy reaction function shown in Section 2. Rather, the strength of the numerical method is that it enables us to verify the qualitative features and, further, enables us to evaluate the optimal policy reaction function quantitatively.

[13] For the pioneering work in taking a numerical approach to the optimal policy reaction function, see Orphanides and Wieland (2000).

[14] For further details and explanations regarding the collocation method adopted in this paper, see the Appendix.

3.1 Parameters

As a preliminary step to implementing the collocation method, we first estimate the parameters of the state transition equations. Table 1 reports the OLS estimation results of eqn (2) and eqn (3) based on Japanese data for the last two decades. In order to control for the influence of the real exchange rate on the output gap, we included the real exchange rate as a regressor in the IS equation. The only potentially controversial matter is the theoretical restriction imposed on the coefficient of π_t, which we call η for convenience. Theory requires η to be equal to one so that monetary policy is assured to be neutral in the long run. This restriction turned out to be reasonable for the Japanese data. As seen in Table 1, the Wald-test did not reject the null hypothesis of $\eta = 1$ at a 5% significance level. Thus, we have restricted η equal to one when estimating the AS equation. The estimation result is reported in the bottom portion of Table 1. Based on this estimation result, we now proceed to implement the collocation method.

3.2 Numerical Results

3.2.1 Optimal Monetary Policy with a Zero Bound

The collocation nodes are set uniformly for both state variables π and y in the range [-10, 10]. As for the benchmark case, we have set the target inflation rate, π^*, to 2%, the central bank's preference parameter, λ, to 1 and the standard deviation of both stochastic disturbance terms υ and ε to 1.5. The central bank's discount rate, β, has been set at 0.6[15].

Figure 1 shows the interpolated optimal policy reaction function with a zero bound on the nominal interest rate. This figure should be interpreted as the numerical counterpart of the optimal policy reaction function, eqn (16), derived in Section 2. A careful inspection of Figure 1 reveals some non-linearity of the policy reaction function, especially when the nominal interest rate is near the zero bound. This nonlinear feature observed in Figure 1 is in sharp contrast to the linear optimal policy reaction function (i.e., Taylor rule), which is implied when there is no zero bound on the nominal interest rate.

Figure 2 shows the interpolated value function with a zero bound which corresponds to the value function in eqn (6). The qualitative feature to be noted here is the asymmetry of the curvature, which implies that the value function is not quadratic.[16] Recall that the value function means the *cost-to-go,* a minimized sum of period-by-period losses given the current state, for the central bank. We can observe that the surface of the computed value function in Figure 2 goes up sharply in the region where both the

[15] At a first glance, the choice of the discount rate may appear to be counter-intuitive. This issue will be further discussed later.

[16] Note that when there is no bound on the interest rate, the value function will be quadratic.

inflation rate and the output gap are negative. This means that stabilizing the economy is costly to the central bank when the economy is in that particular state. Practitioners or central bankers often describe this phenomenon as a "deflationary spiral" in the sense that the economy is almost impossible to stabilize or is on a downward divergent path to a serious recession. This deflationary spiral phenomenon not only seems to be the most notable harm of the zero lower bound *qualitatively*, but also we emphasize that this harm is *quantitatively* a serious obstacle for the actual implementation of monetary policy. We have already mentioned that we set β, the central bank's subjective discount rate, equal to 0.6. While we were computing the numerical results, we found it difficult to reach convergence in the value function when β was set higher than 0.6. This implies that the Contraction Mapping Theorem for the Bellman equation (6) does not hold when β is set larger than 0.6. In this regard, recall that β is a component of the roots (see eqn (17)) of the characteristic equation (i.e., eqn (13)) presented in Section 2. We need a small θ_2 so that the stationary optimal policy reaction function could exist. A sufficiently small β ensures that this necessary condition for the existence of a stationary optimal policy reaction function is met, otherwise the economy diverges so rapidly that an optimal reaction function does not exist. Putting it another way, in order to keep the optimal policy reaction function and the value function visible under both deflationary and non-deflationary spiral regimes, we need a sufficiently low value of β. Although it may not make sense to pass a value judgement on β, 0.6 seems to imply that given that all those parameters in the transition matrix are correctly estimated, the central bank in this economy is confronted with an extremely hard task to accomplish.

For comparison, we depict the optimal policy reaction function without a zero bound (i.e., Taylor rule) and the reaction function with a zero bound under both deterministic (i.e., σ set to zero) and stochastic (i.e., σ set to 1.5) environments in Figure 3 and Figure 4. Figure 3 shows the relationship between the nominal interest rate and the output gap, holding the inflation rate constant. Figure 4 shows the relationship between the nominal interest rate and the inflation rate, holding the output gap constant.

Figures 3 and 4 vividly show the key features of the optimal reaction function characterized by the propositions in Section 2. First of all, the level of the optimal policy reaction function with a zero bound is positioned lower than that of the Taylor rule. Note that this very gap between the optimal policy reaction function and the Taylor rule represents the term Θ in eqn (16). This gap implies that it is optimal for a central bank to pursue more expansionary monetary policy (relative to the Taylor rule) in the presence of the zero bound constraint, which is consistent with Proposition 1 proved in Section 2. The intuition is as follows. As a side-effect of the presence of the zero bound constraint, the deflationary spiral region emerges in the state space. With a threat of a deflationary spiral present, it is in the interest of the central bank to lean toward expansionary monetary policy than in the case where the threat is non-existent.

The second observation is related to Proposition 2 proved in Section 2. As can be seen from Figure 4, the slope of the optimal policy reaction function with respect to the inflation rate is steeper than that of the Taylor rule (provided that the zero lower bound constraint is non-binding).[17] Further, we observe the optimal policy reaction function to be concave in the state variable (provided that the zero lower bound constraint is non-binding). This concavity is closely related to the concept of "aggressive" monetary policy conduct. In other words, as the nominal interest rate approaches the zero bound, the threat of a deflationary spiral becomes more realistic so that it is in the interest of a central bank to pursue even more expansionary monetary policy. This particular behavior by a central bank can be interpreted as "aggressive" monetary policy conduct. Moreover, provided that the risk of a deflationary spiral is monotonically increasing as the nominal interest rate is decreasing toward zero, it is natural to conceive the optimal policy reaction to be concave with respect to the state variables.[18]

Finally, one can also observe the violation of the certainty-equivalence property[19] of optimal monetary policy by comparing the optimal policy reaction function under a deterministic environment and a stochastic environment. The reaction function under a stochastic environment is positioned lower than that in a deterministic environment. The basic question here is: why does the uncertainty change the qualitative shape of the optimal policy reaction function? It is reasonably intuitive to think that the larger the variance of the state variables, the higher the probability for them to get caught in the deflationary spiral region. Accordingly, it is optimal for the central bank to take expansionary policy at an early stage, so that it can keep the economy under its control with a higher probability in the future.

3.2.2 Robustness Check

The qualitative features observed in Figures 3 and 4 may depend upon the numerical values of the parameters. In order to test the robustness of these qualitative features, we conduct a sensitivity analysis in this subsection. In particular, we are interested in the sensitivity of the policy reaction function with regard to the central bank's unobservable parameters: the inflation target, π^*, and the preference parameter, λ.[20]

[17] Of course, when the zero lower bound constraint is binding, the optimal interest rate is trivially equal to zero – i.e., $i^* = 0$. It should be noted that, as discussed in Section 2, Proposition 2 is only valid for a state where the nominal interest rate is non-binding – i.e., $i^* > 0$.

[18] Although this concavity feature of the optimal policy reaction function can be clearly seen from the numerical demonstration, we were not able to provide the analytical proof of concavity in this paper.

[19] It should be noted that when there is no constraint on the nominal interest rate, the optimal policy reaction function will be exactly the same for both deterministic and stochastic cases. This is known as a certainty-equivalence property of the Taylor rule.

[20] It should be noted that since we regard eqns (2) and (3) as the *true* economy and the parameter estimates in Section 3.1 as given, we did not conduct sensitivity analysis with regard to the economy's parameters ρ, δ, and α.

Figures 5 shows the results of the sensitivity analysis in a deterministic environment. Figure 5 depicts the contours of the optimal policy reaction functions with and without a zero bound. The solid line represents the level curve of the optimal policy reaction function without a zero bound. The dashed line represents the level curve of the optimal policy reaction function in the presence of a zero bound. In the sensitivity analysis, inflation targets of 1% and 3% have been considered, while for the preference parameter, the values of 0.5, 1 and 2 have been considered. As can be observed for each parameter value for π^* and λ, the optimal policy reaction function is more expansionary than the Taylor rule, especially when the inflation rate is low and the output gap is high. The difference is as much as 7%.

Figures 6 shows the results of a sensitivity analysis in a stochastic environment. The standard deviation of the stochastic disturbance term has been set at 1.5. As can be observed from Figure 6, for each parameter considered, the differences in implied nominal interest rates are pervasive throughout the state space and especially apparent when the nominal interest rate is near the zero bound. Thus, for any parameters considered, the concavity of the policy reaction function seems to be unaltered. By observing Figure 6, one can again identify the clear non-linearity of the level curve, which supports the concavity of the policy reaction function in the state variables.

Finally, a very careful comparison of Figures 5 and 6 reveals the non-certainty-equivalence property of the optimal policy reaction function. While the level curves of the policy reaction function in Figure 5 are more kinked, the level curves in Figure 6 are much smoother – an evidence of the violation of the certainty-equivalence. Thus, in the reasonable range of those parameters, the basic qualitative features of the policy reaction function seem to be preserved.

4 Empirical Analysis: Japanese Monetary Policy in the 1990s

4.1 Brief Review of the Japanese Economy in the 1990s

Thus far we have discussed the theoretical and numerical implications regarding optimal monetary policy when the nominal interest rates are bounded at zero. In this section, in order to verify those implications empirically, we take the case of Japanese monetary policy in the 1990s – so-called the "lost decade."[21] Before we actually conduct the empirical analysis, it is useful to briefly review the Japanese economy in the 1990s.

[21] For a thorough review regarding Japan's 'lost decade' and the important lessons learned, see, for instance, Posen (1998), Fujiki et al. (2001), Mori et al. (2001), Ahearne *et al.* (2002) and Callen and Ostry (2003).

Figure 7 shows the time series plot of the uncollateralized overnight call rate,[22] the inflation rate,[23] and the output gap[24] from 1983:Q2 to 2002:Q3. As can be seen from the figure, Japanese economy plunged into a serious recession accompanied by a sharp decline in both output and inflation after the so-called "bubble" boom in the late 1980s. During the first half of the 1990s, the output gap decreased by 6-7 percentage points, with inflation going down from approximately 4% per year to almost zero by the mid 1990's. In the meantime, the Bank of Japan (BOJ) successively cut the call rate from about 8% down to 0.5% in 1995 to stimulate the economy. Turning to the latter half of the 1990s, the Japanese economy basically remained stagnant, except for the temporary economic recovery during 1996-97. In the face of another deep recession in the late 1990s accompanied by mild deflation, the BOJ adopted an unprecedented zero interest rate policy on February 12, 1999.[25] Since then, with the exception of a short-lived recovery period during the IT boom, which occurred from 2000 to 2001, the BOJ has been committed to the zero interest rate policy.

4.2 Qualitative Analysis

The qualitative features of the optimal monetary policy reaction function implied in Section 3 are that it is increasing and concave (provided that the zero bound constraint is not binding) in π and y, especially when the nominal interest rate is near zero. In this

[22] Data of the overnight call rate are from the Bank of Japan's *Financial and Economics Statistics Monthly*. We computed the quarterly average of the call rate to construct the data set for the nominal interest rate.

[23] The inflation rate is defined as the percent change of the Consumer Price Index (CPI; ex-fresh food, seasonally adjusted, quarterly average) from the same quarter of the previous year. It should be noted that the CPI during the period contains the effects of the introduction of the 3% consumption tax (a variant of a value-added tax) in April, 1989, and its successive 2 percentage point raise in April, 1997. We have adjusted these effects following Kitagawa and Kawasaki (2001). Basically, the adjustment equalizes the inflation rate at the month directly before (i.e., March) and directly after (i.e., April) the introduction (or raiseing) of the consumption tax and adjusts the level of the CPI accordingly. For more details regarding the adjustment method, see Kitagawa and Kawasaki (2001). The time series data of CPI is available from the Statistics Bureau's homepage; (`http://www.stat.go.jp/english/data/cpi/index.htm`).

[24] The estimates of the output gap during the period are due to Hirose and Kamada (2002). Basically, they estimate the potential GDP growth rate taking into account the innovation in technology, the growth rate of the production factors, and the time-variant Non-Accelerating Inflation Rate of Unemployment (NAIRU). For more details regarding their estimation methodology, see Hirose and Kamada (2002). It should be noted that the output gap in Hirose and Kamada (2002) is defined as the deviation of the current output from its maximum potential level. Thus, by construction, their estimates of the output gap always take a negative number. In order to suit their data to our definition, we have simply demeaned their estimates of the output gap. We would like to acknowledge Yasuo Hirose for graciously providing the data to us. Their estimates of the output gap are available to the public upon request.

[25] As for the zero interest rate policy, the Bank of Japan (1999) officially states: "Since the launch of the zero interest rate policy in February, the Bank has continued to provide the financial market with ample funds to guide the overnight call rate as low as possible, currently at virtually zero percent $\cdots$."

subsection, using the time series data shown in Figure 7, we test these qualitative implications by estimating the monetary policy reaction function of the BOJ from 1983:Q2 to 2002:Q3.

Several issues need to be addressed in estimating the policy reaction function. If the closed form expression of the policy reaction function is known, it is ideal to specify the regression form accordingly. Unfortunately, however, since the closed form expression of the policy reaction function does not exist, we adopt a polynomial regression to capture the increasing and concave nature of the policy reaction function. The choice of polynomial regression can be justified as an approximation of the policy reaction function via Taylor series expansion. Further, since the goal of this subsection is to detect the qualitative nature of the policy reaction function, polynomial regression is deemed sufficient for our purpose. Under polynomial regression, we expect that the coefficients on π_t^2 and/or y_t^2 to be significantly different from zero, indeed negative, if Japanese monetary policy conduct was consistent with our model's implication.

The benchmark specification is in the spirit of the Taylor rule.

Benchmark Specification:

$$i_t^* = c_0 + c_1\pi_t + c_2 y_t + e_t \tag{19}$$

where e_t is assumed to be independently and identically normally[26] distributed with variance σ^2.

For alternative specifications, we consider the following polynomial form.

Alternative Specification:

$$i_t^* = c_0 + c_1\pi_t + c_2 y_t + c_3\pi_t^2 + c_4 y_t^2 + c_5\pi_t y_t + e_t \tag{20}$$

where e_t is, again, assumed to be independently and identically normally distributed with the variance σ^2. The alternative specification – which can be interpreted as a second order Taylor series approximation of the policy reaction function – is capable of capturing the concavity of the policy function with respect to π and y due to the presence of square-terms. Some caution needs to be exerted in estimating the above specifications. Due to some zero interest rate observations of the call rate,[27] if we simply conduct least squares estimation such as OLS, the coefficient estimates will likely be biased (see, for instance, Cheung and Goldberger, 1984). In order to obtain unbiased estimates, we

[26] The assumption of normal distribution is purely auxiliary. However, this assumption is necessary for the following Tobit analysis which calculates the exact probability of observing the positive nominal interest rate.

[27] It should be noted that, in practice, perfect fine-tuning of the overnight call rate is not feasible since the BOJ's open market operation can only be implemented in a discrete manner. Indeed, the quaterly-average of the call rate during the zero interest rate policy was slightly above zero (0.03% to be specific). Nevertheless, taking into account the BOJ's intention during the zero interest rate policy, we assumed that the call rate was binding at zero during the period.

therefore conduct Tobit analysis. Also, as can be seen from Figure 7, the inflation rate and output gap move in a synchronous fashion. Thus, empirically speaking, there is an issue of multicollinearity among the regressors π^2, y^2, and πy. In order to check the robustness of the alternative specification in the presence of multicollinearity, we estimate and test four alternative specifications.

Following the standard procedure in Tobit analysis, we assume that "latent" interest rate i^* is observed only if it is greater than zero. Otherwise, it is left-censored at zero. This is expressed mathematically as follows,

$$i_t = \left\{ \begin{array}{cc} 0 & \text{if } i_t^* \leq 0 \\ i_t^* & \text{if } i_t^* > 0 \end{array} \right\}.$$

Thus, under this Tobit model, the probability of the observed nominal interest rate to be zero is given as $\Pr(i_t = 0) = 1 - \Phi\left(\mathbf{c}'\mathbf{x}_t/\sigma\right)$ and the likelihood of an observed positive nominal interest rate is given as $f(i_t) = \phi\left((i_t - \mathbf{c}'\mathbf{x}_t)/\sigma\right)/\sigma$, where $\mathbf{c}$ stands for the vector of coefficients, $\mathbf{x}_t$ is the vector of regressors, $\Phi(\cdot)$ is the standard normal *c.d.f.* and $\phi(\cdot)$ is the standard normal *p.d.f.* Therefore, the log-likelihood function of the observed interest rates $(i_1, ..., i_T)'$ can be expressed as

$$\ln L = \sum_{t=1}^{T} \left\{ \mathbf{1}\left(i_t > 0\right) \left[-\ln(2\pi)/2 - \ln\sigma - \frac{\left(i_t - \mathbf{c}'\mathbf{x}_t\right)^2}{\sigma^2} \right] + \mathbf{1}(i_t = 0) \ln \left[1 - \Phi\left(\frac{\mathbf{c}'\mathbf{x}_t}{\sigma}\right) \right] \right\} \tag{21}$$

where $\mathbf{1}(\cdot)$ is an indicator function which takes the value of one, if the condition inside the parenthesis is true, and zero otherwise. The result of the Maximum Likelihood (ML) estimation is reported in Table 2.

The top portion of Table 2 reports the estimates of the coefficient vector $\mathbf{c}$ and σ for the benchmark and four alternative specifications. The benchmark specification is in the spirit of the Taylor rule. As can be seen, the coefficient estimates for all the regressors were statistically significant at the 1% level. Further, the coefficient estimates for the inflation rate (which was 1.585) and output gap (which was 0.570) were remarkably similar to those estimates by Taylor (1993). Alternative 1 is the most general specification including the square terms π^2 and y^2 and the cross-products πy. Alternative 2 allows for the concavity both in π and y, but excludes the cross-products πy. Alternative 3 allows for the concavity in π, but not in y. Finally, Alternative 4 allows for the concavity in y, but not in π. The coefficient estimates on π^2 and y^2 were negative for all the specifications, but the results were not statistically significant for Alternatives 1 and 2. Even for Alternatives 3 and 4, which are deemed relatively free from the multicollinearity problem, the coefficient estimates for π^2 and y^2 were statistically significant at the 10% significance level, but marginally insignificant at the 5% level. Although the evidence was not too strong, the results from the Tobit estimation are basically in line with the qualitative implication that the policy reaction function is (locally) concave.[28]

In addition, in order to test the overall performance of the benchmark specifications against the alternative specification, we conducted a log-likelihood ratio test. The null hypothesis is that, for all the second order terms in each alternative specification, the coefficients are jointly equal to zero. By restricting the coefficient of second order terms as such, each alternative specification reduces to the benchmark specification. Therefore, under the null hypothesis, the test can be motivated as a performance test of the benchmark specification, which is a proxy for the Taylor rule, against the alternative specification, which allows for the policy reaction function to be concave. The result of the log-likelihood ratio test is reported at the bottom portion of Table 2. As can be seen, the test rejects the null hypothesis for all the cases at the significance level of 10%, which is evidence (albeit weak) against the benchmark specification. This test result is additional evidence that supports the concavity of the policy reaction function when the nominal interest rate is near zero.

4.3 Quantitative Analysis

In the previous subsection, we have empirically verified the qualitative implications from Section 3, especially the concavity of the policy reaction function. However, qualitative analysis is unindicative about the optimality of the *level* of the nominal interest rate. In order to verify whether the level of the actual nominal interest rate was consistent with the optimal monetary policy discussed in this paper, a quantitative analysis is indispensable. In this subsection, we quantitatively analyze Japanese monetary policy conduct during the 1990s by comparing the actual call rate path with the simulated path implied by the optimal policy reaction function computed in Section 3.[29]

Figure 8 shows the actual call rate, the predicted path from the estimated Taylor rule (i.e., the benchmark case in Table 2), and the simulated path implied by the optimal policy reaction function in Section 3. Let us first turn to the estimated Taylor rule. As can be seen from the figure, the estimated Taylor rule matches the actual movement of

[28] From August 2000 to March 2001, the BOJ adopted a controversial monetary policy. Despite the weak movement in output with no sign of deflation abating, the BOJ raised the overnight call rate during that period. Although not reported in the main section, we have also conducted a counterfactual estimation assuming that the BOJ kept the zero interest rate policy during this period. Indeed, under this counterfactual estimation, the coefficient estimates for all the alternative specifications were significantly negative. Thus, one of the reasons that we were only able to find weak qualitative evidence regarding the concavity of the policy reaction function stems from the controversial monetary policy conduct during this period. Although we have no intention to pass any value judgment here, we can at least say that the actual monetary policy conduct from August 2000 to March 2001 deviated from the optimal monetary policy reaction function considered in this paper.

[29] Parameter values used here are those of the benchmark case in Section 3.1.

the call rate fairly well until the mid 1990s, which is consistent with the findings reported in McCallum (1999) and Muto and Kamada (2000). This can be considered evidence that Japanese monetary policy was following the naive Taylor rule until the mid 1990s. In contrast, turning to the period from 1995 to 1998, the estimated Taylor rule suggests that the call rate should have been raised to as much as 4 percent in response to the temporary expansion of the Japanese economy, while, in reality, the BOJ maintained an almost zero interest rate during that period – an anomaly which remains unexplained by the naive Taylor rule. Taking the implication of the estimated Taylor rule at face value, this implies that the actual monetary policy conduct during this period was too expansionary.

Now, let us turn to the simulated path implied by the optimal monetary policy reaction function in Figure 8. By comparing[30] the simulated path and the actual path of the call rate, two main observations can be made. The first observation concerns the first half of the 1990s. In response to sharply declining output accompanied by disinflation, the BOJ consecutively cut the call rate from 8 percent to 0.5 percent during the period from 1991 to 1995. At first glance, especially in light of the estimated Taylor rule, this monetary policy conduct by the BOJ does not seem to be too problematic. However, turning to the simulated path implied by the optimal policy reaction function during the period, we observe an aggressive decline in the simulated call rate reaching 0 percent as early as 1993.[31] This implication from the optimal policy reaction function stands in sharp contrast to the naive Taylor rule. Taking the implication from the simulated path for granted, this implies that the actual conduct of Japanese monetary policy was too contractionary and too slow in responding to the falling output and inflation rate during the early 1990s. In particular, in comparison to the aggressive rate-cut implied by the simulated path from 1992 to 1995, the gradual reduction of the call rate during this period reveals the BOJ's hesitation in breaking the psychological floor of 2%. According to our model's implications, this hesitation – which can be considered as a "preserving the ammunition" policy – was clearly sub-optimal and possibly destructive in the sense that the bank was applying contractionary pressure when the Japanese

[30] The simulated path has been calculated assuming a time-invariant inflation target of 2 percent. Some caution needs to be exerted when comparing the simulated path and the actual path. First of all, it may be possible that the BOJ's implicit inflation targets were different in the early 1990s and late 1990s. Indeed, during the early 1990s, the average Japanese inflation rate was essentially higher than that in the late 1990s, which might imply that there was a break in the implicit inflation target. Further, it may be possible that the BOJ might be pursuing more complex targets, not confined to the inflation rate or output gap. When comparing the actual path and simulated path, these possibilities must be kept in mind.

[31] This aggressive decline in the nominal interest rate relative to the Taylor rule is consistent with what Reifschneider and Williams (2000) called "forward-looking adjustment" strategy. They suggest policymakers cut the nominal interest rate pre-emptively if they anticipate the zero lower bound to bind in the near future.

economy was already on the way to a liquidity trap. Finally, by the end of 1995, the BOJ aggressively decreased the call rate from 2% to 0.5%, although, in retrospect, this stimulus seems to have been "too-little, too-late" in preventing a liquidity trap. One possible interpretation is that the BOJ may have underestimated the risk caused by the zero lower bound and therefore did not dare to take aggressive action in cutting the call rate in order to pre-empt a possibility of being caught by the liquidity trap in the future.

The second observation concerns the latter half of the 1990s. Despite the temporary economic recovery from 1996 to 1997, the BOJ did not react to this boom, but maintained a low interest rate policy (the actual call rate was fixed at 0.5 percent during the period). As we have seen, the predicted path from the estimated Taylor rule suggests the raising of the call rate during this period, implying that actual monetary policy was too expansionary. In contrast, turning to the simulated path during this period, we observe that the simulated call rate was kept at zero percent, except for a marginal rise in 1997.[32] Taking the indication by the simulated path for granted, this suggests that the policy conduct by the BOJ during this period was basically consistent with the optimal monetary policy reaction considered in this paper. One possible interpretation is that the policy stance of the BOJ was sufficiently "precautionary" that it kept the nominal interest rate low during this period in order to reduce the risk of falling into a deflationary spiral. Being precautionary to avoid a deflationary spiral gives enough reason for the BOJ to conduct more expansionary monetary policy than the naive Taylor rule. This interpretation is essentially consistent with the nonlinear reaction function advocated by Blinder (2000), Reifschneider and Williams (2000), and Orphanides and Wieland (2000).

In sum, Japanese monetary policy seemed to have closely followed the naive Taylor rule during the early 1990s, showing no sign of aggressiveness in pre-empting the risk of falling into a deflationary spiral. In contrast, the low interest policy during the late 1990s turned out to be fairly consistent with the optimal monetary policy reaction considered in this paper, showing some signs of precautionary[33] monetary policy conduct. In this sense, the BOJ's monetary policy conduct during 1990s seems to have followed two different types of policy regimes - i.e, the naive Taylor rule in the first half of the decade and the "precautionary" monetary policy in the latter half of the decade. Then, the

[32] This belated rise of the nominal interest rate relative to the Taylor rule is consistent with what Reifschneider and Williams (2000) called the "backward-looking adjustment" strategy. They suggest policymakers hold down the nominal interest rate longer than the Taylor rule during the recovery period from a liquidity trap. Thus, the optimal monetary policy reaction function considered in this paper is successful in replicating the "forward-looking" and "backward-looking" strategies proposed by Reifschneider and Williams (2000).

[33] The terms "aggressive" and "precautionary" may sound confusing for some readers. In this paper, both terms are essentially refering to the same concept proved in Proposition 2. However, since the term "aggressive" sounds awkward to be used in the recovery phase of the economy, we instead use the term "precautionary" in refering to the same concept.

question is what prompted the BOJ to change its monetary policy stance before and after the mid 1990s? One possible explanation comes from the balance sheet channel advocated by Bernanke and Gertler (1989). After the bursting of the asset price bubble in the beginning of the 1990s, corporate balance sheets have gradually, but severely deteriorated, weakening the financial position (i.e., the net worth) of many borrowers in Japan. From the viewpoint of the balance sheet channel, which assumes some kind of a friction in the credit market, this reduction in net worth will in turn raise the external finance premium, making it harder for a firm to borrow. As such, the bursting of the asset bubble – which can be considered an exogenous shock to the Japanese economy – could have possibly weakened the monetary policy transmission mechanism, prompting the BOJ to take a precautionary stance after the mid 1990s.[34] Although this explanation is convincing, since the focus of this paper is on the standard nominal interest rate channel, we will exclude the balance sheet channel from the scope of this paper and will not pursue it further.

5 Concluding Remarks

In this paper, we have studied the optimal policy reaction function where the zero lower bound of nominal interest rates might interfere with the conduct of monetary policy. The main contribution of this paper is that we have derived an analytical expression of the optimal monetary policy reaction function in the presence of the zero lower bound and proved the key properties that it is more expansionary and more aggressive than the Taylor rule. Although preceding research has pointed out or simulated these properties, to the best of our knowledge, none has derived an analytical expression of the optimal policy reaction function in the presence of the zero lower bound or formally proved the above properties. Further, in order to verify these analytical implications, we have numerically approximated the optimal policy reaction function using the method known as the collocation method. Conforming with the earlier numerical and simulation results demonstrated in Reifschneider and Williams (2000) and Orphanides and Wieland (2000), we have verified that, indeed, the reaction function is more expansionary and aggressive than the Taylor rule and concave (provided that the nominal interest rate is not binding at zero) in the inflation rate and the output gap. Based on this numerically approximated optimal reaction function, we have empirically tested the qualitative and quantitative implications taking the case of Japanese monetary policy conduct in the 1990s. According to our Tobit estimation results, we found some empirical evidence

[34] Bernanke and Gertler (1999) conduct a simulation study using the Japanese data. According to their result, on the contrary to our conjecture, they report the simulated path of the call rate to be higher than the actual monetary policy conduct during 1996-97. However, it should be noted that their dynamic general equilibrium model does not take into account the zero lower bound constraint on the nominal interest rate.

that BOJ's monetary policy conduct in the 1990s was qualitatively consistent – i.e., concave in inflation rate and output gap – with the optimal policy reaction function implied in the paper. Finally, in order to evaluate the BOJ's monetary policy conduct quantitatively, we have compared the actual path of the nominal interest rate with the simulated path implied by the numerically approximated policy reaction function. According to our quantitative analysis, we found the BOJ's monetary policy conduct in the first half of the 1990s to be too contractionary and too slow in responding to disinflation and declining output, while, in the latter half of the 1990s, we found the BOJ's low interest rate policy to be fairly consistent with the optimal policy reaction function implied in our model.

One important remark should be made. The economy assumed in this paper is the simplest case, in the sense that state variables in the transition system are bivariate VAR(1) of the inflation rate and the output gap. In general, it is likely that the state variables in the "true" transition system of the economy are not limited to the inflation rate and the output gap. Generally speaking, the transition system may be multivariate VAR(P) and not limited to the state variables of the inflation rate and the output gap. Yet, it is still notable that the optimal policy reaction function derived based on the simple VAR(1) transition system was able to capture the qualitative character of the BOJ's monetary policy conduct in the 1990s fairly well, despite the mixed evidence revealed by the quantitative analysis. It remains to be seen whether a more general specification of the transition system can characterize both the qualitative and quantitative features of Japanese monetary policy in the 1990s. Although this extension is interesting and important, this will be left for future research.

A Appendix: Proof of Propositions

A.1 Proposition 1

First, let us note from Svensson (1997a) that

$$i^{Taylor} = \pi_t + \left(\alpha + \frac{\rho\theta_1 + \theta_1 - 1}{\delta\theta_1}\right) y_t + \left(\frac{\theta_1 - 1}{\alpha\delta\theta_1}\right)(\pi_t - \pi^*).$$

Then by eqn (16) and by $\delta, \theta_1 > 0$, suffice to show

$$\sum_{i=0}^{\infty} \theta_2^i E_t \Psi_{t+i} \leq 0,$$

where $\Psi_{t+i} = \left(\rho\beta\psi_{t+2+i} - (1+\rho+\alpha\delta)\psi_{t+1+i} + \beta^{-1}\psi_{t+i}\right)\delta^{-1}$. Expanding the terms and multiplying δ on both sides will yield the following expression.

$$\begin{aligned}
\delta\sum_{i=0}^{\infty}\theta_2^i E_t\Psi_{t+i} &= (\beta^{-1}\psi_t - (1+\rho+\alpha\delta)E_t\psi_{t+1} + \rho\beta E_t\psi_{t+2}) \\
&\quad +\theta_2\left(\beta^{-1}E_t\psi_{t+1} - (1+\rho+\alpha\delta)E_t\psi_{t+2} + \rho\beta E_t\psi_{t+3}\right) \\
&\quad +\theta_2^2\left(\beta^{-1}E_t\psi_{t+2} - (1+\rho+\alpha\delta)E_t\psi_{t+3} + \rho\beta E_t\psi_{t+4}\right) \\
&\quad +\cdots \\
&\quad +\theta_2^i\left(\beta^{-1}E_t\psi_{t+i} - (1+\rho+\alpha\delta)E_t\psi_{t+i+1} + \rho\beta E_t\psi_{t+i+2}\right) \\
&\quad +\cdots. \\
&= \frac{1}{\beta}\psi_t - \left[\rho + \alpha\delta + \frac{1}{\beta}(\beta-\theta_2)\right]E_t\psi_{t+1} \\
&\quad -\left[\frac{1}{\beta}(\beta-\theta_2)(\theta_2-\rho\beta) + \alpha\delta\theta_2\right]E_t\sum_{j=0}^{\infty}\theta_2^j\psi_{t+2+j} \\
&= \frac{1}{\beta}\psi_t - \left[\rho + \alpha\delta + (1-\frac{1}{\theta_1})\right]E_t\psi_{t+1} \\
&\quad -\left[\theta_2(\theta_1-\theta_2)(\frac{1}{\theta_1}-\rho) + \alpha\delta\theta_2\right]E_t\sum_{j=0}^{\infty}\theta_2^j\psi_{t+2+j} \qquad (22)
\end{aligned}$$

But since i^* is strictly greater than zero and the constraint is not binding at the current period t, it follows that $\psi_t = 0$. Otherwise the optimal interest rate i_t^* is trivially equal to zero. Since the rest of expected Lagrange multipliers, $\left\{E_t\psi_{t+i}\right\}_{i=1}^{\infty}$, are all non-negative, it suffices to show that the coefficients on the second term and third term of eqn (22) inside the brackets are positive. Since $\theta_1 > 1$, the coefficient inside the brackets of the second term is positive. Finally, since $1 > \theta_2 > 0$, the coefficient inside the brackets of the third term is assured to be positive under the regularity condition such that $\theta_1^{-1} + \alpha\delta/(\theta_1 - \theta_2) > \rho$.

A.2 Proposition 2

Since technically it is equivalent to choose i_t and $E_t y_{t+1}$, the Bellman equation (eqn (6)) can be rewritten as an alternative expression which represents a one-state (i.e., $s_t = \pi_t + \alpha y_t$) and one-control (i.e., $E_t y_{t+1}$) variable problem as follows,

$$\widetilde{V}^C(s_t) = \min_{E_t y_{t+1} \leq c_t} E_t \left[\frac{1}{2} \left\{ y_{t+1}^2 + \lambda (\pi_{t+1} - \pi^*)^2 \right\} + \beta \widetilde{V}^C(s_{t+1}) \right],$$

where $c_t = (\rho + \alpha\delta) y_t + \delta\pi_t$. Note that the newly defined value function $\widetilde{V}^C(s_t)$ corresponds to our previous notation $V(y_t, \pi_t)$ in eqn (6). As a result, the first order condition is also rewritten in the following equation,[35]

$$E_t y_{t+1} + \alpha\beta E_t \widetilde{V}^{C\prime}(s_{t+1}) = 0.$$

Combining this first order condition with the IS equation, we can obtain another closed form expression of the optimal reaction function (i.e., eqn (16)),

$$i_t^* = \left(\alpha + \frac{\rho}{\delta} \right) y_t + \pi_t + \delta^{-1} \alpha\beta E_t \widetilde{V}^{C\prime}(s_{t+1}), \tag{23}$$

where $s_{t+1} = \pi_{t+1} + \alpha y_{t+1} = (\pi_t + \alpha y_t + \epsilon_{t+1}) + \alpha((\rho + \alpha\delta) y_t + \delta\pi_t - \delta i_t^* + \upsilon_{t+1})$. Thus this eqn (23) is a closed form expression of the optimal reaction function whose arguments are only current state variables. On the other hand, let the value function of the unconstrained problem be $\widetilde{V}^U(\bullet)$. When there is no zero lower bound, the third term of eqn (23) is replaced by $\delta^{-1}\alpha\beta E_t \widetilde{V}^{U\prime}(s_{t+1})$ for i_t^{Taylor}. Now consider that $i_t^* - i_t^{Taylor}$ can be rewritten in the following expression,

$$i_t^* - i_t^{Taylor} = \delta^{-1}\alpha\beta E_t \left[\widetilde{V}^{C\prime}(s_{t+1}) - \widetilde{V}^{U\prime}(s_{t+1}) \right].$$

Recall that s_{t+1} inside each value function $\widetilde{V}^C(\bullet)$ and $\widetilde{V}^U(\bullet)$ is also a function of each i_t. Hence, we can apply the implicit function theorem to this so that,

$$\frac{\partial i_t^*}{\partial y_t} - \frac{\partial i_t^{Taylor}}{\partial y_t} = \alpha \left(\frac{\partial i_t^*}{\partial \pi_t} - \frac{\partial i_t^{Taylor}}{\partial \pi_t} \right) = \frac{\alpha^2\beta}{\delta} E_t \left[\frac{\widetilde{V}^{C\prime\prime}(s_{t+1})}{1 + \alpha^2\beta\widetilde{V}^{C\prime\prime}(s_{t+1})} - \frac{\widetilde{V}^{U\prime\prime}(s_{t+1})}{1 + \alpha^2\beta\widetilde{V}^{U\prime\prime}(s_{t+1})} \right].$$

Hence, it suffices to show that,

$$\widetilde{V}^{C\prime\prime} \geq \widetilde{V}^{U\prime\prime}.$$

The rest of the proof is mainly based on Chmielewski and Manousiouthakis (1996). Let us state the lemma provided in Chmielewski and Manousiouthakis (1996) in a modified form so that we can apply it to our problem.

[35] One can verify that this first order condition is equivalent to the one that we derived in Section 2. See Svensson (1997a) for this short-cut notation.

Lemma 1 (Chmielewski&Manousiouthakis (1996)) *Define a twice differentiable continuous function,* h*:*$\Re \rightarrow \Re$*,* $h(s_t) = \widetilde{V}^C(s_t) - \widetilde{V}^U(s_t)$*, where* s_t *is a state variable. Then* $h(s_t)$ *is a convex function, i.e.,* $h'' \geq 0$*.*

Proof. It can be easily verified that the above constrained and unconstrained dynamic programming can be rewritten as

$$\begin{aligned}\widetilde{V}^C(s_0) &= \min_{x_t \leq c_t} \left\{ E_0 \sum_{t=0}^{\infty} \beta^t (\lambda s_t^2 + x_t^2) \right\} \\ s.t.\ s_{t+1} &= s_t + \alpha x_t + \xi_{t+1}\end{aligned}$$

$$\begin{aligned}\widetilde{V}^U(s_0) &= \min_{x_t} \left\{ E_0 \sum_{t=0}^{\infty} \beta^t (\lambda s_t^2 + x_t^2) \right\}, \\ s.t.\ s_{t+1} &= s_t + \alpha x_t + \xi_{t+1}\end{aligned}$$

where $s_t \equiv \pi_t + \alpha y_t,\ x_t \equiv E_t y_{t+1},\ \xi_{t+1} \equiv \varepsilon_{t+1} + \alpha v_{t+1}$. (For notational simplicity, we rewrite the initial period as zero, instead of t.) Let $\{\widehat{x}_t^C\}_{t=0}^{\infty}$, $\{\widetilde{x}_t^C\}_{t=0}^{\infty}$, $\{\widehat{x}_t^U\}_{t=0}^{\infty}$ and $\{\widetilde{x}_t^U\}_{t=0}^{\infty}$ be the optimal path of the control variable for a constrained and unconstrained problem, respectively, given initial states $\widehat{s}_0$ and $\widetilde{s}_0$. Let $\{\widehat{s}_t^C\}_{t=0}^{\infty}$, $\{\widetilde{s}_t^C\}_{t=0}^{\infty}$, $\{\widehat{s}_t^U\}_{t=0}^{\infty}$ and $\{\widetilde{s}_t^U\}_{t=0}^{\infty}$ be the controlled optimal path of the state variable for the constrained and unconstrained problem respectively, given initial states $\widehat{s}_0$ and $\widetilde{s}_0$. Now define $\overline{s}_t = a\widehat{s}_t + (1-a)\widetilde{s}_t$ and $\overline{x}_t = a\widehat{x}_t + (1-a)\widetilde{x}_t$ where $0 \leq a \leq 1$. Then, from Chmielewski and Manousiouthakis (1996), it can be shown that the pair of sequences $\{\overline{x}_t^C\}_{t=0}^{\infty}$ and $\{\overline{s}_t^C\}_{t=0}^{\infty}$ under the constrained problem is feasible, but not necessarily equal to the optimal path given the initial state $\overline{s}_0$. Also, it can be shown that the pair of sequences $\{\overline{x}_t^U\}_{t=0}^{\infty}$, and $\{\overline{s}_t^U\}_{t=0}^{\infty}$ under the unconstrained problem is feasible and optimal given the initial state $\overline{s}_0$. Therefore, the following inequality holds.

$$h(\overline{s}_0) = \widetilde{V}^C(\overline{s}_0) - \widetilde{V}^U(\overline{s}_0) \leq E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda \left(\overline{s}_t^C\right)^2 + \left(\overline{x}_t^C\right)^2 \right] - E_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda \left(\overline{s}_t^U\right)^2 + \left(\overline{x}_t^U\right)^2 \right]$$

Then, it suffices to show that

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\lambda \left(\overline{s}_t^C\right)^2 + \left(\overline{x}_t^C\right)^2 \right) - \left(\lambda \left(\overline{s}_t^U\right)^2 + \left(\overline{x}_t^U\right)^2 \right) \right] \\ \leq\ & aE_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\lambda \left(\widehat{s}_t^C\right)^2 + \left(\widehat{x}_t^C\right)^2 \right) - \left(\lambda \left(\widehat{s}_t^U\right)^2 + \left(\widehat{x}_t^U\right)^2 \right) \right] \\ & +(1-a)E_0 \sum_{t=0}^{\infty} \beta^t \left[\left(\lambda \left(\widetilde{s}_t^C\right)^2 + \left(\widetilde{x}_t^C\right)^2 \right) - \left(\lambda \left(\widetilde{s}_t^U\right)^2 + \left(\widetilde{x}_t^U\right)^2 \right) \right], \end{aligned}$$

which is equivalent to showing that

$$a(a-1)\left[E_0\sum_{t=0}^{\infty}\beta^t\left(\lambda\left(\widehat{s}_t^C-\widetilde{s}_t^C\right)^2+\left(\widehat{x}_t^C-\widetilde{x}_t^C\right)^2\right)-E_0\sum_{t=0}^{\infty}\beta^t\left(\lambda\left(\widehat{s}_t^U-\widetilde{s}_t^U\right)^2+\left(\widehat{x}_t^U-\widetilde{x}_t^U\right)^2\right)\right]\leq 0.$$

Now, define $\overline{\overline{s}}_t \equiv \widehat{s}_t - \widetilde{s}_t$. Then, again, it can be shown that the sequence $\{\overline{\overline{s}}_t^C\}_{t=0}^{\infty}$ under the constrained problem is feasible, but not necessary optimal given the initial state $\overline{\overline{s}}_0$. Similarly, it can be shown that the sequence $\{\overline{\overline{s}}_t^U\}_{t=0}^{\infty}$ under the unconstrained problem is feasible and optimal given the initial state $\overline{\overline{s}}_0$. Therefore, the following is true.

$$\begin{aligned}\widetilde{V}^C(\overline{\overline{s}}_0) &\leq E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda\left(\overline{\overline{s}}_t^C\right)^2+\left(\overline{\overline{x}}_t^C\right)^2\right]\\ \widetilde{V}^U(\overline{\overline{s}}_0) &= E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda\left(\overline{\overline{s}}_t^U\right)^2+\left(\overline{\overline{x}}_t^U\right)^2\right].\end{aligned}$$

But since the cost of the constrained problem is higher or equal to the unconstrained problem for any initial state, it then follows that $\widetilde{V}^C(\overline{\overline{s}}_0) \geq \widetilde{V}^U(\overline{\overline{s}}_0)$, which in turn implies that

$$E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda\left(\overline{\overline{s}}_t^C\right)^2+\left(\overline{\overline{x}}_t^C\right)^2\right]\geq E_0\sum_{t=0}^{\infty}\beta^t\left[\lambda\left(\overline{\overline{s}}_t^U\right)^2+\left(\overline{\overline{x}}_t^U\right)^2\right].$$

This proves the lemma. ■

B Appendix: Collocation Method

In this appendix, we explain the numerical algorithm in approximating the value function and optimal policy reaction function in the presence of the zero bound constraint. Specifically, we employ the numerical method known as the collocation method[36] in solving the functional fixed-point problem posed by the Bellman equation.

For convenience, let us restate the Bellman equation (eqn (6)) suppressing the time subscripts as follows,

$$V(\pi,y)=\min_{x\geq 0}\{f(\pi,y)+\beta EV(g(\pi,y,x,\upsilon,\varepsilon)\},\tag{24}$$

where $f(\pi,y)$ stands for the period-by-period loss function and $g(\pi,y,x,\upsilon,\varepsilon)$ stands for the state transition function. Note that the nominal interest rate, denoted by x in this appendix, is constrained by the zero lower bound. The state transition function is linear in the state variables and the coefficient matrix is time-invariant, i.e.,

$$g(\pi,y,x,\upsilon,\varepsilon)=\begin{bmatrix}\rho+\alpha\delta & \delta\\ \alpha & 1\end{bmatrix}\begin{bmatrix}y\\ \pi\end{bmatrix}-\begin{bmatrix}\delta\\ 0\end{bmatrix}x+\begin{bmatrix}\upsilon\\ \varepsilon\end{bmatrix}.$$

[36] For complete elucidation regarding the collocation method, see Judd (1998, Ch.11 and 12) and Miranda and Fackler (2002, Ch.8 and 9).

Given the above specification of the Bellman equation and the state transition function, our goal is to interpolate the value function $V(\pi, y)$ in the interval of $-10 \leq \pi \leq 10$ and $-10 \leq y \leq 10$.

The collocation method proceeds in the following steps. First, we discretize the state space by the set of interpolation nodes such that $Node = \{(\pi_{n_\pi}, y_{n_y})|\ n_\pi = 1, 2, \cdots, N_\pi$ and $n_y = 1, 2, \cdots, N_y\}$.[37] Thus, we discretize the state space into the total of $N_\pi \times N_y$ interpolation nodes. Then we interpolate the value function $V(\cdot)$ using a cubic spline function[38] over these interpolation nodes as follows.

$$V(\pi_{n_\pi}, y_{n_y}) = \sum_{i=1}^{N_\pi} \sum_{j=1}^{N_y} c_{ij} \gamma_i^\pi(\pi_{n_\pi}) \gamma_j^y(y_{n_y}) \text{ for each } (\pi_{n_\pi}, y_{n_y}) \in Node \tag{25}$$

The basis functions $\gamma_i^\pi(\pi_{n_\pi})$ and $\gamma_j^y(y_{n_y})$ take the form of cubic spline functions and are defined as

$$\begin{aligned}
\gamma_i^\pi(\pi_{n_\pi}) &= \left\{ \begin{array}{cc} \frac{2}{3}(1 - 6q_\pi^2(1 - q_\pi)) & \text{if } q_\pi = \frac{|\pi_{n_\pi} - \pi_i|}{w} \leq 1 \\ \frac{4}{3}(1 - q_\pi)^3 & \text{if } 1 \leq q_\pi \leq 2 \\ 0 & \text{otherwise} \end{array} \right\} \\
\gamma_j^y(y_{n_y}) &= \left\{ \begin{array}{cc} \frac{2}{3}(1 - 6q_y^2(1 - q_y)) & \text{if } q_y = \frac{|y_{n_y} - y_j|}{w} \leq 1 \\ \frac{4}{3}(1 - q_y)^3 & \text{if } 1 \leq q_y \leq 2 \\ 0 & \text{otherwise} \end{array} \right\},
\end{aligned}$$

where $\pi_i = \underline{\pi} + wi$, where w is an equal step from the lower bound of state π (which is -10 in this paper) to the upper bound (which is 10 in this paper). The definition of y_j is similar. Interpolation equations (25) could be expressed compactly using the tensor product notation as follows,

$$\mathbf{v} = [\Gamma_\pi \otimes \Gamma_y] \cdot \mathbf{c}, \tag{26}$$

where $\mathbf{v}$ stands for $N_\pi N_y \times 1$ vector of the values of $V(\pi_{n_\pi}, y_{n_y})$ for each interpolation node, Γ_π stands for $N_\pi \times N_\pi$ matrix of the basis functions $\gamma_i^\pi(\pi_{n_\pi})$ (i.e., each matrix element is defined as $\Gamma_\pi[i, n_\pi] = \gamma_i^\pi(\pi_{n_\pi})$), Γ_y stands for $N_y \times N_y$ matrix of the basis functions $\gamma_j^y(y_{n_y})$, and $\mathbf{c}$ stands for $N_\pi N_y \times 1$ vector of the basis coefficients c_{ij}.

[37] There are several ways to discretize the state space. One example is Chebychev nodes. However, in order to preserve the exact solution of the value function and optimal policy reaction function at the equally distributed states, equally distributed interpolation nodes have been chosen in this paper.

[38] There are several other options for the basis function. One of the most frequently used basis functions is the Chebychev polynomial, which is known to possess superior properties when the curvature of the function to be interpolated is "nice and smooth." In contrast, the cubic spline function is known to possess superior properties when the function contains some "kinks." Since the value function and the optimal policy reaction function are kinked due to the presence of the zero lower bound in this paper, the cubic spline function will be our choice as a basis function. For more details regarding the cubic spline interpolation, see Judd (1998, Ch.6), Cheney and Kincaid (1999), and Miranda and Fackler (2002, Ch.6).

Next, we turn to the right-hand side of the Bellman equation (24). In approximating the expected value function, i.e., $E[V(g(\pi, y, x, \upsilon, \varepsilon)]$, we assume the distribution of the error terms (υ, ε) to be *i.i.d.* multivariate normal. Under the assumption of normal distribution, the expected value function can be approximated by the Gaussian-Hermite quadrature method[39] – a member of the Gaussian quadrature methods which is specifically used when the error terms are normally distributed. The Gaussian-Hermite quadrature method discretizes the random space with the set of quadrature nodes such that $QNode = \{(\upsilon_{h_\upsilon}, \varepsilon_{h_\varepsilon}) | h_\upsilon = 1, 2, \cdots, M_\upsilon \text{ and } h_\varepsilon = 1, 2, \cdots, M_\varepsilon\}$ with corresponding quadrature weights $\omega_{h_\upsilon h_\varepsilon}$. Thus, we discretize the random space into a total of $M_\upsilon \times M_\varepsilon$ quadrature nodes. Then by substituting the interpolation equation (25) for the value function $V(g(\pi, y, x, \upsilon, \varepsilon)$, the right-hand side of the Bellman equation can be approximated as

$$RHS_{n_\pi n_y}(\mathbf{c}) = \min_{x \geq 0} \left\{ f(\pi_{n_\pi}, y_{n_y}) + \beta \sum_{h_\upsilon=1}^{M_\upsilon} \sum_{h_\varepsilon=1}^{M_\varepsilon} \sum_{i=1}^{N_\pi} \sum_{j=1}^{N_y} \omega_{h_\upsilon h_\varepsilon} c_{ij} \gamma_{ij} (g(\pi_{n_\pi}, y_{n_y}, x, \upsilon_{h_\upsilon}, \varepsilon_{h_\varepsilon})) \right\} \tag{27}$$

for each $(\pi_{n_\pi}, y_{n_y}) \in Node$ where γ_{ij} stands for the cross products of the basis function. The minimization of the above problem with respect to x can be attained using a standard Quasi-Newton optimization method. It should be noted that when implementing this minimization problem, one should pay attention to the corner solution of the minimization problem due to the zero lower bound constraint on the control variable x.

Finally, by equating eqn (25) and eqn (27) for each interpolation node, we obtain the following approximation of the Bellman equation (24);

$$\sum_{i=1}^{N_\pi} \sum_{j=1}^{N_y} c_{ij} \gamma_i^\pi (\pi_{n_\pi}) \gamma_j^y (y_{n_y}) = RHS_{n_\pi n_y}(\mathbf{c}) \text{ for each } (\pi_{n_\pi}, y_{n_y}) \in Node. \tag{28}$$

Using the tensor product notation, the above equation can be compactly expressed as

$$[\Gamma_\pi \otimes \Gamma_y] \mathbf{c} = \mathbf{RHS}(\mathbf{c}), \tag{29}$$

where $\mathbf{RHS}(\mathbf{c})$ stands for $N_\pi N_y \times 1$ vector of the values of $RHS_{n_\pi n_y}(\mathbf{c})$. Now the task is to find the unknown basis coefficient vector $\mathbf{c}$ from the above nonlinear equation system (29). The nonlinear equation system can be solved using an iterative nonlinear root-finding technique such as the Functional Iteration method, Newton's method or a

[39] For more details regarding the Gaussian Quadrature method, see Judd (1998, Ch.7) and Miranda and Fackler (2002, Ch. 5).

Quasi-Newton method.[40] For computational ease, we adopt the Functional Iteration method as the solution algorithm.

Algorithm 1 *(Functional Iteration method)*
Step 1: Choose the degree of approximation N_π, N_y, M_υ, and M_ε. Then set the appropriate interpolation nodes and quadrature nodes for the state space and random space, respectively. Guess the initial basis coefficients vector $\mathbf{c}_0$*.*
Step 2: Update the basis coefficient vector by the following functional iteration;

$$\mathbf{c}_{k+1} \leftarrow [\Gamma_\pi^{-1} \otimes \Gamma_y^{-1}] \cdot \mathbf{RHS}(\mathbf{c}_k).$$

Step 3: Check for convergence. If $|c_{ij,k+1} - c_{ij,k}| < \tau$ for any i and j, where τ is a convergence tolerance parameter, then stop. Otherwise, repeat step 2.

Once convergence has been reached, the interpolation of the value function $V(\pi, y)$ is now attained. Of course, as a by-product of interpolating the value function, the approximation of the optimal policy function $x^*(\pi, y)$ will also be attained at the same time. It should be noted that one can attain the desired level of approximation by controlling the degree of interpolation nodes, quadrature nodes and convergence tolerance parameter τ with a trade-off of convergence speed.[41]

[40] For more details regarding the nonlinear root-finding technique, see Judd (1998, Ch.5) and Miranda and Fackler (2002, Ch.3).

[41] In our paper, we have set the parameter values as follows; $N_\pi = 20$, $N_y = 20$, $M_\upsilon = 3$, $M_\varepsilon = 3$ and $\tau = 10^{-8}$. With these parameter values, the maximum absolute approximation error of the value function was smaller than 10^{-3}. Using the Pentium III computing environment, convergence was attained within 10 minutes in most cases.

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Table 1: OLS Estimation Results for Eqn (2) and (3)

$y_{t+1} = const + \rho y_t + \delta RINT_t + a_1 SRDR_t + \upsilon_{t+1}$			
Regressor	Coeff.	S.E.	t-stat
$const.$	-0.035	0.076	-0.459
y_t	0.754	0.073	10.206**
$RINT_t$	-0.445	0.134	-3.331**
$SRDR$	1.050	0.411	2.552*
$\pi_{t+1} = const + \eta\pi_t + \alpha y_t + \varepsilon_{t+1}$			
Specification Test	$H_0 : \eta = 1$		
Wald Statistics	2.842		
Regressor	Coeff.	S.E.	t-stat.
$const.$	-0.115	0.078	-1.473
π_t	1	- -	- -
y_t	0.086	0.050	1.708

Note: The data source for the price level is Statistics Bureau & Statistics Center's *Consumer Price Index Japan Monthly*, the output gap is from Watanabe (1997) and the real interest rate, RINT, is from the Bank of Japan (1996). The inflation rate, π, was constructed as the percent change in the consumer price index from the same month of the previous year and has been converted into quarterly data by taking a quarterly average. Finally, y was constructed by estimating the aggregate production function. For the detailed methodology, see Watanabe (1997). SRDS is the real exchange rate from Kamada and Muto (2000). This real exchange rate data is normalized so that its average equals zero. Rejection of the null hypothesis at the significance level of 5% and 1% is indicated by (*) and (**) respectively.

Table 2: Tobit Estimation Results and Log-Likelihood Ratio Test

	Benchmark		Alternative 1		Alternative 2		Alternative 3		Alternative 4	
Regressor	Coeff.	z-stat.	Coeff.	z-stat.	Coeff.	z-stat.	Coeff.	z-stat	Coeff.	z-stat.
$cst.$	1.668	7.329***	1.676	5.545***	1.776	6.139***	1.567	6.749***	1.918	7.440***
	(0.228)	[0.000]	(0.302)	[0.000]	(0.289)	[0.000]	(0.232)	[0.000]	(0.258)	[0.000]
π	1.585	8.156***	2.615	3.559***	1.988	4.522***	2.236	5.773***	1.579	8.145***
	(0.194)	[0.000]	(0.735)	[0.000]	(0.440)	[0.000]	(0.387)	[0.000]	(0.194)	[0.000]
y	0.570	5.277***	0.382	1.503	0.619	5.048***	0.557	5.172***	0.650	5.369***
	(0.108)	[0.000]	(0.254)	[0.133]	(0.123)	[0.000]	(0.108)	[0.000]	(0.121)	[0.000]
π^2	- -	- -	-0.492	-1.422	-0.166	-1.035	-0.266	-1.936*	- -	- -
			(0.346)	[0.155]	(0.161)	[0.301]	(0.137)	[0.053]		
y^2	- -	- -	-0.126	-1.534	-0.054	-1.192	- -	- -	-0.077	-1.948*
			(0.082)	[0.125]	(0.045)	[0.233]			(0.040)	[0.051]
πy	- -	- -	0.282	1.061	- -	- -	- -	- -	- -	- -
			(0.266)	[0.289]						
σ	1.176	- -	1.152	- -	1.161	- -	1.158	- -	1.171	- -
	(0.102)		(0.100)		(0.101)		(0.1001)		(0.102)	

Log-Likelihood Ratio Test

	H_o: $c_3=c_4=c_5=0$	H_o: $c_3=c_4=0$	H_o: $c_3=0$	H_o: $c_4=0$
log-LR statistic	6.422*	5.304*	3.805*	4.243**
	[0.093]	[0.070]	[0.051]	[0.039]

Note: The number inside the parentheses denotes the standard error. The number inside the brackets denotes the p-value. In estimating the asymptotic variance of the ML estimator, we used the algorithm proposed by Berndt, Hall, Hall, and Hausman (1974). Rejection of the null hypothesis at the significance levels of 10%, 5% and 1% is indicated by (*), (**), and (***), respectively. Note that under the null of coefficient equal to zero, z-statistic is asymptotically distributed as the standard normal. Also note that the log-LR statistic under the null is asymptotically χ^2 distributed with a degree of freedom corresponding to the number of restrictions imposed.

Figure 1: Optimal Monetary Policy Reaction Function with Zero Bound
(Inflation Target = 2%, l = 1, s = 1.5)

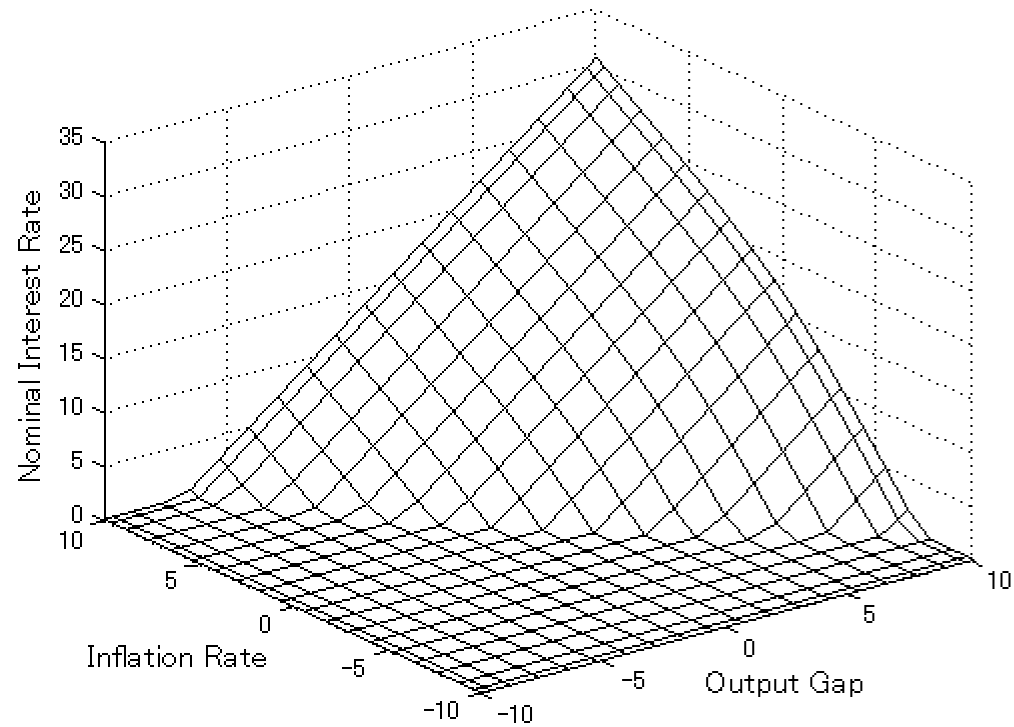


Figure 2: Value Function with Zero Bound
(Inflation Target = 2%, l = 1, s = 1.5)

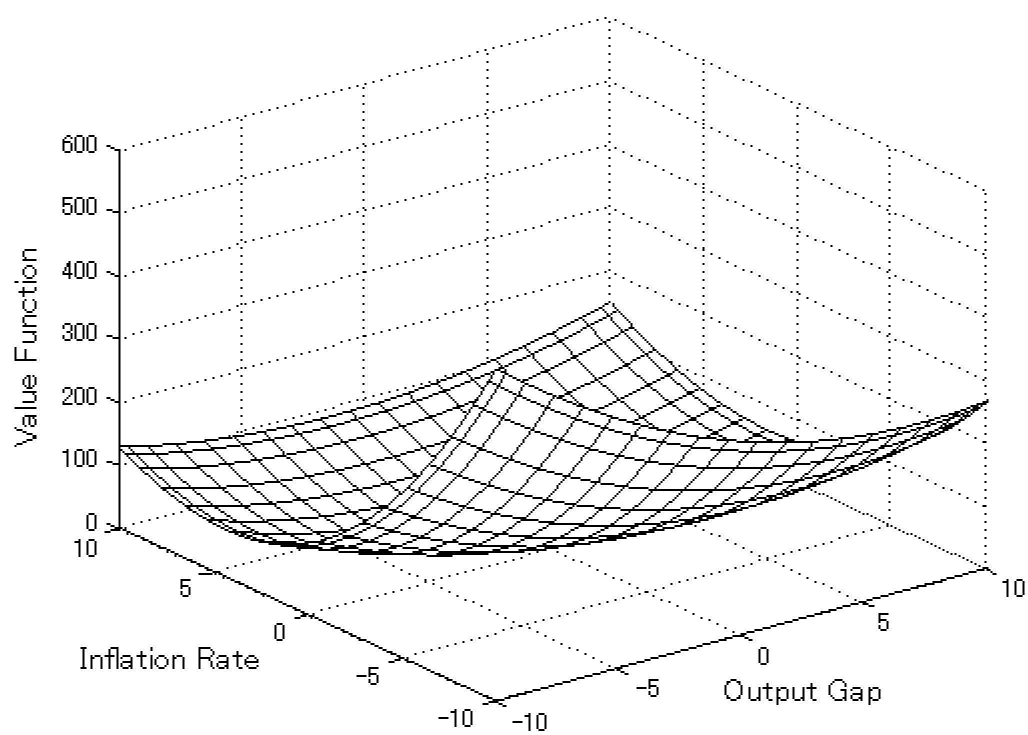


Figure 3: Cross Sectional View of the Optimal Policy Reaction Function
(Inflation rate held constant at –1.77%)

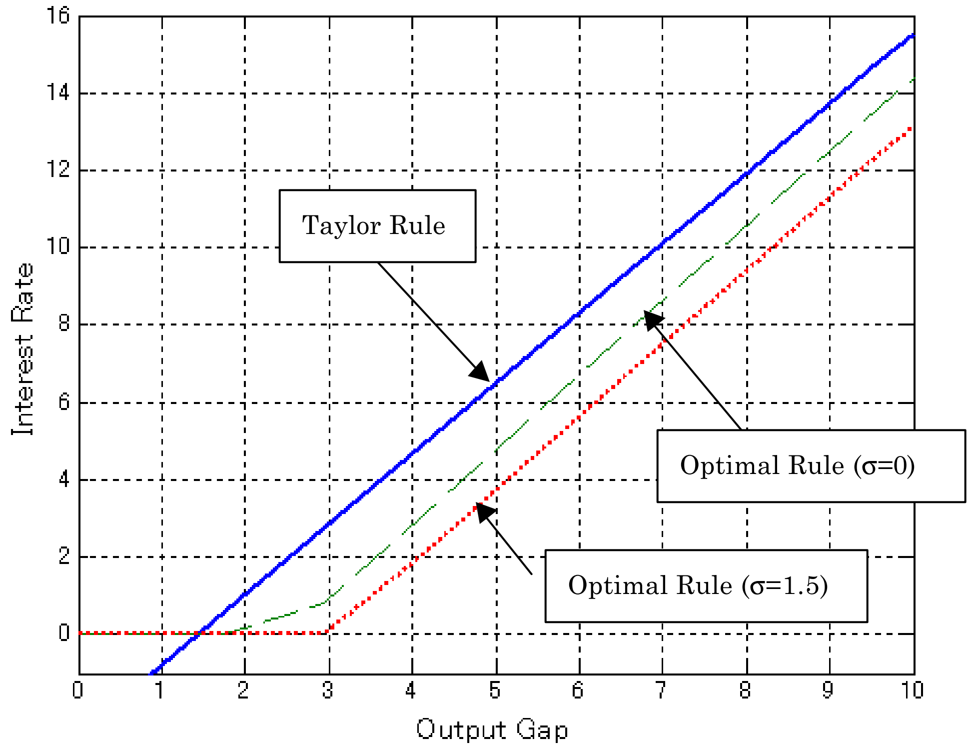


Figure 4: Cross Sectional View of the Optimal Policy Reaction Function
(Output gap held constant at 5.29%)

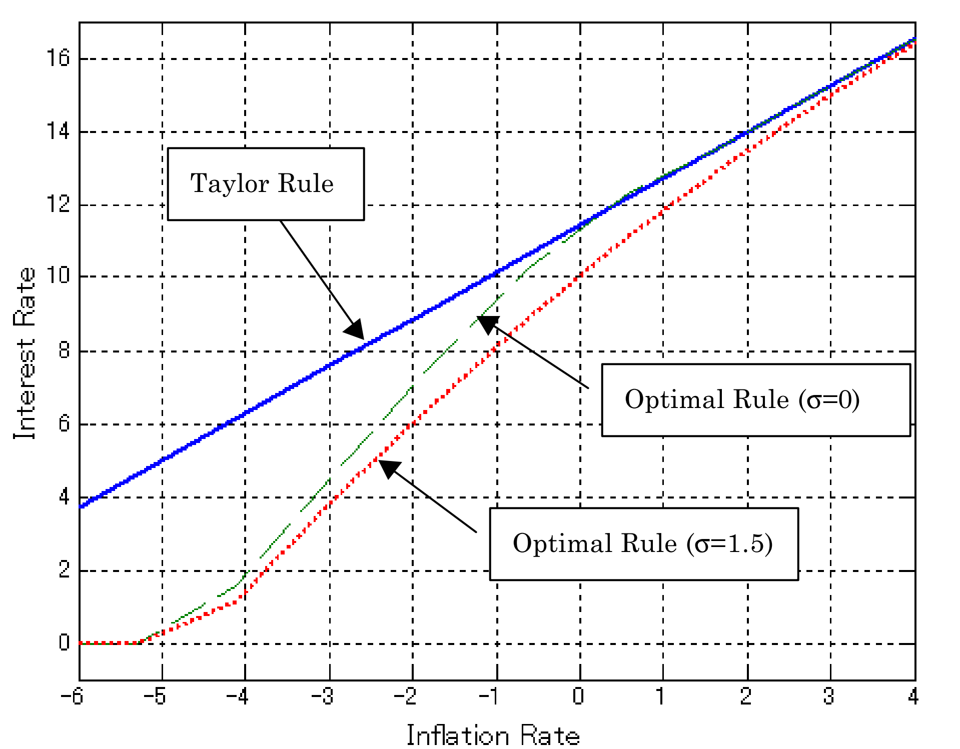


Figure 5: Sensitivity Analysis of the Policy Reaction Function
(Deterministic Environment)

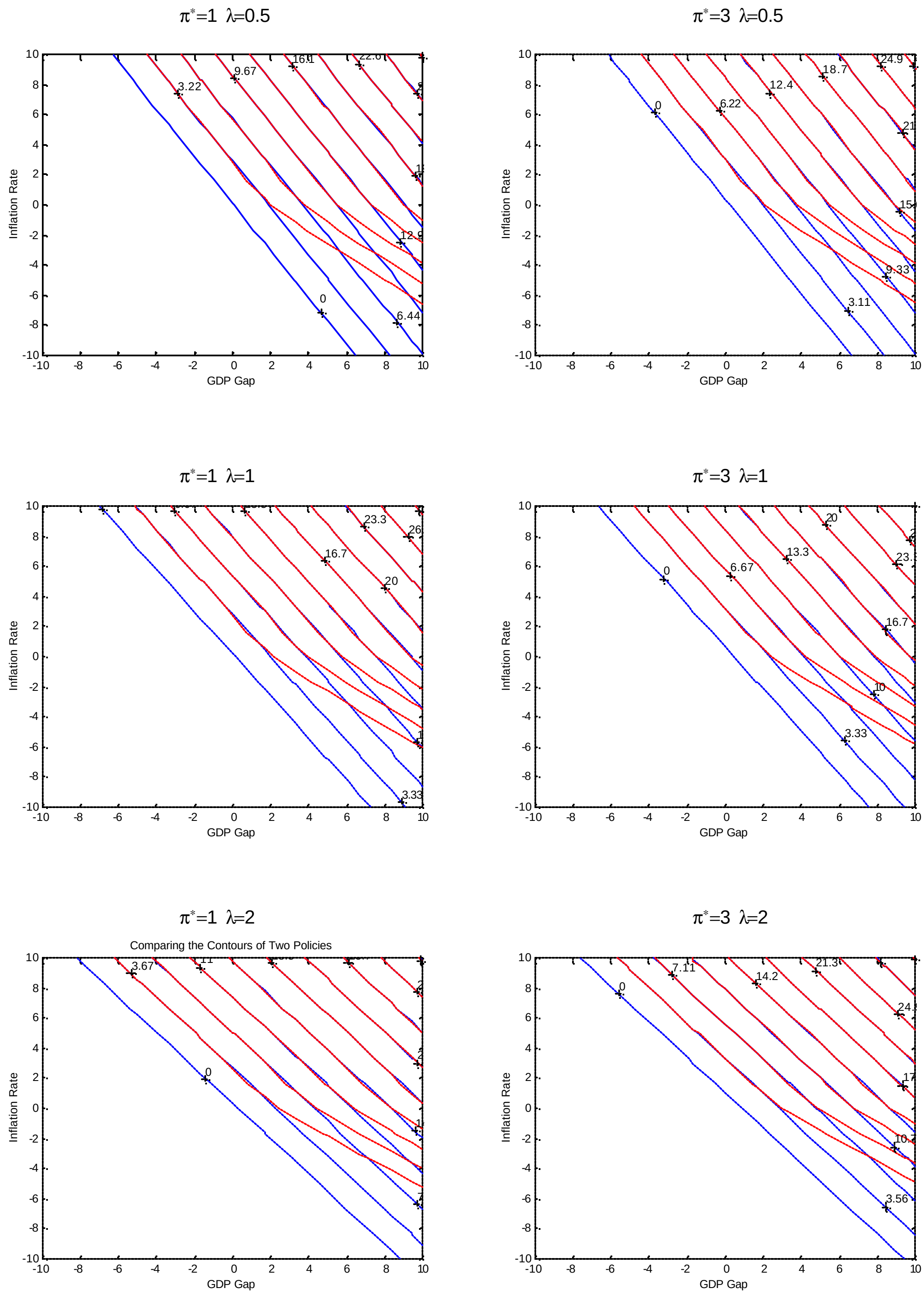


Figure 6: Sensitivity Analysis of the Policy Reaction Function (Stochastic Environment)

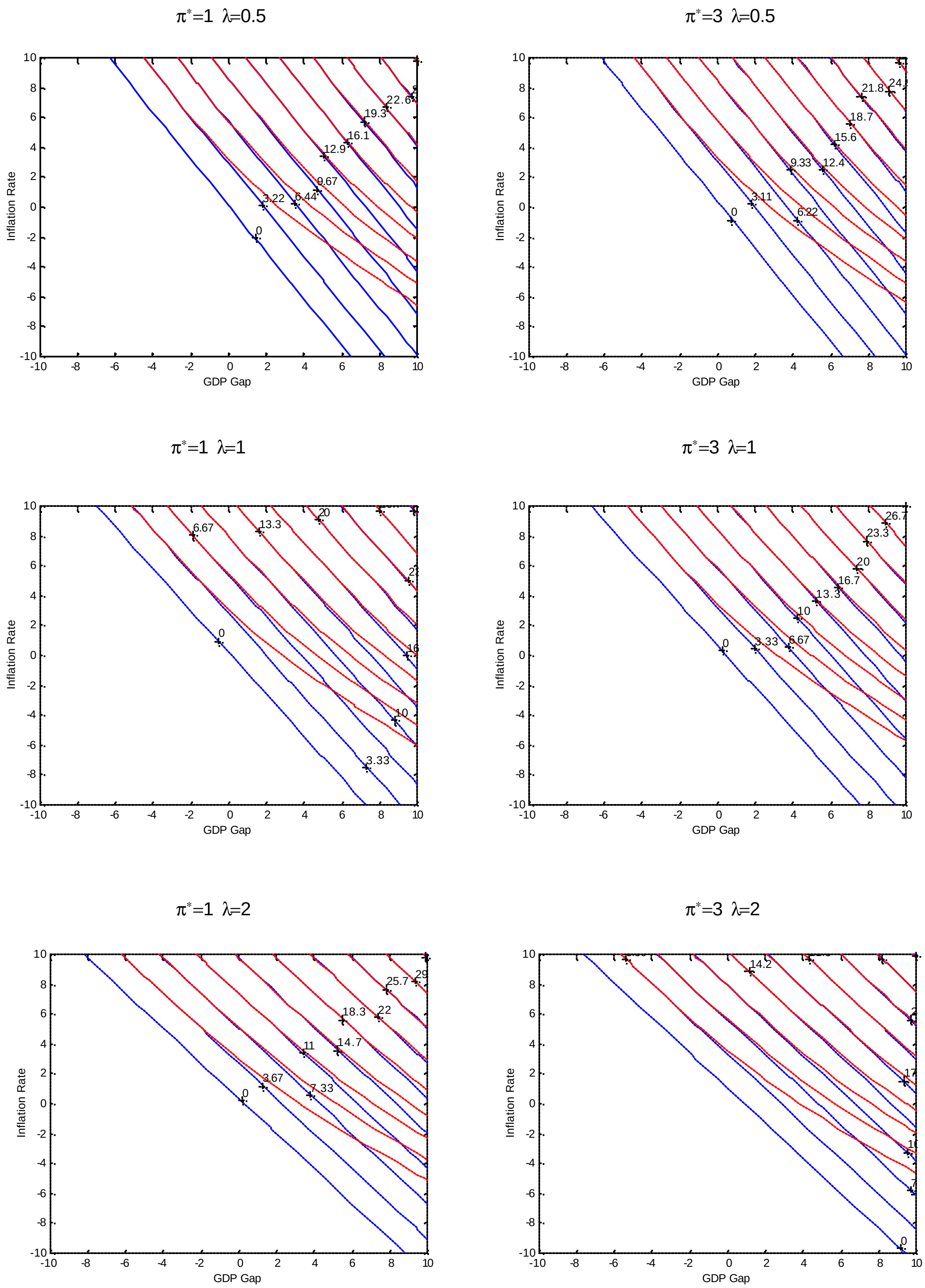
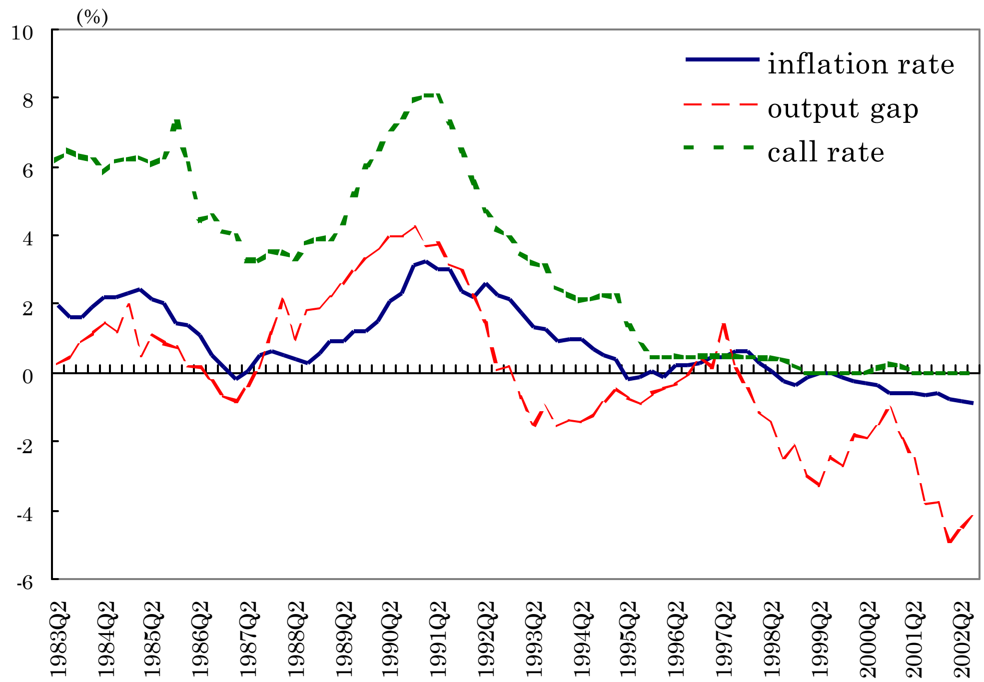


Figure 7: Inflation Rate, Output Gap, and Call Rate: 1983:Q2 - 2002:Q3



Sources: Bank of Japan, Financial and Economic Statistics Monthly; Ministry of Public Management, Home Affairs, Posts and Telecommunications, Consumer Price Index; Hirose and Kamada (2002).

Figure 8: Simulated Path, Estimated Taylor Rule, and Actual Call Rate