Risk Accumulation, Contagion and the Rules for Bank Failure*

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Abstract

This paper examines how we view the cost associated with negative externality, often referred to as "contagion effect," in the financial market and how effectively various types of *ex ante* disciplines can prevent contagion from spilling over to other participants in the market. The ultimate objective of the regulator is to keep the market open, i.e. to attract investors' participation in financial transactions. For that purpose, we aim to identify how the regulator sets the rule which prevents the contagion effect, while at the same time making as many participants as possible take part in the transactions. Our Monte-Carlo simulation suggests that such regulatory rules as early closure and capital enforcement are effective devices for preventing the contagion effect, although contagion itself can never be eliminated fully under the arc-sine feature of the random walk process. The results further suggest that a more practical resolution method is to combine several policy rules.

Almost all human life depends on probabilities.

Voltaire, *Essays*

1 Introduction

This paper examines how we view the cost associated with negative externality, often referred to as "contagion effect," in the financial market and how effectively various types of *ex ante* disciplines can prevent the contagion from spilling over to other market participants.

* The authors are grateful to Jeff Lacker, the discussant in the conference, and participants in the seminar at Osaka University. The views expressed herein are the authors' and not necessarily those of the Bank of Japan.

Suppose that banks are engaged in risky financial transactions, and the overall results of these transactions are summarised by winning and losing scores, which add up to zero (i.e. zero-sum game). If, however, the losers of these transactions fail to pay their losses due to insufficient capital holdings, then the winners might not be able to collect all their gains. Unless the losses are compensated by a third party, the winners will have to start the next round of transactions with less wealth than they would have obtained if the losers had paid their losses in full. If the winners continuously fail to receive their full winning position at the end of each round of transactions, this may generate a higher possibility that they will eventually reach a threshold point of bankruptcy. This is one way of illustrating the contagion effect transmitted from one party to others.[1] Here the contagion cost is measured by the amount of funds which could have been obtained, had all the losses been paid in full or covered by a third party, minus the amount actually received.

In this paper, we are interested in how this type of contagion effect accumulates over the long run for financial transactions. We believe this is a unique interpretation. One often associates the contagion with a situation where one strategy, such as a deposit withdrawal, can spread from a finite set of players to the whole population. Our interpretation is unrelated to the choice of strategies. Instead, we examine how a given strategy (i.e. there is no choice of strategies—players keep playing the game) changes financial positions over time by undertaking different methods of resolution against bank failures. This method establishes our general approach for analysing how the contagion effect is amplified through the course of play.

In order to perform this analysis, we need to identify what resolution signifies. Prior research has focused on two means of resolution mechanism: market discipline and regulatory discipline as mechanisms of preventing the contagious effect. Market discipline has been regarded as discipline imposed on bank managers and stockholders through monitoring debtholders' behaviour. Here, debtholders can be either uninsured depositors or subordinated debtholders, who have a stronger incentive to monitor banking activities than ordinary debtholders.

Empirical research on market discipline, however, shows somewhat mixed results. Park and Peristiani (1998) examine two effects: one being the price effect where uninsured depositors demand risk premium, and the other being the quantity effect where smaller amounts of uninsured deposits are collected. They conclude that both effects are present in the U.S. thrift industry. Avery, Belton and Goldberg (1988) examine market discipline from the perspective of subordinated debtholders, who have a stronger incentive to monitor banks than uninsured depositors, who can withdraw their funds and are likely to receive *de facto* insurance in some cases of resolution. Their empirical evidence, however, suggests no such discipline is observed. Recent studies by Billett, Garfinkel and O'Neal

[1] This may cause a chain reaction—if the liabilities to one institution cannot be covered, this institution may fail to pay its liabilities to another institution, and a sequence of domino effects follows. In a broader sense, a network of debtor-creditor relationships transmits the effects of a debtor's defaults to legally unrelated agents. Such negative externality is called "systemic risk."

(1998) contend that given the complementary nature of market discipline and regulatory discipline, banks faced with more severe market discipline (e.g. downgrading by Moody's), will shift their source of funding toward insured deposits while maintaining their riskiness; hence, market discipline will not have any effect on banks' behaviour regarding moral hazard.

In addition, the existence of market discipline is often undermined because regulatory discretion is often granted. Gorton and Santomero (1990) contend that the presence of market discipline is non-linear and non-monotonic, and liabilities should be priced according to option theory. Nonetheless, liability pricing is based on the assumption that the regulator's closure policy is exogenous, well-defined and well-understood. In reality regulatory discretion is prevalent, and may take several forms, such as a straightforward injection of capital, or purchase and assumption. In any case, pricing tends to be complex. This leads to Gordon and Santomero's conclusion that risk premium cannot be priced accurately, because the regulator's closure policy is not implemented discretion-free.

These potential problems in measuring market discipline have led us to analyse the role of regulatory discipline. We examine the *ex ante* rules on how to resolve the problematic banks. We scrutinise two rules: exit enforcement and capital injection. In the former, banks are asked to terminate operations once they hit a threshold point of exit (e.g. insolvency). We study different scenarios such as implementation of the early closure rule (i.e. banks are asked to leave the financial system once they fail to fulfil minimum capital standards), the capital enforcement rule (i.e. banks hold higher levels of capital to strengthen their financial positions), and the netting operation rule (i.e. changing the frequency of netting that takes place between transactions in order to finalise winning and losing scores). The third scenario demonstrates, as further discussed in footnote 1, that systemic risk is often transmitted through the netting operation or the payment system. In order to prevent systemic risk from spreading, it may be better to increase settlement frequency.

The next rule is capital injection whereby a third party injects a necessary amount of capital in order to prevent bankruptcy. Here the third party plays the role of "lender of last resort." We compare how costly each rule and scenario is for the participants, and try to obtain a clearer view in order to assess the least costly resolution to prevent bank failures.

Finally, we would like to stress that the ultimate objective of market participants, policy makers and the clearing house is to keep the markets open; there is no profit source if the markets were closed and no more transactions were to take place. In this sense, our view is consistent with the alleged short-run objective of market participants, i.e. profit maximisation.[2] This statement implies that participation is the first step to profit maximisation. Consider players initially choosing whether to participate or not, and then choosing strategies to maximise their

[2] What is implicit in this statement is that limited liabilities are adopted and the participation constraints are satisfied.

payoff. As long as they are secured by limited liabilities, they will initially choose to participate in the transactions. In this paper, we do not concern the second stage of the game, i.e. the choice of strategies, but our objective is to compare and contrast ourselves with the ways in which the financial market can continuously operate without the severe contagion effect by altering the rules to identify those players who must leave.

2 Coin-tossing game reflecting banks' investment activities

Let us describe how we analyse the contagion and study the menu of policy options designed to prevent such contagion. We will treat bank investment activity as a coin-tossing game, the simplest random walk process, which will enable us to examine a detailed mechanism of the contagion effect. We will then study where the drawbacks lie in the classical random walk model as we interpret it as financial transactions. We will review the main theorem of the random walk model. We will then show a non-technical explanation of the arc-sine theorem and describe its implication in our model. Lastly, we will explain a formal set-up of the game, especially the importance of the netting activities to settle the winning and losing scores between banks. The simulation results are presented in sections 3 and 4.

2.1 Coin-tossing game revisited

Unlike most bank studies where collecting deposits and making loans are carefully modelled, we will focus on financial transactions that take place between financial institutions, such as derivative transactions. A simple way to illustrate such activities is to assume the outcome of the transactions is summarised in the results of the random walk process, such as a coin-tossing game.[3]

Some may feel that a game so simple will not capture all essential features of today's highly sophisticated financial transactions. Nonetheless, we believe that one crucial feature of such financial transactions is their zero sum nature, namely that the payoff of each participant will total zero, and this feature is easily incorporated in the coin-tossing game. To illustrate, think of a call option, where the buyer of the option has the right to buy stock at a specified exercise price. Suppose further that the market price of the share is higher than the exercise price. Then the buyer can exercise the right and earns profit equivalent to the difference between the market price and the exercise price. On the other hand, the seller of the option suffers an equivalent loss. If we add up the buyer's profit and the seller's loss, we expect a total of zero, because the profit simply reflects the loss borne by the counterpart. In this sense, we are not concerned with the nature of the derivative transactions, but instead we wish to emphasise that buying and

[3] The coin-tossing game has been developed extensively in probability theory. The origin of probability theory can be traced back to Pascal and Fermat who formulated the theory of chance in 17th century France. Since then, the notion of chance has found its way into almost all branches of knowledge.

selling the derivatives will enable them to achieve higher efficiency in terms of risk management, whereby we expect social welfare to improve. The welfare level is not zero sum after the derivative transactions. This is, however, not peculiar to financial derivatives. Any transaction which transfers risk from one agent to another implies that the social welfare level will change as long as the transaction is carried out among agents with differing risk attitudes.[4] For example, recall Diamond and Dybvig (1983), where the bank is assumed to be risk neutral and depositors are risk averse. The demand deposit contract, which insures depositors against liquidity risk, will improve overall efficiency to a level that could not have been achieved without the deposit contract. This implies that as long as the game is measured in terms of Pareto efficiency, we cannot conclude that the game possesses the zero-sum feature any more. In the end, let us refer to Feller (1957), who notes that “the (coin-tossing) model may serve as a first approximation to many more complicated chance-dependent processes in physics, economics, and learning theory.”[5] We now review some stylised features of the coin-tossing game and explain how we depart from the classical set-up of the game in order to bridge the gap between the classical coin-tossing game and the nature of financial transactions.

Let us next explain how the gap is bridged by incorporating several factors which involve further features of the financial practice. We focus on two established types of the random walk: the random walk with absorption barriers and the random walk with reflection barriers. One example of the random walk with absorption barriers is the Gambler’s Ruin problem. Let us call this problem the “classical ruin problem,” where two players play the game and their aggregate endowments are fixed. As soon as one player’s position reaches zero (i.e. he loses his entire endowment after playing the game), the game stops. The main point of the classical ruin problem is to identify the probability of the gambler’s ultimate ruin and to calculate the duration of the game.[6] This set-up, nevertheless, has several drawbacks when we try to interpret the problem as financial transactions.

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[4] This issue is somewhat related to the St. Petersburg Paradox. According to the paradox, the game runs as follows: suppose a coin is tossed repeatedly until the head appears. The gambler will be paid $2n$ dollars if the head appears in the n-th toss. Then the question is, what price is the gambler willing to pay for the privilege of gambling? The expected return from the gamble which the gambler wants to maximise is,

$$2(1/2)+2^2(1/2)^2+\ldots+2^n(1/2)^n=1+1+\ldots+1=\infty;$$

hence, the gambler should be willing to pay any large sum of money in order to enter this gamble. Several suggestions have been made to challenge this paradox, including one by Daniel Bernoulli, who proposes that one maximise the expected non-linear utility rather than the expected return. Taking a finite number for the expected utility, one can derive an upper bound for the price that the gambler is willing to pay. In order to understand why the agents play a zero-sum game (expected return is zero), it is more natural to think that the agents use the expected utility (not the expected return) for the basis of their decision making.

[5] Feller (1957) p.71.

[6] It is worth mentioning well-established results in the classical gambler’s ruin problem. Under a fair game with the same probability of win and loss (i.e. 1/2), the probability that the gambler’s wealth reaches N (combined capital) starting from i is i/N. The expected duration of the game is $D = i(N-i)$. See Feller (1957), Maitra and Sudderth (1996) or Ross (1996) for more details.

On this front, we intend to present the "new ruin problem." In the following paragraphs, we will describe the main departure from the classical ruin problem.

First, the game in practice normally consists of more than two players. This implies that even if one player is forced to quit, there remains a possibility that the game continues among the remaining players. Our game is thus extended by incorporating more than two players in the game. It should be pointed out, however, that the study of the ruin problem deals with the case where there are more than two players. But the study does not address how long each player can survive, nor does it address how each player's position is influenced if the game continues among the remaining players after other players are eliminated.

Second, we believe that the ruin decision essentially inherits a mechanism underlying the closure rule. In other words, whether a player is forced to retire or not depends on such *ex ante* rules as (1) the player is ruined if he fails to pay the liabilities fixed through the netting operation, or (2) the player is ruined if his capital ratio falls below a certain level (prompt corrective action). By incorporating different rules of closure, we can derive some policy implications of the ruin decision. This is another extension we will put forth in the following section.

A related issue in this ruin decision is how to decide the bank's closure. The key lies in the payment operation, through which the settlement amount is calculated, allowing for assessment of the bank's net wealth. In extreme cases, the operation can take place every time the coin is tossed. This is equivalent to the philosophy of the "real time" settlement. In the following section, we will see how the amount of capital necessary for each player changes by alternating the frequency of the settlement interval.

Turning to the random walk with reflection barriers, one normally assumes that when one player reaches either the upper or lower barrier, the other player compensates him for the same amount and lets him stay in the game. Again, this is crucial if the game consists of only two players; the game cannot be played any more if one player does not save the other. This operation does not have any practical meaning, once we allow more than two players in the game. Imagine a gambling house;[7] it does not make sense why an almost ruined player's position has to be covered by the winners. By leaving the almost ruined player without any compensation, winners may not be able to collect all of their winning positions, but they can still keep playing the game had they not saved the ruined player.

Let us summarise our new ruin problem. First, we allow more than two players in the game and we let the settlement operation take place a number of times. Second, we specify the closure rule in various ways; we can let banks go bankrupt if they become insolvent, or we can let them fail if their net wealth position recedes from a certain positive level (i.e. early closure rule). Third, we explicitly introduce, as a second rule, rescue operations by a third party. Note this operation

[7] In practice, most legally approved gambling houses sell tokens or tickets in exchange for cash before admitting people to play the game with these tokens or tickets, which allows for guaranteed payments for winners.

is specifically intended to show the effect of capital injection. We are aware that such a policy may generate moral hazard, but our intention is to present the bottom line of the cost associated with the contagion effect, i.e. the cost can still be generated even if we assume banks do not engage in any risk-taking behaviour.

2.2 Arc-sine theorem

The purpose of this section is to direct the reader's attention to a striking result derived in the coin-tossing game. The theorem was established as part of the probability theory in the 1950s,[8] but has been somewhat neglected mainly because of its nature—it contradicts generally accepted views based on the law of large numbers.

More precisely, the random walk process has two seemingly contradictory features; a mean reversion (recurrent) process and the process summarised in the arc-sine theorem.[9] To show these contradicting features of the policy, we derive a completely different implication.

The mean reversion process implies that a player will return to his initial position if the game is played *ad infinitum*. As a corollary, the wealth positions of all players at any point in time form a normal distribution. This feature is the keystone of modern finance theory.[10] Recent discussion of the value-at-risk (VaR) method of market risk management is also based on this feature.

The arc-sine theorem, on the other hand, implies that it is more likely for players to stay continuously on the winning side or the losing side than to frequently switch between the two sides.[11] This neglected theorem has a remarkable policy implication for the recent financial market crisis. While the mean reversion process tends to imply a wait-and-see policy or a no-intervention policy, namely that no matter how badly banks perform, many of them will return to the mean; hence, they can be left alone. However, the arc-sine theorem justifies an early closure policy and is sceptical of the wait-and-see policy, namely that those who perform poorly have most likely been performing poorly and unless an infinite

[8] The arc-sine theorem was first discovered by Paul Levy in his study of Brownian motion, then developed by Evolos and Kac as a limit theorem for sums of independent random variables with finite second moments in line with Lindeberg conditions (see Loeve (1977) Vol. 1. p. 380). In our daily life, we often come across situations where an unfortunate event causes another unfortunate event. It is such a rare event for one who has been suffering substantial losses in the casino to all of a sudden recover the accumulated losses and start making gains. Similar observations can be seen in the coin-tossing game.

[9] Ross (1996, p.148) cunningly points out that the arc-sine theorem indicates the proportion of time the symmetric random walk is positive does not converge to the constant value 1/2. On the other hand, it is clear by symmetry, the random walk process returns to an initial position (recurrent) if the process continues infinitely. The reason why the above two seemingly contradicting statements are in fact not a contradiction is that the expected cycle time is *infinite*. See Appendix 1 and 2 for more details. As Keynes wrote "in the long run we are all dead" (*A Tract on Monetary Reform*, 1924), recurrence in infinity may not have a practical implication.

[10] In the modern finance theory, most prices of financial assets are assumed to be non-zero, thus to follow geometric Brownian motion (i.e. random walk with reflection barrier at zero). The mean reversion process is reinforced by this assumption. See Dixit (1993) and Duffie (1996).

[11] In terms of the fair coin-tossing game, extreme scores such as large number of heads (or large number of tails) tend to appear quite frequently.

time interval is granted, it will probably take a substantially long time to recover; hence, given a finite time period (e.g. players' performance has to be evaluated every accounting year), they should not be left alone but rather they can be expelled from the system. The most disturbing aspect of this theorem is that even after expelling poor performing players, this financial game will generate endogenously other poor performing players. In other words, this game will not converge to the stable equilibrium where there are no more players who perform poorly, but keep generating divergence between winners and losers. So the game is inherently unstable and is constantly exposed to the contagion risk.

In the following, we will prove this rather counter intuitive theorem and see if we can apply it to our Monte Carlo simulation. To give readers a more concrete feel for the arc-sine theorem, we will borrow a numerical example from Feller (1957). In this example, the game conducts many coin tosses at the rate of one per second, day and night, for 365 days. The total number of tosses will be approximately 31.53 million. The probability that a gambler whose score is eventually negative, but during the course of the game, has a positive score (i.e. more heads than tails at that point) for a total time less than the listed time of lead, is summarised as follows.

Table 1: Time of lead in the coin-tossing game

Probability of The Unfortunate	Time of Lead
0.90	153.95 days
0.80	126.10 days
0.70	99.65 days
0.60	75.23 days
0.50	53.45 days
0.40	34.85 days
0.30	19.89 days
0.20	8.93 days
0.10	2.24 days
0.05	13.5 hours
0.02	2.16 hours
0.01	32.4 minutes

Source: Feller (1957, p.83)

The first row of the table describes that 90% of the losers (i.e. unfortunate gamblers with negative scores at the end of the game) have a positive score throughout the game for only 153.95 days. What is even more surprising is that approximately one in two losers enjoys a positive score for only 53.45 days of the year. This example highlights that it is more likely for an unfortunate event to last long, even in a perfectly fair game such as coin-tossing. Let us now formally present the theorem.

Arc-sine Theorem (Feller (1957, p.82))

The probability that in the time interval from 0 to 2*n* the position spends 2*k* time units on the positive side (i.e. the position lies above the initial level) and 2*n*-2*k* time units on the negative side (i.e. the position lies below the initial level) equals $R\{2k,2n\} \equiv P(2k,0) \times P(2n - 2k,0)$, where

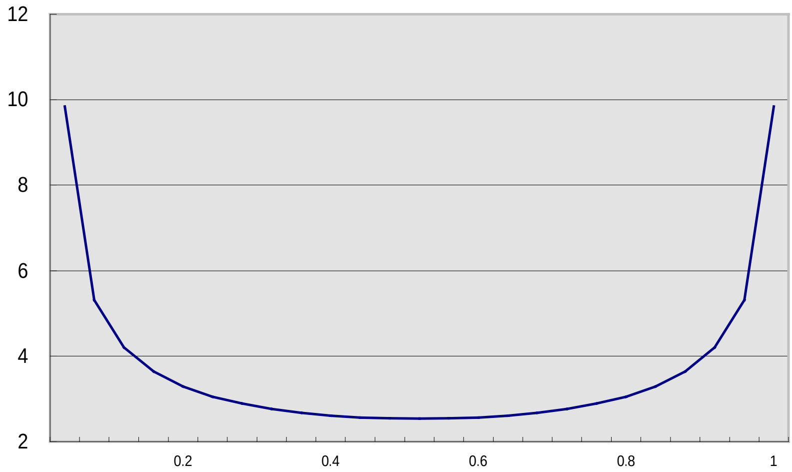
$$P(n,r) = \binom{n}{\frac{n+r}{2}} \times 2^{-n}.$$

Hence,

$$R\{2k,2n\} = \binom{2k}{k} \times \binom{2n-2k}{n-k} \times 2^{-2n}.$$

For a detailed theoretical discussion and proof, see Appendix 3. The following figure graphs the theorem. The x-axis denotes *k*/*n*, i.e. the probability that the position stays positive, and the y-axis denotes $R\{2k,2n\}$ times the number of banks, which is 100 in our simulation.

Figure 1: Graphical presentation of the arc-sine theorem



The figure explicitly shows the main feature of the arc-sine distribution whereby the probability that the position is on the positive side half the time is the least likely outcome, and the probability that the position stays either on the positive or negative side most of the time is the most likely outcome.

The implications of this theorem are rather straightforward. Those who fall on the negative side tend to stay on the same side and therefore have less chance to recover their losses, even if they continue to play the game for a substantially long period of time. Specifically, the only practical solution for those who have huge losses is to leave the game as soon as possible, if feasible.

In the following section, we will see how this feature can be attained in our Monte Carlo simulation, but first let us describe the basic set-up of the game.

2.3 Set-up of the simulation

There are 100 banks and all of them are engaged in investment activities. Through the simulation, we derive a minimum level of wealth required to generate the outcome specified in the scenario. Each scenario will be described in greater detail later in the section. There is also a clearing house whose role is to preside over the netting activities among the banks. We introduce three notations; round, position and score. We define "round" as the duration of investment activities between the netting operations. We assume that there are several rounds of play, and each time a round finishes, the netting takes place until the game completely ends. Next we define "position" as the net-wealth of each bank. The position of each bank before the first round is equivalent to an initial level of wealth, and changes as the game proceeds, according to each bank's performance. In the worst case scenario, if a bank performs very poorly in one round, it may not have a sufficient position to cover the losses incurred during the round, in which case the bank may go bankrupt or seek capital injection from the government. Lastly, we define "score" as the outcome of investment activities during the round.

The game in our simulation proceeds as follows. Each bank tosses the coin and adds + 1 when it obtains heads and – 1 when it obtains tails. It independently tosses the coin and records the number of heads and tails to compute the score. For example, if the bank has 8 heads and 2 tails after 10 tosses, its score is 6. Then at the end of the first round of play, when scores must be settled, the clearing house will undertake the following procedure.[12] First the banks are divided into two categories by their scores. If the score is positive, they are regarded as "winners," whereas if the score is negative, they are regarded as "losers." Among the losers, if a bank's position is sufficient to cover the score, it will give the clearing house an equivalent amount to the score. On the contrary, if its position is not sufficient to cover the score, a problem arises. In the following, we examine different scenarios by incorporating how a bank with insufficient funds to cover losses is treated. In the first scenario, we allow banks to go bankrupt as soon as

[12] The following interpretation of the settlement operation may potentially underestimate the contagion effect. In our simulation, the net debit position of the players is pooled and surviving players will share the loss if any (loss sharing rule). Hence, each player's exposure to a potential loss is limited. If on the other hand each player is fully exposed to the potential loss caused by a bilateral transaction, the impact of the counterpart's default is much more serious, and we may observe a severer degree of contagion effect.

they become insolvent. In the second scenario, we examine the early closure rule, i.e. banks become bankrupt once they fail to fulfil a required standard, such as a solvency ratio. In the third scenario, we change the frequency of netting operation (i.e. the duration of each round) and see if there is any effect on the bank's initial wealth which prevents bankruptcy.

3 Simulation result: Exit enforcement

In our GAUSS programme, the coin toss is based on the uniformly distributed random variable between zero and one. If the variable is greater than (or equal to) 0.5, we assume the coin comes up heads, and add 1 to the bank's score. On the other hand, if the variable is less than 0.5, we assume that the coin comes up tails, and subtract 1 from the score.[13] Each bank is assigned a specific number as a "seed." Each seed contains a stochastic process, known as a *Markov chain*, i.e. the conditional distribution of any future state x_{n+1}, given the past states x_0, x_1, ..., x_{n-1} and the present state x_n, is independent of the past states and depends only on the present state. Having the same seed implies a specific Markov chain is assigned to a specific bank.

3.1 Basic result

In this simulation, we assume that banks which failed to cover their losses in a round of play will go bankrupt after paying off their position to the clearing house. This means that the clearing house is granted authority to seize the bank's assets (position) if the bank fails to cover its losses from the round. After the money is collected from the losers, the clearing house will distribute it to the winners. If there were no banks which failed to pay their losses in full, winners will receive their winnings in full. If the clearing house fails to collect all the money as a result of the bankruptcy of losers, winners will not fully receive their winnings, but will instead be paid their winnings in proportion to others' winnings.

After the money is distributed among the players, the losers who failed to pay their liabilities in full will go bankrupt, and they will not be engaged in the investment activities starting with the next round. Other players, i.e. winners and losers who covered their losses but still managed to sustain a positive position, will proceed to the next round, where the same activities will continue: tossing the coin a certain number of times, counting the score, and executing of the clearing house netting operations. According to these specifications, we obtain the following results:

[13] In this set-up, the scores of winners and losers uniformly move one unit (±1), as we ignore the case where a player bets more than one unit in a game. However, the optimal strategy of attaining a certain level of wealth or of maximising expected playing time is the timid strategy that always bets 1 unit when the probability of win or loss is 1/2. See Ross (1983, p.76-83).

Table 2: Required level of wealth which leads to 10 banks defaulting at various stages of coin-tossing

	sim=1	10	20	50	100	150	200
n=1	0	5	7	11	11	16	18
10	6	13	22	27	43	63	70
20	7	22	28	45	60	79	103
50	13	31	50	71	111	143	154
100	15	49	73	112	151	212	242
150	20	64	98	140	204	273	350
200	22	68	111	168	252	298	383

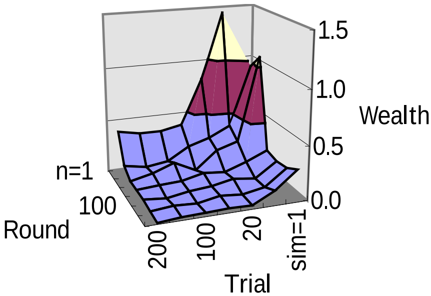
Table 2 shows the minimum amount of wealth each bank needs to hold if the system allows 10 banks to default through the course of play.[14] The row (n) in the table represents the number of coin tosses taking place during a single round. The column (sim) represents the number of rounds. Hence, each bank will toss the coin a number of times equal to the product of row and column (n×sim). At the end of each round, the operation which assesses the winning and losing scores for every bank will take place. This is equivalent to the net settlement. For example, the table says that each bank will need to hold 383 units of wealth if the game is to last 40,000 tosses during which the netting operation will be carried out every time 200 coin tosses have taken place. It is quite obvious as the number of coin tosses increases the amount of initial wealth necessary to create the same number of defaults will increase. For the cases to be discussed later, we shadowed 151 units of wealth as a benchmark.[15]

In Figure 2, we show a 3-dimensional presentation of the wealth per coin toss. In other words, it plots the amount of wealth necessary for each coin toss. For example, 383 units for 40,000 coin tosses in Table 2 is plotted as approximately 0.01. This figure signifies that the wealth per coin toss will generally decrease and stay around 0.01, as the number of tosses increases.

[14] Admittedly the simulation is not sophisticated enough to allow banks to start with different levels of wealth. Our simulation thus far treats all 100 banks homogeneously at the beginning. It is an important extension, because the initial wealth represents the strength of the financial position (i.e. the larger amount of wealth, the less likely that the bank goes insolvent), and the financial system is better represented by players with differing wealth positions. Nevertheless, our simulation allows wealth positions of banks to vary from the second round onwards.

[15] After 10,000 coin tosses, the marginal increase in required levels of wealth becomes substantially small. We believe that the marginal gain of increasing the number of coin tosses becomes relatively small and that small sample bias may disappear after 10,000 times. See Figure 2.

Figure 2: Level of initial wealth per coin toss



(Contagion effect)

We now proceed to show how the existence of the contagion effect affects the financial state of the other players in the game. The best way to see the effect is to measure wealth distribution at the end of the game. To be more precise, we first take the case where the contagion effect exists, namely that banks which failed to pay their liabilities in full go bankrupt and other banks without receiving their full winnings keep playing the game. We choose as a benchmark the situation where banks hold 151 units of initial endowments and play 100 rounds of the game, where each round consists of 100 coin tosses; hence the coin is tossed 10,000 times. This figure is taken from Table 2 which results in 10 bankruptcies. Second, we assume that banks suffering financial difficulties are rescued by having their liability positions covered by third parties.[16] They are rescued by means of capital injection, and the winners in each round of the game receive their full winnings. After they pay the liabilities in full, they can keep playing the game. This coincides with the case where there is no contagion effect and the risk derived from the inability of losers to pay liabilities in full will not take place.

The next table and figure show the summary statistics for the distribution of wealth with and without the contagion effect after 10,000 coin tosses.

[16] This is equivalent to the capital injection case studied in the next section. For a detailed analysis, see section 4.

Table 3: Summary statistics of wealth distribution for the surviving players

	Contagion	No Contagion
Mean	167.8	156.3
Standard Deviation	90.5	96.2
Minimum Value	5.0	0.0
Maximum Value	422.4	424.9
# of Players	90	100

Table 4: Summary statistics of wealth distribution for the winning players

	Contagion	No Contagion
Mean	224.2	226.4
Standard Deviation	64.2	64.1
Minimum Value	151.1	152.9
Maximum Value	422.4	424.9
# of Players	55	55

In the table, “Contagion” refers to the situation where the players toss the coin 10,000 times (n=100 and sim=100) with an initial wealth of 151. As seen from Table 2, this is equivalent to the case where 10 banks go bankrupt; hence, we expect to observe the contagion. On the other hand, “No Contagion” refers to the situation where the players again toss the coin 10,000 times (n=100 and sim=100) with an initial wealth of 151. This time, however, the players who fail to pay their losses have their debt positions covered by a third party; hence, we do not expect to observe the contagion effect.

We have summarised two tables: the first table shows the statistics of wealth distribution for all surviving players, and the second table shows the statistics of wealth distribution for the winners. From Table 3, it is not apparent whether or not “Contagion” really has a contagion effect, because the mean distribution is much higher for “Contagion.” This is, nonetheless, due to the fact that in cases of “No Contagion,” banks which played very badly receive injected capital, but they tend to perform at the very bottom of the wealth distribution.[17] This pulls down the mean wealth.

Table 4 shows the very weak contagion effect.[18] By focusing on the wealth distribution of the winning players, we see that the “Contagion” mean is margi-

[17] Here we assume that the third party covers only debt positions; hence, the wealth of the injected banks right after the injection is zero. We will further develop the analysis where the injection policy is more accommodating, allowing for situations such as injections raising a bank’s wealth position to 8% of the initial level.

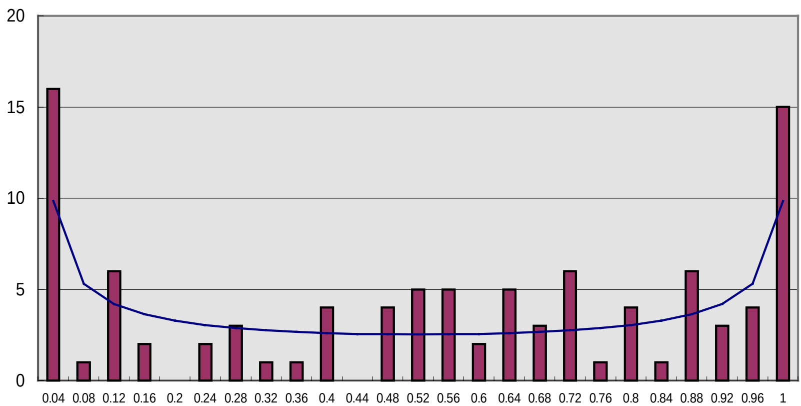
[18] This is basically because the contagion effect is pooled among all winners, so that a marginal impact per player is limited.

nally lower than the “No Contagion” mean, with the same number of winning banks. Both the minimum and maximum wealth positions in “No Contagion” are marginally higher than those in “Contagion.” Although its effect seems rather weak, it seems to be consistent with our proposition that contagion weakens the financial positions of sound players.

(Arc-sine feature)

Next, we present a result that exemplifies the arc-sine nature. We have seen a benchmark case where 100 banks hold 151 units of initial wealth and play 100 rounds of the game with each round consisting of 100 coin tosses. Figure 3 shows the histogram of the first round, where the x-axis represents the probability that the accumulated score of coin tosses lies above zero, and the y-axis represents the number of banks. For example, if the accumulated record of 10 coin tosses follows a path, {1, 2, 1, 0, –1, –2, –3, –2, –1, 0}, then the record is 4, because the intervals $(0 \rightarrow 1)$, $(1 \rightarrow 2)$, $(2 \rightarrow 1)$ and $(1 \rightarrow 0)$ remain above zero, i.e. on the positive side. The line in Figure 3 represents the number of banks theoretically derived from the arc-sine theorem. This is a pure random walk process. Here the higher the probability is, the higher the number of times the score remains afloat.

Figure 3: Graphical presentation of arc-sine after the first round in the benchmark case

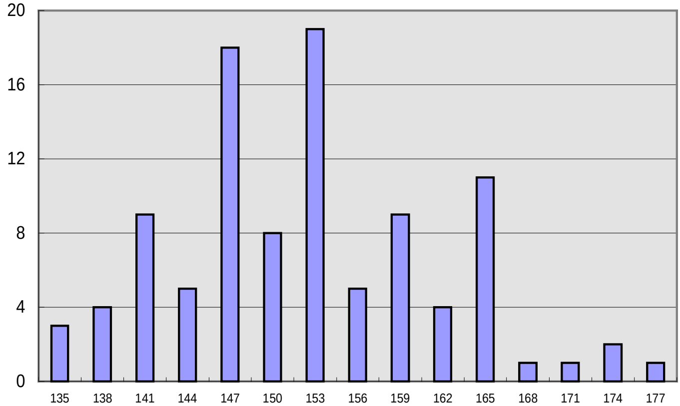


The figure inherits the nature of the arc-sine distribution, where probabilities staying on either the positive side or negative side are more likely to occur than other probabilities.

The next figure presents the wealth distribution at the end of 100 coin tosses.

All the banks start with 151 units of initial wealth.

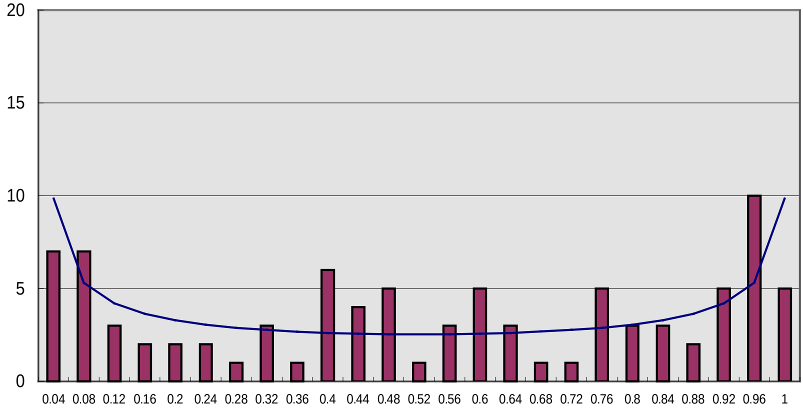
Figure 4: Wealth distribution after 100 coin tosses



By construction of the game, we find the contagion effect is rather weak, although the results still support our proposition that the contagion will weaken the financial positions of the sound players. On the other hand, the arc-sine theorem shows that those banks that are destined to perform poorly continue to perform poorly. From this result, it is quite costly to leave the poor performing banks; a wait-and-see policy will allow these banks to perform even worse and the contagion effect will become more prevalent. Thus it is better not to wait until they recover in the unforeseeable future.

What we are more interested in is to see whether we still observe the differences as suggested above by depicting the observations at the end of each round. Of course, the path which consists solely of the end-of-round observations does not represent the random walk in a genuine sense—a sequence of the outcome after the aggregation of each round's 100 coin tosses, followed by the netting operation in order to make the game zero sum, does not possess a random walk feature, i.e. +1 with probability 0.5 and –1 with probability 0.5. Figure 5 is a histogram that records the number of times that the banks' position stays above the initial wealth. There seems to be less observations under extreme probabilities, but the figure still inherits the arc-sine feature—a tendency which suggests there are more banks which either play poorly (i.e. the probability is close to 0) or nicely (i.e. the probability is close to 1).

Figure 5: Graphical presentation of arc-sine at the end of each round in the benchmark case

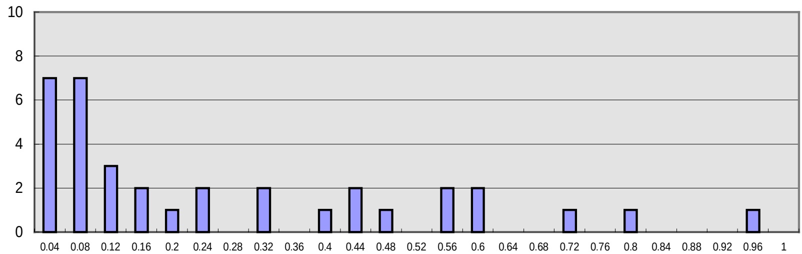


The results in Figures 3 and 5 suggest a general trend tracing an arc-sine. In one sense, the result in Figure 5 is quite striking; the path we follow in the figure is not a purely random walk—the change in wealth position from round to round is not uniform and the settlement process at the end of each round disturbs the smoothness of the arc-sine.

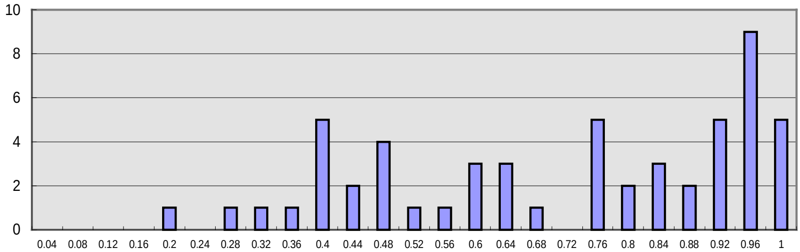
What is more striking is that once we divide banks into two groups: those banks that are ultimately winners (i.e. banks whose position ends above the initial wealth) and those that are ultimately losers (i.e. banks whose position ends below the initial wealth), we find the winners' history of end-of-round wealth tends to be generated by good performance whereas the losers' history of wealth is generated by poor performance. This observation is graphed in Figure 6.

Figure 6: Histogram of winners and losers at the end of each round

Losers



Winners



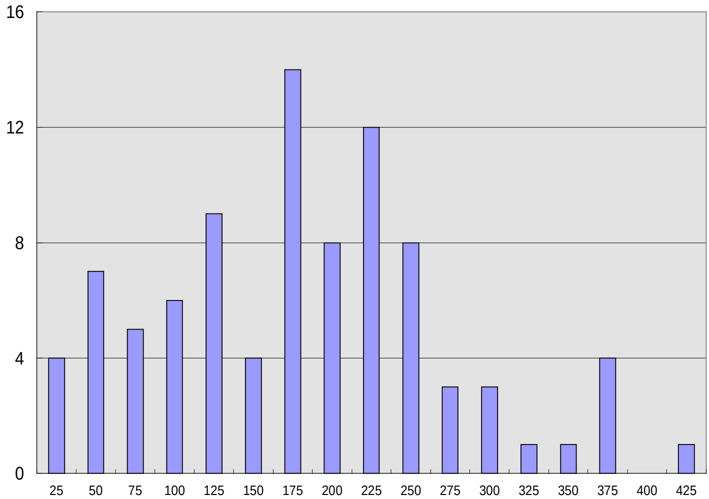
We see that the lower end of the arc-sine graph generally comes from losing banks. On the contrary, the second figure is the histogram based on the number of winners. Here we understand that the upper end of the arc-sine graph consists of these winners. In sum, Figure 6 summarises our conjecture that the upper end of the arc-sine histogram comes from the winners and the lower end of the histogram comes from the losers.

One implication we derive from this analysis is that those banks that are destined to perform poorly (i.e. to stay on the negative side) will continue to perform poorly. From this point of view, it will be quite costly to leave the poor performing banks; a wait-and-see policy will allow these banks to perform even worse and the contagion effect will become more prevalent. Thus it is better not to wait until they recover in the unforeseeable future. Owing to this observation, the next section concerns several policy options and evaluates which policy seems appropriate.

We examine three scenarios. First we study what if banks are asked to close operation before they become insolvent. We study the case where the threshold level of closing its operation is 8% of the initial wealth. We compare the performance of players and see how effective this early closure policy is to isolate the contagion. Second we examine what if they start with higher levels of initial wealth by raising additional capital. We identify four cases that generate no bankruptcy, 1 bankruptcy, 5 bankruptcies and 10 bankruptcies, and the minimum levels of initial wealth to generate an assigned number of bankruptcies are computed. Third we examine what if the clearing house changes the frequency of the netting operation. In an extreme case, the operation takes place every time banks toss the coin. We calculate how many banks will go bankrupt if we change the frequency of the operation while maintaining both the total number of coin tosses and the initial wealth constant.

Lastly Figure 7 presents the wealth distribution at the end of the game (i.e. 100 rounds).

Figure 7: Wealth distribution at the end of the game



3.2 Policy: Early closure rule

In the last section, we showed a possibility that the contagion effect results in a weakening of the winners' positions. In this section, we extend the analysis to study the scenario where the banks are forced into bankruptcy even if their net wealth position remains positive.

The last section assumed that banks fail if they cannot cover their debt positions with their wealth. In other words, if the summation of the wealth and the score of the game in each round is below zero, the banks are asked to leave the game. Here we assume that they are forced to shut down their operation if the summation of the wealth and the score fails to reach a certain level above zero, say 8% of the initial wealth position. In essence, this is in line with the early closure rule, often referred to as "prompt corrective action."

Prompt corrective action provisions are proposed as a possible approach to overcome time-inconsistency and eliminate any forbearance in financial regulation. For example the U.S. Federal Deposit Insurance Corporation Improvement Act (FDICIA 1991) mandates that any bank whose tier 1 capital ratio falls below 2% is subject to mandatory provisions such as the appointment of a receiver/conservator within 90 days, suspension of payments on subordinated debt, and restrictions on certain activities.

In the context of our game, the early closure maintains the following feature—due to the game's losers being asked to leave before their net-wealth position reaches zero (strictly speaking, they are not yet insolvent), the winners are able to collect larger sums; hence, the severity of the contagion effect is limited.

Although we may still observe some degree of the contagion effect unless full losses can be covered, we expect the effect to lie somewhere between the “Contagion” and “No Contagion” cases in the last section. The statistical results of a numerical example which enables us to run a comparative study with the benchmark cases is shown below.

Table 5: Summary statistics of the wealth distribution for the surviving players

		Reference	
	Early Closure	Contagion	No Contagion
Mean	177.6	167.8	156.3
Standard Deviation	85.2	90.5	96.2
Minimum Value	34.8	5.0	0.0
Maximum Value	424.1	422.4	424.9
# of Players	85	90	100

Table 6: Summary statistics of the wealth distribution for the winning players

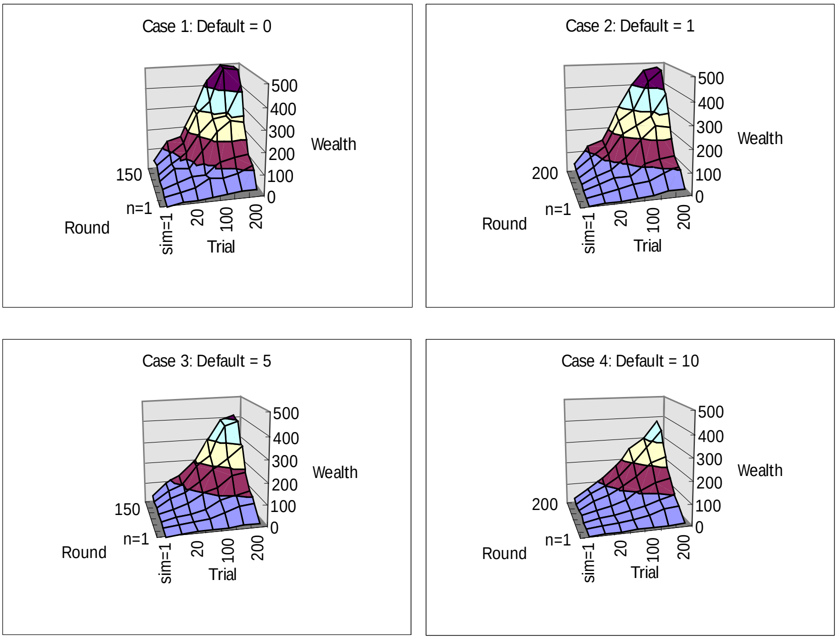
		Reference	
	Early Closure	Contagion	No Contagion
Mean	225.3	224.2	226.4
Standard Deviation	64.2	64.2	64.1
Minimum Value	152.5	151.1	152.9
Maximum Value	424.1	422.4	424.9
# of Players	55	55	55

From Table 5, we see that “Early Closure” results in more bank closures (15 banks), but the average wealth position of the surviving banks further improves. Table 6 reveals, as expected, that the average wealth position of the winners in “Early Closure” is higher than the position of the winners in “Contagion.” This seems to suggest that the early closure policy is effective in the sense that the contagion effect is restricted; hence, the policy would be of benefit to the financial system as a whole. The argument states that because losing banks tend to do worse, it is best to adopt a non-discretionary policy and let these banks fail through adopting harsher standards, such as the early closure rule.

3.3 Policy: Capital enforcement rule

One way to avoid the risk from contagion is to implement a tighter capital enforcement rule.[19] We know from Table 1 the minimum level of initial endowments which would allow 10% of the total banks to go bankrupt. If the regulator asks each bank to hold more endowments before the start of the game, how much would each bank have to increase its level? This study is conducted and summarised in the following figure. The figure represents the initial level of wealth necessary for no default (Case 1, i.e. none of 100 banks goes bankrupt), one bank's default (Case 2), five banks' default (Case 3) and ten banks' default (Case 4) which is the same result obtained in the previous sub-section.

Figure 8: Minimum level of initial wealth needed to generate a specified number of defaults



The general trend is as banks hold more wealth, they are less likely to face a threat of default. In order to achieve a "risk free system," banks have to hold a substantial amount of wealth, perhaps by means of raising additional capital, as shown in the figure as Case 1: Default=0. None of the banks having sufficient

[19] Note in this paper what we mean by "capital enforcement" is to raise additional capital at the beginning of the game. Hence the degree of capital enforcement is not related to the performance of banks. As we will examine in the next section, we distinguish it from "capital injection," which is another resolution mechanism whereby additional capital is injected if banks perform poorly.

wealth will go bankrupt, and the rescue operation as specified in the last section will not need to be implemented.

An interesting observation is to see how the capital enforcement rule and the early closure policy will interact with each other in our model. Their fundamental philosophy concerning how to resolve the systemic risk is, however, different.

In the capital enforcement rule, the policy guide is to raise additional capital, so that the players have sufficient wealth to avoid bankruptcy beforehand. Thus the more capital the banks raise, the less banks face the threat of going bankrupt. The policy aims to prevent market risk by strengthening the financial position of the players.

In the closure policy, the policy guide is how to pinpoint the poor performing players in the early stage and to get rid of them from the financial system before systemic risk becomes more prevalent. Thus the tighter the policy is, the greater the number of banks that will face bankruptcy. Here the policy does not aim to strengthen the financial position, but rather it takes the initial position as given and implements the mechanism according to which contagion does not spill over to other players.

In this sense, we claim that the former is ultimately contagion-free, whereas the latter more realistically lets the contagion happen but concerns how to minimise its effect.

In the following, we show a table where each cell identifies the improvement of wealth for survivors at the end of the game. In the table, the row represents the closure policy and the column represents the capital enforcement.

	151	186	245	278
Exit by Insolvency	+16.8	+9.8	+2.5	0
Early Closure	+26.6	+18.4	+7.6	+2.8

Each level of wealth in the first row represents respectively 10 bankruptcies, 5 bankruptcies, 1 bankruptcy and 0 bankruptcy, under the benchmark case with the exit rule specified in the first column. Obviously, by adopting the early closure rule, the number of bankruptcies in each level would rise. The general trend is, the larger the level of wealth banks hold, the less magnitude of improvement they experience. Also, the early closure rule, because it excludes a larger number of ailing banks, will improve the financial position of the survivors even further.

3.4 Policy: Frequency of netting operation

The next possibility we will explore is how the netting operation will affect the number of defaulting banks. Let us first show how the number of defaults is altered by changing the frequency of netting while maintaining not only the total number of coin tosses but also the initial amount of wealth in 151 units.

Our conjecture is that the number of bankruptcies will rise as netting takes place more frequently (i.e. larger values for n indicate that netting of the scores will take place for a lower number of coin tosses; for example, sim=5 and n=2,000 means that netting is done every 5 tosses). This follows basic intuition—the more frequently the banks net their scores, the more often they face a situation where they must pay their liabilities if they end up with negative scores after a round, the higher level of wealth they must hold *ex ante* in order to avoid bankruptcy.[20]

Table 7: Number of defaults for 151 units of wealth

sim	n	Number of Defaults
5	2,000	13
10	1,000	15
20	500	14
50	200	15
100	100	10
200	50	11
500	20	10
1,000	10	10
2,000	5	8

However, Table 7 shows results which are somewhat mixed. In the figure, we expected the number of defaults to fall as sim rose. The difference in the number of defaults between 13 in sim=5 and n=2,000 and 15 in sim=10 and n=1,000 may be marginal, but we are not inclined to make a concrete statement on how to interpret this result. The table does not seem to support our conjecture that the number of defaults will rise as netting takes place more frequently. In the next table, we show another result.

[20] In an extreme case, netting can take place every time the banks toss the coin (sim=1 and n=10,000 represent the case). This “real time” netting could not be addressed in this paper, because it exceeds our computer capacity.

Table 8: Minimum amount of wealth for various combinations of sim and n

		Number of Defaults			
sim	n	0	1	5	10
5	2,000	290	250	194	164
10	1,000	288	279	190	169
20	500	285	267	191	156
50	200	283	271	188	168
100	100	278	245	186	151
200	50	269	245	186	154
500	20	269	245	178	151
1,000	10	269	245	179	147
2,000	5	269	245	175	139
5,000	2	269	245	165	130

The table presents the minimum amount of wealth that causes four different numbers of default under specified numbers for rounds and coin tosses. Again, our benchmark (sim=100 and n=100 with 151 units of wealth) is shadowed. In order to reduce the number of bankruptcies, it is best to increase wealth to a level where there are no more defaults. This is the same policy implication we derived in the last section. We know, however, from Table 7, that there may be another route, namely that if additional capital accumulation is too costly (for example, each bank is asked to add 127 units of capital *ex ante* to reach a wealth position of 278 if they want to avoid any bankruptcy in the sim=100 and n=100 situation), they could prolong the interval of netting.

Of course, our numerical result in the table did not produce a case where there were no bankruptcies with an initial wealth equivalent to 151. The best they could achieve is to allow 8 bankruptcies by extending the interval of netting from one netting out of sim=100 to one netting out of sim=2,000 (see Figure 7). Nonetheless, our analysis in this section seems to suggest that the interaction between the capital enforcement rule and the frequency of settlement has to be considered. In other words, it is easy to see a paradigm where the capital is strongly enforced and the netting operation takes place in “real time” (for example, 290 units of wealth in sim=5 and n=2,000 results in no bankruptcy). Yet if we face the situation where capital is limited or is simply too costly for banks to hold the required amount of wealth by raising additional capital derived in the simulation, then a more feasible solution is to find a compromise which requires less wealth, but provides a lower number of defaults. Our simulation seems to have this feature; it suggests that it is not just increasing the amount of initial level of wealth, but once combined with the frequency of settlement procedure, it may achieve better results.

We also carried out the simulation with different seed numbers for banks to see if there were any differences in our results. The next table is based on the same GAUSS programme, but it features different seed numbers (we added 5,000 to each seed). The overall result of the simulation does not seem to show significant change.

Table 9: Minimum amount of wealth for various combinations of sim and n

		Number of Defaults			
sim	n	0	1	5	10
5	2,000	285	256	171	151
10	1,000	284	248	179	153
20	500	280	244	185	149
50	200	291	244	185	152
100	100	281	244	190	168
200	50	270	245	190	161
500	20	282	244	186	140
1,000	10	270	244	153	140
2,000	5	270	244	162	136
5,000	2	270	244	153	130

4 Simulation: Capital injection

In this section, we will add another dimension to the fundamental set-up of the game. In the previous section, it was simply assumed that a bank is driven out of the system once the loss in a round of play reaches a level that cannot be covered by the available funds (position). This is consistent with the classical Gambler's Ruin problem. Then we extended the analysis by adopting the early closure rule: if the summation of the bank's wealth and the resulting score of the game fail to reach the minimum standard the regulator has set in advance, then the bank is closed. We saw how this mitigates the contagion effect because the winners manage to cover larger shares of their winning positions. Nonetheless, what we have recently seen is not just how to get rid of the broke players, but also how to contain them in the system without causing moral hazard. Of course, our framework is too primitive to incorporate any incentive to assume an additional risk factor in investment activities, but our point is, even without generating moral hazard, the system is exposed to further risk by an additional injection policy of the regulator.

Here we ask the following question, what if a bank, faced with the situation where the loss exceeds the position, has been injected with capital rather than been forced to go bankrupt? We will see that, although the new capital is injected, the bank will repeatedly face the threat of going bankrupt, where in each case the regulator needs to add new capital. We believe this is due to the fact that as long as the bank maintains the same portfolio (the same Markov chain in our programme), it is more likely that the bank will continue to produce poor scores. Perhaps, a more reasonable solution is to close these banks before injecting additional capital.

4.1 Capital injection at insolvency

In the following tables, we summarise the total amount of capital injection that generates a specific number of bankruptcies (10 bankruptcies in Table 10 and 5 bankruptcies in Table 11). Here the amount of injection is defined as a net-debit position; hence, in each case the bank's position will become zero after both the capital injection and the settlement of the scores with other players take place.

Table 10: Total amount of capital injection with an initial wealth position generating 10 bankruptcies (as specified in Table 2)

	sim=1	10	20	50	100	150	200
n=1	n.a.	14.4	16.3	18.9	33.4	51.0	75.7
10	9.5	38.6	57.7	112.3	104.3	112.3	221.0
20	21.8	59.4	92.8	64.9	330.5	276.4	177.5
50	12.7	84.5	97.0	294.8	340.2	455.5	461.7
100	22.5	85.4	224.4	371.6	533.6	326.6	796.0
150	35.8	128.2	144.2	514.5	453.1	970.6	664.1
200	60.4	275.1	153.2	423.7	714.8	922.5	495.9

We need to compare the results in Table 10 with those in Table 8 where the minimum amount of wealth to generate different numbers of default is shown. In the benchmark case, we know from Table 8 that if the system has to generate the non-bankrupt case, each bank has to raise 127 (i.e. 278–151) additional units of capital; hence, the total cost for the system to reach a risk-free environment would be 12,700. In Table 10, however, we derive that the total cost represented by the amount of capital injection is 533.6. Although we should be careful in comparing the cost derived in these two figures,[21] our result seemingly suggests that any *ex ante* measure against bankruptcy is more costly than an *ex post* measure, because the latter pinpoints the exact amount of funds necessary to avoid bankruptcy, while the former requires a large risk premium.

The next table also conducts the same simulation which results in 5 bankruptcies. Because banks initially hold higher levels of wealth, the total cost of injection is much less, compared with the amounts specified in Table 10.

[21] The comparison requires care because the simulation has not incorporated heterogeneity of banks. If each bank is treated differently (e.g. riskier banks are required to raise additional capital to achieve higher levels of wealth while sound ones are not), the total cost is expected to decline.

Table 11: Total amount of capital injection with an initial wealth position generating 5 bankruptcies

	sim=1	10	20	50	100	150	200
n=1	n.a.	3.6	5.5	7.9	19.4	34.5	41.6
10	4.0	5.0	32.2	74.1	43.9	29.6	93.9
20	8.7	33.2	50.6	29.3	87.0	77.3	70.7
50	6.4	61.7	33.8	101.0	281.9	219.9	209.1
100	4.4	31.6	108.0	241.3	222.0	140.1	421.9
150	29.9	37.7	111.2	168.4	191.6	270.2	306.1
200	25.6	117.4	69.2	269.1	417.9	212.8	262.9

4.2 Capital injection at 8% level

We have also carried out another simulation. In section 4.1, the amount of capital necessary for injection is designed to cover a net-debit position. Here we adopt a more generous rule, namely that the amount of capital injection not only covers the net-debit position but also raises the next round's position equivalent to 8% of the initial wealth. Under this rule, banks which accepted the capital injection will have some buffer to start with from the next round of play.

Table 12: Total amount of capital injection with an initial wealth position generating 10 bankruptcies

	sim=1	10	20	50	100	150	200
n=1	n.a.	17.2	19.6	24.2	45.7	63.8	88.6
10	13.4	48.6	71.7	134.1	137.4	155.4	275.6
20	27.1	75.0	108.9	94.4	378.5	334.6	247.5
50	15.9	100.4	130.7	343.8	394.9	548.2	568.8
100	34.7	118.1	276.0	434.7	608.2	459.7	904.9
150	44.1	174.3	202.6	597.5	572.8	1,068.5	876.5
200	72.2	296.5	201.4	511.5	852.3	1,115.4	760.0

Table 12 summarises the total amount of capital injection that generates 10 bankruptcies. Because each time capital is injected, it raises the bank's position to 8% of the initial wealth, the total amount is much higher than the one in Table 10.[22]

[22] Admittedly, the analysis in this section is still incomplete. Several further studies remain to be done. We would like to know what the optimal level of capital injection is and how efficient is the capital injection.

5 Summary of simulation results

This section sums up broad simulation results. By construction of the game, we find the contagion effect is rather weak although our results still support our proposition that the contagion will weaken the financial positions of the sound players. On the other hand, the arc-sine theorem shows that those banks which are destined to perform poorly continue to perform poorly. From this result, it is quite costly to leave the poor performing banks; a wait-and-see policy will allow these banks to perform even worse and the contagion effect will become more prevalent. Thus it is better not to wait until they recover in the unforeseeable future.

5.1 Ex ante measures

Three types of ex ante policy measures are considered. Although we cannot provide an overall evaluation of which policy rule is the optimal, we show ample evidences that combined use of these rules would enforce the policy implementation more effectively.

(a) Early closure rule

We study what if banks are asked to close operation before they become insolvent. We study the case where the threshold level of closing its operation is 8% of the initial wealth. We compare the performance of players and see how effective this early closure policy is to isolate the contagion. From the benchmark case, we find the early closure results in more bank closures, but the average wealth position of the surviving banks improves further. This seems to suggest that the early closure policy is effective in the sense that the contagion is restricted. Comparing the early closure rule with the exit by insolvency rule, the former, because it excludes more ailing banks, would improve the financial position of the survivors even further than the latter. We therefore conclude that the policy would be beneficial to the financial system as a whole.

(b) Capital enforcement rule

We examine what if banks start with higher levels of initial wealth by raising additional capital. We identify four cases that generate no bankruptcy, 1 bankruptcy, 5 bankruptcies and 10 bankruptcies, and the minimum levels of initial wealth to generate an assigned number of bankruptcies are computed. In short, the general trend is that the larger the amount of wealth banks hold, the less likely they face the threat of going bankrupt. In order to achieve the “risk free system,” they have to hold a substantial amount of wealth, perhaps by means of raising additional capital. In this system, none of the banks having sufficient capital would go bankrupt, and the rescue operation as specified in the last section would not have to be implemented. Needless to say, this rule imposes a high cost on the participants.

(c) Frequency of netting operation

We examine what if the clearing house changes the frequency of the netting operation. In an extreme case, the operation takes place every time banks toss the coin. We calculate how many banks will go bankrupt if we change the frequency of the operation while maintaining both the total number of coin tosses and the initial wealth constant. In general, the more frequently the banks net out their scores, the more often they face the situation where they are asked to pay their liabilities if they end up with negative scores in the round of the game, and the higher the level of wealth they need to hold *ex ante* in order to avoid bankruptcy. Our simulation did not provide strong support to this conjecture, but the combination of this measure with others seems to be a promising as well as practical solution to the contagion effect. If we face the situation where the capital is limited or simply it is just too costly for banks to hold the required amount of wealth by raising additional capital derived in the capital enforcement rule, then a more feasible solution is to find a compromise which requires less amount of wealth with more frequent settlement.

5.2 Ex post measures

Although the new capital is injected, the bank will face the threat of going bankrupt repeatedly, where in each case the regulator needs to add new capital. This is due to the fact that as long as the bank maintains the same portfolio (the same Markov chain in our programme), it is more likely that the bank will continue to produce poor scores.

Our result also contends that any *ex ante* measure against bankruptcy is more costly than any *ex post* measure, because the latter pinpoints the exact amount of funds necessary to avoid bankruptcy, while the former requires a large risk premium.

6 Conclusion

This paper examined how we view the cost associated with negative externality, often referred to as “systemic risk” in the financial markets and how effectively *ex ante* discipline can prevent this externality from spreading to other participants in the market. The ultimate objective for the market participants, the policy makers and the operators of the clearing house is to keep the market open, and for that purpose we identified how the regulator sets the rule which prevents the contagion effect, while making as many participants as possible take part in the transactions.

By construction of the game, we find the contagion effect is rather weak although our results still support our proposition that the contagion will weaken the financial positions of the sound players. Moreover, the arc-sine theorem shows that those banks that are destined to perform poorly continue to perform poorly. This result has a remarkable policy implication that no matter how well policy rules are implemented, the finanical market game will not converge to any stable equilibrium where no player wants to play the game any longer. By construction, the game generates big winners and big losers constantly. It is the big losers that make the financial system unstable. As long as this financial market game keeps going, potential prevalence of systemic risk or contagion exists.

Practical solutions to prevent the contagion effect from spreading are twofold: market discipline and regulatory discipline. Nonetheless, the sheer existence of market discipline has been under controversy in various studies.

In this paper, we focused on regulatory discipline. We examined *ex ante* rules on how to resolve the problematic banks. We scrutinised three scenarios: early closure rule, capital enforcement policy and frequent settlement system. In the first scenario, banks which have performed very poorly but are not yet insolvent, are asked to close down. In the second scenario, the participants are asked to hold additional capital at the beginning of the game in order to prevent insolvency of the participants. In the third scenario, we examine how the amount of wealth necessary to generate the same number of exits is altered by changing the frequency of settlement operations.

Our simulation results indicate that each policy rule works with its own costs. Although we do not intend to identify which policy rule is most appropriate, through our analysis under the arc-sine theorem, our results seem to suggest that early closure and capital enforcement are effective devices for preventing the contagion effect. In addition, the combination of various policy rules seems to derive a more practical resolution mechanism for preventing the contagion effect.

We also studied the capital injection rule whereby the clearing house injects money into a participant facing potential bankruptcy, in order to improve the bank's debt position. In such a situation, the clearing house is a "lender of last resort." We concluded that the *ex post* measure is less costly than any *ex ante* measures, because the former pinpoints the exact cost necessary to prevent contagion.

Appendix 1: Diffusion processes of random walk

If the length of one-step (d) is made smaller and each step is spaced so close in time, the resultant change appears practically as a continuous Brownian motion.

Suppose that an unrestricted random walk starts at the initial endowment position x_0 and that the n-th step leads to the position w_n where $w_n = x_0 + x_1 + x_2 + ... + x_n$ is the sum of n-independent random variables each assuming the values $\pm d$ with equal probability of 1/2.

The variance of w_n is expressed as $v(w_n) = tD$ ($D=d^2/T$, d= length of one-step, T = unit time interval), and D is a diffusion coefficient. The mean of w_n is $E(w_n)=tC$ where C is a drift coefficient.

Considering the position of the particle at the n-th and the (n+1)-st trial, it is obvious that the stochastic process $p(z, t+T)$ satisfies the difference equations,

$$p(z,t+T)=\frac{1}{2}p(z-1,t)+\frac{1}{2}p(z+1,t)$$
$$p(0,0)=1,\ p(z,0)=0(Z\neq 0) \tag{A1-1}$$

Since p has continuous derivatives we can use the Taylor series expansion at the limit,

$$\frac{\partial p(z,t)}{\partial t}=-C\frac{\partial p(z,t)}{\partial z}+\frac{1}{2}D\frac{\partial^2 p(z,t)}{\partial z^2} \tag{A1-2}$$

This is a special diffusion equation known as *the Fokker-Plank equation* for diffusion. In our system, drift coefficient C is zero and thus (A1-2) becomes,

$$\frac{\partial p(z,t)}{\partial t}=\frac{1}{2}D\frac{\partial^2 p(z,t)}{\partial z^2} \tag{A1-3}$$

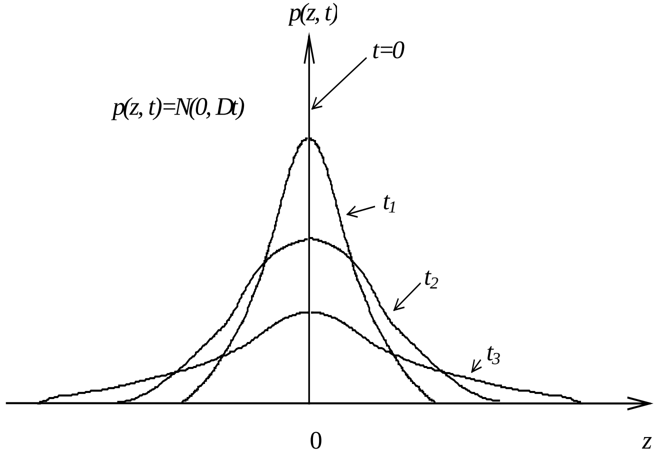
This is known as Brownian motion or Wiener process.

Stochastic process $p(z,t)$ can be obtained by integration over $(-\infty, \infty)$, hence by Fourier transformation with initial condition $p(0,0)=1$,

$$p(z,t)=\frac{1}{(2\pi Dt)^{1/2}}e^{-\frac{z^2}{2Dt}}\equiv N(0,tD) \tag{A1-4}$$

This turns out to be the normal distribution with mean = zero and variance = tD. Figure A1-1 shows changes in the probability density of Brownian motion over time ($t_1 < t_2 < t_3$).

Figure A1-1: The probability density of Brownian motion without drift



Appendix 2: Relationship between the mean reversion process and the arc-sine theorem

Under a fair and infinitely repeated game, a player will return to the initial position with probability one. This is known as the mean reversion process (see Feller (1957), pp.347-348). The arc-sine theorem, on the other hand, implies that it is more likely for players to stay continuously on the winning side or the losing side than to frequently switch between the two sides.

The former states that a player will return to the initial position for sure, while the latter states that a player tends to stay on one side for most times in the game or the rounds of games.

These seemingly contradictory phenomena occur in the same diffusion process of a random walk as shown in Appendix 1 and they are, in fact, not contradictory at all, if we understand the nature of a random walk, i.e. non-stationarity.

We can restate the random walk process discussed in Appendix 1 in terms of time series econometrics. According to Granger and Newbold (1986, pp.38-41), a random walk without drift has the following characteristics.

If the process starts at t=0 with initial endowment, x_0, then

$$x_t = x_0 + \sum_{j=0}^{t-1} \varepsilon_{t-j} \tag{A2-1}$$

where ε_t is a zero-mean white noise process.

$$\mu_t = E(x_t) = x_0 \tag{A2-2}$$

$$\lambda_{0,t} = \mathrm{var}(x_t) = t\sigma_\varepsilon^2 \tag{A2-3}$$

$$\lambda_{\tau,t} = \mathrm{cov}(x_t, x_{t-\tau}) = (t-\tau)\sigma_\varepsilon^2, \tau \geq 0 \tag{A2-4}$$

where σ_ε^2 is the variance of ε_t and is finite. Thus,

$$\rho_{\tau,t} = corr(x_t, x_{t-\tau}) = \frac{(t-\tau)}{\sqrt{t(t-\tau)}} = \sqrt{\frac{t-\tau}{t}} \tag{A2-5}$$

Provided t is large compared to τ, all $\rho_{\tau,t}$ approximate *unity*. The sequence of x_t is smooth but nonstationary since its variance is increasing with t as shown in (A2-3). This result implies that the x_t process tends to stay on one side (above or below the initial position x_0) and thus it satisfies the arc-sine theorem. Of course, this process will reverse the course. That is, if it continues *infinitely*, the process will return to the initial position. The statement “all roads lead to Rome” is justified in a random walk of less than two dimensions. The mean reversion is fulfilled.

Nevertheless, within a practical length of period, the arc-sine process dominates the mean reversion process. Furthermore, if the random walk is defined in more than two dimensions, the mean reversion process is not satisfied (Feller (1957), pp.359-360).

Appendix 3: Coin tossing game and arc-sine theorem

The purpose of this appendix is to formally derive the arc-sine theorem in a mathematical context, which nevertheless contradicts a general belief based on the law of averages.

The starting point to resolve the misunderstanding based on the law of averages is stating the derived result that a large number of independent coin tosses at one time does not share the same statistical nature as the results from a single coin toss played over a long time period. The former game possesses the characteristics of the law of averages, while the latter is what we will pursue in the arc-sine theorem.

With these differences in mind, imagine the following situation. An individual is involved in a coin-tossing game. If he gets heads, he adds + 1 to his score, and if he gets tails, he adds – 1 to his score. Now the question is, how likely it is that his score reverts to zero, namely that the number of heads and tails up to the time of counting is identical, as he continuously plays the game over a long period of time. The intuitive answer to this question is, because the game itself is fair in the sense that both heads and tails appear with the same probability (i.e. probability 0.5), the resulting score tends to stay around zero. The arc-sine theorem, however, states that this is not the case. The startling message of the theorem is, unlike basic intuition, that the score breaking even is the least likely outcome.

To demonstrate, we need to follow several steps.[23] We first examine the ballot theorem based on the reflection principle, from which we will learn the number of paths which do not cross the break-even point (position value of zero). Second, we show the lemma which states that the probability that the paths will not go through position 0 at all is identical to the probability that the paths will return to position 0 eventually. Third, we explain the arc-sine theorem which can be derived from the ballot theorem and the lemma.

Notation

Let us denote $P\{(0,0)\rightarrow(n,r)\}$ as the probability that the path leads to (n,r). In the following analysis, we write each position as (n,r) which corresponds to the position r after n coin tosses. The starting position is obviously (0,0). For simplicity any probability starting from (0,0) will be written as $P(n,r)\equiv P\{(0,0)\rightarrow(n,r)\}$. Let S_i be the position after time period i. Let us denote

[23] We owe deep gratitude to Cox and Miller (1965), Feller (1957, 1966), Loeve (1977) and Ross (1970, 1983, 1996).

$N(n,r)$ as the number of paths with $S_n = r$, namely after n tosses, the player's position is r.

Suppose now coin tossing results are p heads and q tails. Consequently, we derive the following relationship,

$$p + q = n, \quad p - q = r \tag{A3-1}$$

and we write

$$N(n,r) = \binom{n}{p} = \binom{n}{\frac{n+r}{2}} \tag{A3-2}$$

and its probability is

$$P(n,r) = \binom{n}{\frac{n+r}{2}} \times 2^{-n}. \tag{A3-3}$$

Ballot Theorem

The number of paths finishing with $S_n = r$ after n coin tosses is proportional to the difference between the number of paths reaching r–1 in n–1 tosses and the number of paths reaching r+1 in n-1 tosses:

$$N(n,r) = \frac{n}{r} \times [N(n-1,r-1) - N(n-1),r+1)].$$

Sketch of Proof

Let us define (n,–r) as the "reflection position" of (n,r). It implies that n coin tosses will yield an opposite outcome, which can be achieved by reversing the numbers of heads and tails.

First, we know that the number of paths from (1,1) to (n,r) which nonetheless do not go through (t,0) where t=1,..., n-1 is equal to the number of paths from (0,0) to (n,r) which do not go through (t,0) either. Second we also know that the number of paths from (α,β), where $0 \le \alpha < n$ and $\beta > 0$, to (n,r), which cross (t,0) where $t = \alpha+1$,..., n-1 is equal to the number of paths from (α,–β), which is the reflection position of (α,β), to (n,r).

With these features, the number of paths from (0,0) to (n,r), which do not cross (t,0) where t=1,..., n-1, is given as $N(n-1,r-1)$, the number of paths from (1, 1) to (n,r), minus $N(n-1,r+1)$, the number of paths from (1, –1) to (n,r). From equations (A1-1) and (A1-2),

$$N(n-1, r-1) - N(n-1, r+1) = \frac{(p+q-1)!}{(p-1)!q!} - \frac{(p+q-1)!}{p!(q-1)!}$$
$$= \frac{(p-q)}{(p+q)} \times \frac{(p+q)!}{p!q!}$$
$$= \frac{r}{n} \times N(n, r).$$

Lemma

The probability that the paths will not go through the position 0 at all during the period between 1 and $2n$ is identical to the probability that the paths will return to the position 0 at period $2n$.

Next, consider $P(2n,2r)$, the probability that paths from (0,0) to $(2n,2r)$ stay on the positive side, or in other words, the position will always remain positive during the period between 0 and $2n$. The ballot theorem yields the equation:

$$P(2n, 2r) = \frac{N(2n-1, 2r-1) - N(2n-1, 2r+1)}{2^{2n}}$$
$$= \frac{1}{2}[P(2n-1, 2r-1) - P(2n-1, 2r+1)].$$

If we add all the paths with respect to r in order to derive the probability that the path will return to $(t,0)$, where t=1,..., $2n$-1, during the periods we are concerned with, we have

$$\sum_{r=1}^{\infty} P(2n, 2r) = \frac{1}{2}[P(2n-1, 1) - P(2n-1, 3) + P(2n-1, 3)$$
$$- \cdots + P(2n-1, 2r-1) - P(2n-1, 2r+1)]$$
$$= \frac{1}{2}P(2n-1, 1).$$

Transforming the equation, we have

$$P(2n-1, 1) = \binom{2n-1}{n} \times 2^{-(2n-1)} = \binom{2n}{n} \times 2^{-2n} = P(2n, 0).$$

This implies that the probability that paths from (0,0) remain positive is equal to half the probability that the path will return to position 0 after a $2n$ time period. The same applies when the position is negative. If we add them together, we will derive the probability that the paths will not go through $(t,0)$ while t=1,..., $2n$-1 is equivalent with $P(2n,0)$.

Arc-sine Theorem (Feller (1957, p.79))

The probability that in the time interval from 0 to $2n$ the position spends $2k$ time units on the positive side and $2n$–$2k$ time units on the negative side equals $R\{2k,2n\}\equiv P(2k,0)\times P(2n-2k,0)$, where

$$P(n,r) = \binom{n}{\frac{n+r}{2}} \times 2^{-n}.$$

Hence,

$$R\{2k,2n\} = \binom{2k}{k} \times \binom{2n-2k}{n-k} \times 2^{-2n}.$$

Sketch of Proof

For simplicity, let us use the equation:

$$u_{2v} = P(2v,0) = \binom{2v}{v} \times 2^{-2v}, \tag{A3-4}$$

which states the probability that the return to the origin occurs at period $2v$. We also let f_{2v} be the probability that the *first return* to the origin occurs at period $2v$, or in other words that $S_1\neq0,\ldots,S_{2v-1}\neq0$, but $S_{2v}=0$. From these two notations, we derive the following relationship:

$$u_{2n} = f_2 u_{2n-2} + f_4 u_{2n-4} + \cdots + f_{2n} u_0. \tag{A3-5}$$

In other words, a return to the origin at period $2n$ may be the first return, or otherwise it occurred some time ago, such as $2k$ (k=1,..., n–1), followed by a renewed return at period $2n$. Here, the probability of the latter is denoted by $f_{2k}u_{2n-2k}$, because there are $2^{2k}f_{2k}$ paths ending with a first return at period $2k$, and $2^{2n-2k}u_{2n-2k}$ paths from ($2k$, 0) to ($2n$, 0).

Next we denote $b_{2k,2v}$ as the probability that the position lies above the initial position for a time period of $2k$ for a length of time of $2v$. What we have to prove is:

$$b_{2k,2v} = R\{2k, 2v\}. \tag{A3-6}$$

We know $b_{2v,2v}=u_{2v}=P(2v, 0)\times P(0, 0)=R\{2v, 2v\}$. Assume now a time period of $2k$ is spent on the positive side. Then there are two possibilities.

(1) A first return to the origin occurs at some time $2r < 2n$. If the $2r$ time period is spent on the positive side, then $2k$–$2r$ must stay above the initial position as well as during the remaining time period. The number of such paths is:

$$\frac{1}{2}(2^{2r} f_{2r} \times 2^{2n-2r} b_{2k-2r,2n-2r}). \tag{A3-7}$$

(2) A first return occurs at some time $2r < 2n$. If the $2r$ time period is spent on the negative side, then $2k$ must stay above the initial position during the remaining time period. The number of such paths is,

$$\frac{1}{2}(2^{2r} f_{2r} \times 2^{2n-2r} b_{2k,2n-2r}). \tag{A3-8}$$

Adding up these numbers of paths in (A3-7) and (A3-8), we derive

$$b_{2k,2n} = \frac{1}{2}\left[\sum_{r=1}^{k} f_{2r} b_{2k-2r,2n-2r} + \sum_{r=1}^{n\text{-}k} f_{2r} b_{2k,2n-2r}\right]. \tag{A3-9}$$

We know (A3-6) holds when v=1. By induction, suppose (A3-6) is also true for $v \leq n$–1. This leads to:

$$\begin{aligned} b_{2k,2n} &= \frac{1}{2}\left[\sum_{r=1}^{k} f_{2r} R\{2k-2r, 2n-2r\} + \sum_{r=1}^{n-k} f_{2r} R\{2k, 2n-2r\}\right] \\ &= \frac{1}{2}\left[\begin{array}{l} \sum_{r=1}^{k} f_{2r} P(2k-2r,0) \times P(2n-2k,0) \\ \qquad\qquad + \sum_{r=1}^{n-k} f_{2r} P(2k,0) \times P(2n-2r-2k,0) \end{array}\right] \\ &= \frac{1}{2}\left[\sum_{r=1}^{k} f_{2r} u_{2k-2r} u_{2n-2k} + \sum_{r=1}^{n-k} f_{2r} u_{2k} u_{2n-2r-2k}\right] \\ &= \frac{1}{2}\left[u_{2n-2k} \sum_{r=1}^{k} f_{2r} u_{2k-2r} + u_{2k} \sum_{r=1}^{n-k} f_{2r} u_{2n-2r-2k}\right] \end{aligned} \tag{A3-10}$$

From (A3-5), the equation above can be written as:

$$b_{2k,2n} = \frac{1}{2}[u_{2n-2k} u_{2k} + u_{2k} u_{2n-2k}] = u_{2k} u_{2n-2k} = R\{2k, 2n\}. \tag{A3-11}$$

We therefore conclude (A3-6) also holds when v=n.

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