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Abstract

This paper proposes a new term structure model to generalize the Gaussian affine model and the Black model with an efficient and accurate solution method. The new model assumes that arbitrage between money or reserves and government bonds works but not perfectly. The new model enables us to quantify the effects of forward guidance, quantitative easing, and the negative interest rate policy. Estimation results for Switzerland, Germany, and Japan show that the new model outperforms both the Gaussian affine model and the Black model. Moreover, the results indicate that the power of arbitrage moves in tandem with basis swap spreads.

Keywords: Term Structure Model; Monetary Policy; Negative Interest Rate; Basis Swap Spreads
JEL classification: E43, E52, G12

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1. Introduction

Nominal interest rates had been assumed to be always non-negative as long as people have the option to hold currency. The amount of negative-yielding sovereign debt, however, stood at an estimated $10.4 trillion as of 31 May 2016 (Fitch, 2016). The coexistence of negative-yielding government bonds with paper money suggests that the power of arbitrage between bonds and currency is not strong to prevent bond yields from falling below zero.

Negative interest rates have appeared in the process of central banks' struggle to stimulate economies and stabilize inflation rates in the aftermath of the global financial crisis. Central banks in Europe and Japan have lowered interest rates on excess reserves (IOERs) into negative territory. This has transmitted directly to other interest rates, in particular government bond yields. Following the introduction of negative IOERs, government bond yields not only at the short end but also for longer terms have fallen below zero.

The negative interest rate policy has been conducted in conjunction with other unconventional monetary policy measures such as quantitative easing and forward guidance. This combination of unconventional policy measures makes it difficult to assess the effects of each single policy measure. Moreover, since the possibility of nominal negative interest rates had been considered to be unrealistic, theoretical work and the development of models dealing with negative interest rates are inevitably being lagged.

The aim of the current study is to develop a model to assess the effects of unconventional monetary policy measures, including a negative interest rate policy, on government bond term structures. The model proposed in this paper generalizes two widely used models, the Gaussian affine model and the Black model. The main difference between the two models is the strength of the assumption that nominal interest rates are always non-negative. Arbitrage between bonds and non-yielding cash works to different extents in the two models. In other words, the two models make different assumptions regarding the holding cost of cash primarily for storage and insurance.

In the Gaussian affine model, nominal interest rates can be negative without any lower bound. This assumption is far from realistic, but it makes the model tractable. In this model, there are linear relationships between government bond yields and the factors which determine the dynamics of yield curves. This linearity makes it easy for researchers using the model. Duffie (2001) highlights the usefulness of the Gaussian affine term structure model provided that actual interest rates are high enough for the fitted model to assign a low likelihood to negative interest rates. Gaussian affine term structure models have been intensively examined in the literature (Dai and Singleton, 2002; Duffee, 2002; Duffie and Kan, 1996; Joslin *et al.*, 2011; and Kim and Wright, 2005). These researches indicate that the Gaussian affine model provides a useful analysis tool in a positive interest rate environment.

However, using Japanese data, Kim and Singleton (2012) point out that affine models work poorly in a situation in which short-term rates are stuck at the zero lower bound. If the Gaussian affine term structure model is fitted to yield curves at the zero lower bound, the model definitely generates a high likelihood of negative interest rates. This contradicts the condition that Duffie (2001) indicates to justify the use of the Gaussian affine model.

Considering this weakness of the Gaussian affine term structure model, many researchers have given recognition to the model proposed in Black (1995). In this model, called the Black model, there is a single number describing the state of the world, namely, so-called “shadow rate”. This rate is equivalent to the short-term interest rate when it is positive and equivalent to what the short-term interest rate would be without the zero lower bound when it is negative. The shadow rate is observable whenever it becomes positive and equal to the short rate. On the other hand, the shadow rate is unobserved when the short rate is zero and needs to be estimated. This feature of the Black model implies that arbitrage between cash and government bonds works perfectly or that holding cash is costless.

This non-linearity produces a computational burden, so that pricing and estimation procedures take much more time than in the Gaussian affine model. However, the model provides a much better fit at the zero lower bound. Many studies find that the Black model performs better at the zero lower bound than the Gaussian affine model in terms of fitting yield curves, extracting market participants’ expectations with regard to monetary policy, and estimating term premia (Bauer and Rudebusch, forthcoming; Christensen and Rudebusch, 2015; Gorovoi and Linetsky, 2004; Ichiue and Ueno, 2006, 2007, 2013, 2015; Kim and Priebsch, 2013; Krippner, 2013, 2014; Priebsch, 2013; and Wu and Xia, 2016). However, as will be shown in this paper, the Black model performs poorly under negative interest rates.

Against the background described above, this study proposes a new model in which arbitrage between bonds and cash still works, but it is not so powerful as to prevent bond yields from falling below the interest rate on cash. The assumption of limited arbitrage allows the model to perform well at any interest rate level.

In addition, this study develops an efficient and accurate approximate solution technique that can be applied to both the Black model and the newly proposed model, since in both models there is no exact solution for the price of bonds due to non-linearity. To the best of my knowledge, this technique offers a superior balance in the trade-off between accuracy and computational burden to other techniques in the literature.

Applying this approximate solution technique and using a single-stage iteration filter – a type of non-linear filter – and a standard Kalman filter, this study examines the performance of the three models, the Black model, the newly proposed, and the Gaussian affine term structure model, using yield curve data from Switzerland, Germany, and Japan. The results taking various criteria into account indicate that the newly proposed model performs better than the other models.

Next, using the estimation results of the new model, the effects of forward guidance, quantitative easing, and the negative IOERs on government bond yields are quantified. These effects work to quantitatively different extents for each maturity. Overall, the reductions of the IOER have a larger influence on yields at shorter maturities. On the other hand, the decline in yield term premia mainly lowers yields at longer maturities.

The power of arbitrage between cash or reserves and government bonds estimated in the proposed model has recently weakened, which coincides with current divergence in monetary policy between the three areas examined and the United States. The introduction of negative interest rate policies in the three areas occurred more or less at the same time as the Federal Reserve in the United States started to normalize monetary policy. Shedding light on the mechanism behind the synchronization between the weakening power of arbitrage and monetary policy divergence, the present study relates the estimated results to developments in the cross-country basis swap market, which measures the extent to which covered interest rate parity is violated.

Arai *et al*. (2016) show that negative basis swap spreads between various currencies and U.S. dollars have increased, mainly due to the violation of covered interest parity under the divergence of monetary policy. Under these circumstances, as reported in the Nikkei Asian Review (2016b), despite negative yields, Japanese government bonds still attract foreign investors since the use of dollar funds to invest in Japanese government bonds offers an opportunity to benefit from basis swap spreads. Therefore, even though Japanese government bonds have a negative yield on a yen-denominated basis, they actually bring higher yields on a dollar-denominated basis than U.S. Treasuries. As a result, U.S. investors, who do not have a current account at the Bank of Japan, buy Japanese government bonds, even if the interest rates on them are below the IOER, to take advantage of the opportunity to earn large premia from the cross-currency swap market.

This mechanism holds as well in the cases of Switzerland and Germany, so that the estimated power of arbitrage moves in tandem with basis swap spreads in the three areas. In the proposed model, there exists room for such activities by investors. In this regard, the assumption of the Black model that arbitrage between cash or reserves and government bonds works perfectly is too strong to describe the actual movements in government bond markets shown in the present study.

The remainder of the study is organized as follows. Section 2 explains the models and the solution method, while Section 3 describes the data and estimation methods employed. Section 4 then presents the estimations results, and Section 5 considers the implications for monetary policy. Finally, Section 6 concludes.

2. Models and solution method

The proposed model consists of the spot rate i_t, the shadow rate s_t and its components x_t, the IOER y_t, and the proxy variable for the power of arbitrage φ_t:

$$i_t = s_t 1_{\{s_t \geq y_t\}} + \{\varphi_t s_t + (1-\varphi_t) y_t\} 1_{\{s_t < y_t\}}, \quad (1)$$

$$dx_t = \kappa_x(\theta_x - x_t)dt + \sigma_x dW_{x,t}^Q, \quad (2)$$

$$dy_t = \sigma_y dW_{y,t}^{\mathrm{Q}}, \quad (3)$$

$$\varphi_t = \varphi_t^* 1_{\{0 \leq \varphi_t^* \leq 1\}} + 1_{\{1 \leq \varphi_t^*\}}, \quad (4)$$

$$d\varphi_t^* = \sigma_{\varphi^*} dW_{\varphi^*,t}^Q \ . \quad (5)$$

Following studies such as Wu and Xia (2016) and Krippner (2014), the shadow rate is given by $s_t \equiv x_{1t} + x_{2t}$. $x_t = [x_{1t}, x_{2t}, x_{3t}]'$ is the vector of factors of which the shadow rate consists. x_t follows a multi-factor Ornstein-Uhlenbeck process. ${W_t}^Q = \left[W_{x,t}^Q, W_{y,t}^Q, W_{\varphi,t}^Q\right]'$ is five dimensional Brownian motion. $W_{y,t}^Q$ and $W_{\varphi,t}^Q$ are independent of each other and from $W_{x,t}^Q$. The normalization proposed in Joslin *et al.* (2011) is used. As in Wu and Xia (2016) and Krippner (2014), κ_x in equation (2) has two distinct eigenvalues. φ_t represents the strength of arbitrage between cash and government bonds. y_t is equal to the IOER whenever the IOER is introduced. If not, y_t is set to zero. In the case that an IOER is introduced, φ_t fluctuates between zero and one as the arbitrage relationship between reserves and bonds changes. To keep this conceptual notion meaningful, φ_t is modeled as in equation (4). When φ_t approaches to zero, the IOER becomes the strict lower bound of the spot rate i_t. On the other hand, when φ_t approaches to one, the spot rate i_t can fall below the IOER and the IOER cannot play any role as the lower bound. y_t and φ_t^* are assumed to follow random walk processes as shown in equations (3) and (5).[1] The reason for employing this modeling strategy is that the use of IOERs as a monetary policy measure is a relatively recent phenomenon, so that not enough time has passed to observe how y_t and φ_t^* behave.

The model described above can be regarded as a generalized version of the Gaussian affine model and the Black model. When $\varphi_t = 1$, the model reduces to the Gaussian affine model. On the other hand, when $\varphi_t = 0$, it reduces to the Black model. This property of the proposed model is why it can be regarded as a generalization of the other two models, and below the new model will be referred as the "extended model," since it can be thought of as an extension of the Gaussian affine model and the Black model in the direction such that the restriction on φ_t (i.e., $\varphi_t = 1$ or $\varphi_t = 0$) is removed. Figure 1 provides a graphic representation of the relationship between the shadow rate and the short rate in the three models. As can be seen, in the extended model proposed here, the relationship is kinked at $y_t = 0$. This implies that the relationship changes at the point where the shadow rate becomes larger than the interest rate on cash or reserves, y_t. This assumption is based on the presumption that the relative attractiveness of reserves and government bonds changes at that point. Specifically, government bonds with a positive interest rate are preferred to reserves with a zero or very small positive interest rate, but if the interest rate on government bonds were below the IOER, an investor with a central bank current account would be less willing to buy government bonds. The number of financial institutions which have a current account at the central bank is limited. Therefore, the shadow rate

[1] Ichiue and Ueno (2013) and Kortela (2016) assume y_t to be a constant parameter, but time variant. If y_t is time variant, the uncertainty related to y_t should be taken into consideration to price bonds. In this study, y_t is assumed to follow a random walk process to consider the impact of the uncertainty since the IOERs were changed in the three areas.

describing the state of the world has less influence on government bond yields, since reserves act as an obstacle to investing in government bonds if the shadow rate is deeply negative.

From a probabilistic viewpoint, the value of φ determines the likelihood that the short rate will fall below y_t. Figure 2 shows the cumulative probability distribution functions of the Gaussian affine model, the Black model, and the extended model with constant φ_t and $y_t = 0$. In the Black model, arbitrage works perfectly or there are no costs for holding cash, so the short rate is always non-negative. On the other hand, in the Gaussian affine model the short rate can take negative values given the assumption that there is no arbitrage between cash or reserves and government bonds. The probability distribution function of the extended model approaches that of the Black model as φ_t goes to zero and that of the Gaussian affine model as φ_t goes to one. Thus, the Gaussian affine model and the Black model are the two special cases of the extended model.

The number of factors in the Gaussian affine model, the Black model, and the extended model is the same. The only difference between the models is the value of $\varphi_{[t]}$. Therefore, the validity of each model can be tested using likelihood ratio tests. Using data on government bond yields in this way, it is possible to judge the adequacy of the assumptions regarding the power of arbitrage between cash or reserves and government bonds assumed in the Gaussian affine model and the Black model using formal statistical criteria.

To quantify the term premia in bond yields, the market price of risk λ_t, which links the actual or subjective probability measure P and the risk neutral measure Q, is introduced. The market price of risk is affected only by x_t and is given by

$$\lambda_t = \begin{bmatrix} \lambda_0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_1 \\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ \varphi_t^* \end{bmatrix},$$

$$dW_t^P = \lambda_t dt + dW_t^Q,$$

$$dW_t^{P\ or\ Q} = \left[dW_{x,t}^{P\ or\ Q}, dW_{y,t}^{P\ or\ Q}, dW_{\varphi,t}^{P\ or\ Q}\right]'.$$

The reason for employing this modeling strategy is that the use of IOERs as a monetary policy measure is a relatively recent phenomenon, so that not enough time has passed to observe how y_t and φ_t^* have influence on the market price of risk.

The government bond price and yield for maturity τ are defined as follows:

$$P_\tau \equiv E^Q\left[exp\left(-\int_0^\tau i_t dt\right)\right], \tag{6}$$

$$R_\tau \equiv -\log(P_\tau)/\tau. \tag{7}$$

As shown in equation (1), the relationship between the short rate i_t and the shadow rate s_t is non-linear in the Black model and the extended model. This non-linearity causes the difficulties in computing bond prices. Although Black (1995) proposed the use of Monte Carlo simulation to price government bonds, this is not realistic in the case of estimating the parameters and shadow rate using historical data because of the huge computational burden. Therefore, studies typically employ some kind of approximation for bond prices to overcome this computational burden, with a variety of methods having been proposed (see, e.g., Ichiue and Ueno, 2013; Kim and Singleton, 2012; Krippner, 2014; Priebsch, 2013; and Wu and Xia, 2016).

Probably the best approximation method to balance the trade-off between accuracy and computational burden in the literature is that proposed by Priebsch (2013). He uses the following 2nd order approximation of bond yields:

$$R_\tau \approx \frac{1}{\tau}\left(E^Q\left[\int_0^\tau i_t dt\right] - 0.5Var^Q\left[\int_0^\tau i_t dt\right]\right).$$

When i_t follows a normal distribution, this approximation method provides the exact solution. In the Black model and the newly proposed model, i_t does not follow a normal distribution. This causes approximation errors for bond yields if this method is used.

Taking the skew of the distribution of i_t into account, the following approximation method is employed here: $I_\tau \equiv \int_0^\tau i_t dt$ is approximated by $\widetilde{I_\tau} \equiv \alpha_0 + \alpha_1 i_{\frac{1}{4}\tau} + \alpha_2 i_{\frac{3}{4}\tau}$.[2] The reason why $\widetilde{I_\tau}$ is chosen as the approximation device is that $\widetilde{I_\tau}$ can be skewed to the same extent as I_τ. The parameters α_0、α_1、α_2 are determined by minimizing the mean squared error as follows:

$$\min E^Q\left[\left(I_\tau - \widetilde{I_\tau}\right)^2\right]$$

$s.t.$

$$E^Q[I_\tau] = E^Q\left[\widetilde{I_\tau}\right],$$

$$Var^Q[I_\tau] = Var^Q\left[\widetilde{I_\tau}\right].$$

If I_τ follows a normal distribution, this method also delivers the exact solution. The bond prices and yields are approximated as follows:

$$R_\tau \approx -\log\left(E^Q\left[exp\left(-\widetilde{I_\tau}\right)\right]\right)/\tau,$$

$$P_\tau \approx E^Q\left[exp\left(-\widetilde{I_\tau}\right)\right].$$

Details of the derivation of $E^Q\left[exp\left(-\widetilde{I_\tau}\right)\right]$ are provided in the Appendix A. The

[2] In the case of variable φ_t^*, $\widetilde{\widetilde{I_\tau}} \equiv \alpha_0 + \alpha_1 i_{\frac{1}{2}\tau}$ is used in place of $\widetilde{I_\tau}$ to speed up the estimation procedure at the cost of a minor loss of approximation accuracy. α_0 and α_1 are set to match the mean and variance. Even if $\widetilde{\widetilde{I_\tau}}$ is used, the approximation accuracy is superior to the method in Priebsch (2013), since $\widetilde{\widetilde{I_\tau}}$ has to some extent the same skewness as I_τ.

closer the skew of $\widetilde{I_\tau}$ is to that of I_τ, the more accurate this approximation method is. Figure 3 presents an example which shows the accuracy of the newly proposed approximation method. In this case, the following simple setting is used. x_t consists of a one factor Ornstein-Uhlenbeck process with a mean reversion level θ of 0.01, a speed of convergence κ of 0.1, and instantaneous volatility σ of 0.2. The initial level of the shadow rate is zero. In addition, φ_t and y_t stay at zero and are treated as if constant. The figure shows three cumulative probability density functions. The thick line is that of the true I_τ with τ equal to 10 years. The thin line is that of $\widetilde{I_\tau}$ obtained using the newly proposed method. The broken line is that of a proxy variable corresponding to $\widetilde{I_\tau}$ in Priebsch (2013). The probability density function of the true I_τ is generated using Monte Carlo simulation with 100,000 paths. In the case of the extended model, $\widetilde{I_\tau}$ is calculated for each generated path based on the optimal values of α_0, α_1 and α_2, and the probability distribution function is then derived. The distribution based on Priebsch (2013) is obtained by generating a normal random variable with mean $E^Q[I_\tau]$ and variance $Var^Q[I_\tau]$ 100,000 times, since Priebsch (2013) approximates I_τ as if it follows a normal distribution. The figure clearly shows how skewed I_τ is and how accurate $\widetilde{I_\tau}$ is.

Even if the mean and variance are equal to the true distribution, the approximated bond prices may contain errors without matching the skew of the true distribution. Table 1 shows the exact and approximated bond prices in a simple example which is almost the same as that shown in Figure 3. The only difference is that Table 1 includes the case in which the initial level of the shadow rate is 1%. The exact prices are given in Gorovoi and Linestky (2004), who show that in the one-factor Black model the exact bond prices can be derived through eigenfunction expansion. In this simple example, their method is effective. Therefore, the exact solutions are used as the benchmark. The example shows that the approximation errors in Priebsch (2013) increase as the maturity of bonds extends, while the errors in the newly proposed method remain small. This difference in approximation accuracy arises since the skew of the probability density function for I_τ becomes larger for longer maturities. Priebsch (2013) shows that the popular approximation method which uses the model explained in Krippner (2014), named by "K-GATSM[3]", produces larger approximation errors than the method proposed in Priebsch (2013). Therefore, the newly proposed method works best among various methods proposed so far in the literature.

3. Data and estimation method

3.1 Data

Data on government bond yields in Switzerland, Germany, and Japan are used for the estimation.[4] The central banks in the three areas have recently lowered the IOER below zero and yields in these areas have become negative. The experience of these areas serves as a useful natural experiment to examine how well the different term structure models work.

For the analysis, government bonds with maturities of three months, six months, one year, two years, three years, five years, seven years, and ten years are selected. Following other studies (Ichiue and Ueno, 2006, 2007, 2013, 2015; Joslin *et al.*, 2011;

[3] Krippner (2014) shows that the Wu and Xia (2016) model is a discrete-time version of the K-GATSM.

[4] Zero coupon yields are used. The data are obtained from Bloomberg and the Swiss National Bank for Swiss government bond yields from January 1989 to November 1994. Moreover, since three-month and six-month government bond yields for Switzerland are not available, Libor rates are used instead.

Krippner, 2013; and Wu and Xia, 2016), monthly data are used and yields are as of the end of the month.

Data for the longest observation period possible are used, since, as Bauer *et al.* (2012) point out, there is a risk of estimation bias in the parameters when short observation periods are used. Specifically, the observation period for Switzerland is January 1989–June 2016, that for Germany October 1991–June 2016, and that for Japan April 1989–June 2016.

Typical yield curves are shown in Figure 4. As described in the literature (e.g., Ichiue and Ueno, 2006, 2015), yield curves tend to be convex when a zero interest policy is conducted. In the figure, the panels on the left show the yield curves for the three areas when three-month rates are positive but close to zero. Furthermore, the convexity of the curves has increased since the negative interest policy started. In addition, government bond yields tend to be falling below the IOER.

3.2 Estimation method

Four different specifications are estimated to examine their validity at various interest rate levels. The first specification is the Gaussian affine model, the second is the Black model, and the third and fourth are two different versions of the extended model with the time invariant and time variant power of arbitrage. The difference between the two versions of the extended model lies in whether φ_t is constant or variable. The model with a constant φ_t is called the "fixed extended model," while the model with a variable φ_t is called the "variable extended model."

In all the specifications, x_t and its components x_{1t}, x_{2t}, and x_{3t} are assumed to be unobservable. In addition to x_t, in the extended model, φ_t is also assumed to be unobservable factor. Therefore, these latent factors should be filtered in estimation. Following Joslin *et al.* (2011), the Kalman filter is used for the Gaussian affine model. For the Black model and the extended models, the single-stage iteration filter is used. In the Black model and the extended models, the relationships between factors and bond yields are not linear, so a non-linear filtering method should be used. In many studies – such as Wu and Xia (2016), Ichiue and Ueno (2006, 2007, 2013, 2015) and Kim and Singleton (2012) – the Extended Kalman filter is used. In the Extended Kalman filter procedure, observation equations are approximated with a first-order Taylor expansion. After linearization, latent factors are filtered following the algorithm of the Kalman filter. However, as shown by Tanizaki (1996) using Monte Carlo simulations, estimation biases arise in the Extended Kalman filter if there is high non-linearity in the estimated systems. On the other hand, if non-linearity exists in the observation equations but not in the state transition equations, the single-stage iteration filter produces less biased results than other non-linear filtering techniques. This applies to the system employed in this study and is the reason why this method is selected for estimating the Black model and the extended models. Lastly, the square root filtering technique is employed for each filtering method to avoid that rounding errors result in estimation bias. Applying the filtering methods mentioned above and (quasi-)maximum likelihood estimation, each parameter set of the four specifications is estimated.

4. Estimation results

4.1 Estimated parameters

The estimated parameters are shown in Tables 2, 3, and 4. Except for φ and σ_φ, there are no large differences in the parameter estimates obtained using the different models. In this sub-section, the estimation results for φ and σ_φ newly introduced in this study are focused.

Starting with φ, the value of this parameter by definition is fixed and given in the Gaussian affine and Black models. That is, it is assumed to be equal to one in the Gaussian affine model, while in the Black model it is by definition zero. In the fixed extended model, the value is fixed but to be estimated. The value of this parameter indicates how strongly arbitrage between cash or reserves and bonds works. The more closely this value approaches to zero, the more strongly arbitrage works. From the viewpoint of identification, φ is determined primarily by the shapes of the yield curves in a low interest rate environment. If yields at various maturities are less than zero, or they are below the IOER when shadow rates are negative, φ should be greater than zero. On the other hand, if yields are stuck at zero even when the shadow rates are deeply negative, φ is identified to be zero.

Reflecting actual developments in yield curves, with shorter- and medium-term interest rates having fallen below the IOER, the estimated values for φ are 0.055 for Switzerland, 0.100 for Germany, and 0.015 for Japan. Further, it is worth noting that by removing the restriction on φ, the average of the log likelihood of the fixed extended model is higher than those of the Gaussian affine and Black models for all areas. The log-likelihood statistic for the null that $\varphi = 1$ calculated using the averages of the log likelihood of the fixed extended model and the Black model is 199.2 for Switzerland, 253.5 for Germany, and 161.5 for Japan. Similarly, the log-likelihood statistic for the null that $\varphi = 0$ calculated using the averages of the log likelihood of the fixed extended model and the Gaussian affine model is 246.8 for Switzerland, 198.9 for Germany, and 809.4 for Japan. These results imply that the assumptions underlying the Gaussian affine model and the Black model are not empirically supported by the data for the three areas. Hence, arbitrage between cash or reserves and bonds does not work perfectly but still works.

Turning to σ_φ in the variable extended, this parameter controls the variability of φ_t. The power of arbitrage between cash and government bonds depends on the money and bond market environment. If changes in the money and bond market environment are frequent and the power of arbitrage is strongly affected by these changes, σ_φ can be expected to be large. σ_φ can be identified mainly from changes in the shape of yield curves in the low interest rate environment. If yields are lower than the IOER and the distance between them is not constant, σ_φ should be greater than zero. Looking at the results, the estimated values of σ_φ are 0.053 for Switzerland, 0.051 for Germany, and 0.016 for Japan. For all three areas, the variable extended model has the highest log likelihood and the lowest Bayesian Information Criteria (BIC) among all the models. Since Joslin *et al.* (2011) point out that their five factor model causes over-fitting problem, the out of sample performance of the variable extended model should be examined. The model-implied yields on bonds with maturities beyond those included in estimation, specifically fifteen and twenty years, are now plausible in line with actual yields and the yields implied by other models. Therefore, the variable extended model is the model that is most strongly supported by the government bond data from the three areas. This result is primarily due to the better fitting performance of the variable extended model with regard to the yield curve data for the three areas, as explained below. In addition, the restriction of the dynamics of φ_t brings the goodness of the out of sample performance comes from.

4.2 Root mean squared errors (RMSEs) and volatility

The reason why the log likelihood average takes the highest value for the variable extended model is that the one-period-ahead yield forecasts by the model are the most accurate; that is, the probability distributions of one-period-ahead yields predicted by the model are closest to the actual probability distributions of yields.

Tables 5 to 7 and Figures 5 to 7 show the goodness of fit of the different models. The first fitting results consist of the root mean squared errors (RMSEs) between the actual yields and the yields computed by the model using the fitted latent factors estimated by the filtering methods. If the RMSE of a model is large, this implies that the model cannot recover the actual shapes of the yield curves and presumably the probability distributions of yields. The second fitting results consist of the volatility of yields. Actual volatilities are calculated following Kim and Singleton (2012). Specifically, the standard deviations of daily yield changes over a period of 60 days are computed at first. As in Kim and Singleton (2012), these standard deviations are used as proxies for the volatility of yields. Next, these standard deviations during three periods, the positive interest rate period, the zero interest rate period, and the negative interest rate period, are averaged.

Tables 5 to 7 show that the RMSEs of the Gaussian affine model and the variable extended model are smaller than those of the other models. While the variable extended model has the smallest RMSEs in the estimation for Germany and Japan, the Gaussian affine model has the smallest RMSE in the estimation for Switzerland. It is worth noting that the RMSEs of the Black model for shorter-term interest rates during the negative interest rate period are quite large compared to those of the other models. This is caused by the fact that actual shorter-term interest rates during this period were below the IOER, but this cannot be generated by the Black model with the IOER set to the lower bound, as is the case in this study. Ichiue and Ueno (2013) and Wu and Xia (2016) similarly assume that y_t is equal to the level of the IOER, which yields a more definite interpretation of y_t rather than estimating the level of y_t without the restriction. If y_t was assumed to be an unobservable latent factor, the goodness of fit would improve, but the interpretation of y_t would be unclear. In this case, the IOER is not explicitly introduced in the model as a monetary policy measure, so that counterfactual simulation to examine the impact of a change in the IOER on government bond yields is not feasible.

Figures 5 to 7 show the fitting results for the term structure of volatility. The diamonds, squares, and triangles in the figures represent the averages of the actual standard deviations during the positive, zero, and negative interest rate periods. The yield volatility at maturity τ given by a particular model is computed as follows. For the yield at maturity τ in period t, $z_t(\tau)$, the instantaneous volatility, $\sigma_{z,t}(\tau)$, as shown by Kim and Singleton (2012), can be represented as follows:

$$\sigma_{z,t}(\tau) = \sqrt{\frac{\partial f\left(x_{t|t}, y_{t|t}, \varphi^*_{t|t}\right)}{\partial[x', y, \varphi^*]} \begin{bmatrix} \sigma_x \sigma_x' & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_{\varphi^*}^2 \end{bmatrix} \frac{\partial f\left(x_{t|t}, y_{t|t}, \varphi^*_{t|t}\right)'}{\partial[x', y, \varphi^*]'}} [\tau, \tau],$$

where $[\tau, \tau]$ indicates the position in the matrix. Figures 5 to 7 also show the averages of the instantaneous volatilities during each period on an annual basis. Looking at the results in the figures the term structure of volatility in the Gaussian affine model, as already indicated by the goodness of fit discussed earlier, does not capture the actual term structure of volatility in the zero and negative interest rate periods well. The reason is that the relationship between factors and yields are linear and the volatilities of factors are constant in the Gaussian affine term structure model, so that the yield volatilities in the model remain constant even if the levels of yields change. Since the observation periods with a positive interest rate are longer than the other periods, the volatility term structures in the model fit the actual volatilities during the positive interest rate periods well at the cost of a bad fit during the zero and negative interest rate periods. On the other hand, the volatility term structures in the extended models capture the downward shift in the actual curve in line with the decline in the interest rate well, keeping the cost of the bad fit during the positive interest rate period to a minimum. It is worth noting that the volatility term structures in the variable extended model capture the actual term structure of volatility in the zero and negative interest rate periods better than the Black model.

4.3 Shadow rates, expected interest rates, and yield term premia

This sub-section presents the estimation results for shadow rates, expected interest rates, and yield term premia in the variable extended model.

Figures 8 to 10 show developments in the shadow rate, expected interest rate, and yield term premium in Switzerland, Germany, and Japan. It is immediately clear that the large decline in the expected interest rate is the main cause of the decline in the market rate in the three areas. Furthermore, the shadow rate also declined in tandem with the expected rate in the three areas. Finally, the figures show that yield term premia also fell in the three areas, and that this made the largest contribution to the decline in interest rates from around year 2000.

Comparing these three areas, it is observed that the shadow rate in Japan has been consistently lower than those in Switzerland and Germany. At the end of the observation period in June 2016, the shadow rate in Switzerland and Germany is around -1%, while in Japan it is -4%. If one regards, as in Xia and Wu (2016), the shadow rate as a measure of the monetary policy stance, the figures suggest that the Bank of Japan, in its attempt to overcome deflation, has taken the most aggressive stance in terms of monetary stimulus. Focusing on the period since the adoption of the negative interest policy, the decline in shadow rates in the three areas has gained momentum. This implies that the adoption of the negative interest rate policy is perceived as indicating a more aggressive policy stance by central banks, since the effects of the negative interest rate policy itself are already factored in in the model through the reduction in the IOER, y_t.

5. Implications for monetary policy

To consider the implications for monetary policy of the estimation results obtained using the variable extended model, the following four different kinds of analysis are conducted.

5.1 Sensitivity of yield curves to the IOER

The first analysis focuses on the sensitivity of yield curves to the IOER. Figure 11 shows the impact of an increase in the IOER on interest rates at different terms and periods. For example, looking at the panel for Switzerland, the red line shows that, in June 2010, an increase in the IOER of 1% results in an increase in the three-month rate by 0.6% and in the ten-year rate by 0.3%. The reason why the sensitivity of short- and long-term interest rates differs is that there is some likelihood that the IOER would not acts as a (loose) lower bound on interest rates; that is, the future expected short-term rate is rather higher than the IOER. As a result, longer-term rates are less influenced by the IOER than short-term rates.

Staying with the example of Switzerland, Figure 11 also shows that the link between the IOER and interest rates has changed over time, as indicated by the difference between the red and green lines. Specifically, this difference shows that in December 2011 the IOER had a stronger effect on the term structure than in June 2010. Why does the sensitivity of interest rates change over time? To explore this point, Figure 12 provides a detailed explanation using a simple example. The green curve displays the cumulative probability of the shadow rate. The mean of the shadow rate at this future date is set to -1% with a standard deviation of 2%, and it is assumed that the IOER will not be introduced. Similarly, the blue curve represents the case when the mean of the shadow rate at this future date is set to -3%. If an IOER of -1% is introduced, the cumulative probability on the vertical axis shifts from 70% to 50% on the green curve and from 95% to 85% on the blue curve. Therefore, the expected short term interest rates in the case of the green curve declines by 0.6 percentage points from 0.4% to -0.2% $(0.3\% + 0.5 \times -1\%)$, with φ_t assumed to be always equal to zero. In the case of the blue curve, it declines by 0.95 percentage points from 0.1% to -0.85% $(0\% + 0.85 \times -1\%)$. This simple example clearly illustrates that if the zero or negative interest rate policy are expected to continue for a long time, the effect of the reduction of the IOER becomes large.

The above analysis shows the condition where a reduction of the IOER has the greatest impact on the entire term structure. Turning to reality, the Bank of Japan adopted the negative interest rate policy in January 2016, three years after it had started Quantitative and Qualitative Easing. At that time, the effect of forward guidance[5] strongly worked, which in the model are represented by the deeply negative shadow rate. This is the situation that a reduction of the IOER has the largest effect on the entire term structure. The purple curve in Figure 11 shows this graphically.

The results shown in the Figure 11 are based on the limited experiences of the negative interest rate policy. This inevitable smallness of the sample might induce the mismeasurement of the sensitivity of yield curve to the IOER. The estimated results, however, seem to be valid in that the estimated results are consistent with the actual development of yield curves when the Bank of Japan adopted the negative interest rate policy. The change of yield curves on that day was the largest among the similar episodes for different areas. In addition, the change of the Japanese Government Bond's ten years yield is almost same as the estimated influence of the reduction of the IOER. This confirms the validity of estimated results since the reduction was reported to be unanticipated by the market participants in the Nikkei Asian Review (2016a).

5.2 Relationship between quantitative easing and yield term premia

Next, the relationship between quantitative easing and yield term premia is examined. Figures 13 to 15 show that there are considerable differences in the relationship across

[5] Okina and Shiratsuka (2004) call this the "policy-duration effect."

the three areas.

In the case of Switzerland, the Swiss National Bank (SNB) has not purchased domestic securities as part of its monetary policy measures since 2010. Nevertheless, yield term premia mainly at the shorter end have moved in tandem with the increase in total assets resulting from the SNB's interventions in the foreign exchange market. These findings are in line with the results obtained by Christensen and Krogstrup (2016). They argue that the issuance of central bank reserves per se can influence long-term interest rates through a reserves-induced portfolio balance channel which is independent of the assets purchased.

In the case of Germany, there does not seem to be a strong link between term premia and quantitative easing. The total asset holdings of the European Central Bank (ECB) have fluctuated as a result of the long-term refinancing operations (LTROs) from 2011 to 2014. This massive expansion of the ECB's balance sheet through the LTROs has been called the equivalent to the quantitative easing undertaken in other areas. Pisani-Ferry and Wolf (2012) argue that the impact of the LTROs on yield curves has been much less significant for issues with a high credit rating such as Germany. The strong heterogeneity within the Eurozone reduces the effectiveness of the instruments on the German government bond market. They also point out that the LTROs have played the role of substituting for the dysfunctional interbank market in southern Europe rather than propping up total bank credit. The finding here of a weak link between term premia and the total asset holdings of the ECB is in line with the above argument by Pisani-Ferry and Wolf (2012). In addition to the LTROs, the Eurosystem started the public sector purchase programme (PSPP) in March 2015. Since then, the increase in total assets and the assets purchased for monetary policy purposes have been followed by declines in term premia.

The most striking results are those for Japan. The Bank of Japan's holdings of Japanese government bonds have increased recently, in particular since the start of Quantitative and Qualitative Easing. Yield term premia in Japan track well the path of holdings, showing a clear relationship between quantitative easing and yield term premia.

5.3 Decomposition of yield curves

The third analysis consists of decomposing yields in the three areas into three components: the part representing the effect of the expected interest rate, the term premium part, and the part quantifying the effect of the IOER. The following equations show how the interest rate can be decomposed into these three parts:

$$
\begin{aligned}
R_\tau \quad &\equiv -\log(P_\tau)/\tau \\
&\equiv -\log \frac{\left(E^Q\left[exp\left(-\int_0^\tau i(x_t, y_t, \varphi_t)dt\right)\right]\right)}{\tau} \\
&\equiv f^Q(x_t, y_t, \varphi_t) \\
&= E^P\left[\int_0^\tau i(x_t, 0, \varphi_t)dt\right] + \left\{f^Q(x_t, 0, \varphi_t) - E^P\left[\int_0^\tau i(x_t, 0, \varphi_t)dt\right]\right\} \\
&+\{f^Q(x_t, y_t, \varphi_t) - f^Q(x_t, 0, \varphi_t)\}\,.
\end{aligned}
$$

In the above decomposition, the first term represents the contribution of the expected interest rate to yields. The next term represents the contribution of the yield term premium. The first and second terms are calculated as if the IOER is not adopted and are obtained through counterfactual simulation. The last term quantifies the effect of the change in the IOER.

Figures 16 to 18 show the decomposition results for developments in the yield curve in the three areas in recent years. The results for the three areas have three things in common. First, the reduction in the IOER plays a greater role in the decline of yields into negative territory at shorter maturities than longer maturities. Second, yield term premia play a greater role at longer maturities than shorter maturities. And third, the contribution of the decline in the expected interest rate at shorter maturities gained momentum primarily after the reduction of the IOER. The last finding suggests that the negative interest rate policy has reinforced forward guidance. The kind of decomposition presented here is useful to examine the trade-offs between monetary policy measures. Each measure helps to lower the entire yield curve to stimulate the economy; however, the quantitative impact of each measure should be examined to find out, for example, which measure is the most effective, or when a specific measure works more effectively than other measures. To determine which monetary policy measure should be prioritized, up-to-date estimates on the size of the impact of each measure on the yield curve are necessary. The analysis presented here is potentially useful for this purpose.

5.4 What explains developments in the power of arbitrage?

In the variable extended model, the factor φ_t^* represents the power of arbitrage between cash or reserves and bonds. Why does φ_t^* vary? By definition, φ_t^* is influenced by the limit to arbitrage arising from cost of holding cash and market segmentations. Figure 19 plots φ_t^* and cross-currency basis swap spreads. A cross-currency basis swap is an agreement between two counterparties trading floating rate payments in their respective currencies. Without frictions in financial markets, a cross-currency swap should have a zero value with no spread on either side. However, funding costs in the different currencies may differ and relative funding costs may vary over the lifetime of the swap. The market charges a premium for transferring assets or liabilities from one currency to another. U.S. investors can transfer their cash flow from foreign government bonds to U.S. dollars through the basis swap market receiving premia. Even if foreign government bond yields are negative, large basis swap spreads can attract U.S. investors. The larger the spread is, the more foreign government bond yields fall below zero or the level of the IOER. While the basis swap market lies outside the model, φ_t^* represents the above mechanism, since it weakens the power of arbitrage between cash or reserves and bonds. In appendix B, using a simple model, the mechanism behind the limitation of arbitrage between government bonds and reserves is described. The profit maximizing behaviors of financial intermediaries under financial frictions such as financial regulations lead to the linear relationship between the effective lower bound of government bond yield and the violation of covered interest parity. The model analysis explains the rationale behind that the power of arbitrage moves in tandem with basis swap spreads.

6. Conclusion

This paper proposes a new type of term structure model label as the "extended model." The model generalizes the Gaussian affine model and the Black model. This paper also proposes an efficient and accurate solution method that can be applied to both the Black model and the extended model. The estimation results using data on government bond term structures in Switzerland, Germany, and Japan show that the new model is superior to the Gaussian affine model and the Black model. Applying the new model, the effects of forward guidance, quantitative easing, and the negative policy interest rate can be individually quantified. The results indicate that a change in the IOER has a larger effect on the term structure when the effect of forward guidance strongly works. The results further suggest that the power of arbitrage between money or reserves and government bonds moves in tandem with basis swap spreads.

This last finding points in the direction of possible future research. In the literature, the effects of unconventional monetary policy measures are quantified using models focusing only on the term structure of one specific country. However, the findings obtained in this study highlight the need to take the influence of other markets on the term structure into account. Broadening the analysis to the term structures of two areas with foreign exchange rates or basis swap spreads provides a promising avenue for future research. It enables us to identify the effects of the actions of a foreign central bank on home country interest rates.

References

Arai, Tadahiko, Yoshibumi Makabe, Yasunori Ookawara, and Teppei Nagano. 2016. "Recent Trends in Cross-currency Basis," Bank of Japan Review Series 2016-E-7.

Bauer, Michael D., Glenn D. Rudebusch, and Jing Cynthia Wu. 2012. "Correcting Estimation Bias in Dynamic Term Structure Models." Journal of Business and Economic Statistics, 30(3), 454–467.

Bauer, Michael D., and Glenn D. Rudebusch. 2016. "Monetary Policy Expectations at the Zero Lower Bound." Journal of Money, Credit, and Banking. Forthcoming.

Black, Fischer. 1995. "Interest Rates as Options," Journal of Finance, 50(5), 1371–1376.

Christensen, Jens H. E., and Glenn D. Rudebusch. 2015. "Estimating Shadow-Rate

Term Structure Models with Near-Zero Yields.” Journal of Financial Econometrics, 13(2), 226–259.

Christensen, Jens H. E., and Signe Krogstrup. 2016. “Transmission of Quantitative Easing: The Role of Central Bank Reserves.” Federal Reserve Bank of San Francisco Working Paper 2014-18.

Dai, Qiang, and Kenneth J. Singleton. 2002. “Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure.” Journal of Financial Economics, 63(3), 415–441.

Duffee, Gregory R. 2002. “Term Premia and Interest Rate Forecasts in Affine Models.” Journal of Finance, 57(1), 405–443.

Duffie, Darrel. 2001. Dynamic Asset Pricing Theory. Princeton, N.J.: Princeton University Press.

Duffie, Darrell, and Rui Kan. 1996. “A Yield-Factor Model of Interest Rates.” Mathematical Finance, 6(4), 379–406.

Fitch. 2016. “Negative-Yielding Sovereign Debt Grew to $10.4 Trn in May.” Online: https://www.fitchratings.com/site/pr/1005505.

Greenspan, Alan. 2005. “Testimony of Chairman Alan Greenspan: Federal Reserve Board’s Semiannual Monetary Policy Report to the Congress.” Washington: Board of Governors of the Federal Reserve (February 16).

Gorovoi, Viatcheslav, and Vadim Linetsky. 2004. “Black’s Model of Interest Rates as Options, Eigenfunction Expansions, and Japanese Interest Rates,” Mathematical Finance, 14(1), 49–78.

Ichiue, Hibiki, and Yoichi Ueno. 2006. “Monetary Policy and the Yield Curve at Zero Interest: The Macro-finance Model of Interest Rates as Options.” Bank of Japan

Working Paper Series 06-E-16.

Ichiue, Hibiki, and Yoichi Ueno. 2007. “Equilibrium Interest Rates and the Yield Curve in a Low Interest Rate Environment.” Working Paper 2007-E-18, Bank of Japan.

Ichiue, Hibiki, and Yoichi Ueno. 2013. “Estimating Term Premia at the Zero Bound: An Analysis of Japanese, US, and UK Yields.” Working Paper 2013-E-8, Bank of Japan.

Ichiue, Hibiki, and Yoichi Ueno. 2015. “Monetary Policy and the Yield Curve at Zero Interest.” Journal of the Japanese and International Economies 38, 1–12.

Joslin, Scott, Kenneth J. Singleton, and Haoxiang Zha. 2011. “A New Perspective on Gaussian Dynamic Term Structure Models.” Review of Financial Studies, 25(7), 926–970.

Kim, Don H., and Marcel A. Priebsch. 2013. “Estimation of Multi-Factor Shadow-Rate Term Structure Models.” Unpublished manuscript, Federal Reserve Board 2013.

Kim, Don H., and Kenneth J. Singleton. 2012. “Term Structure Models and the Zero Bound: An Empirical Investigation of Japanese Yields.” Journal of Econometrics, 170(1), 32–49.

Kim, Don H., and Jonathan H. Wright. 2005. “An Arbitrage-Free Three-Factor Term Structure Model and the Recent Behavior of Long-Term Yields and Distant-Horizon Forward Rates.” Finance and Economics Discussion Paper 2005-33. Washington: Board of Governors of the Federal Reserve System.

Kortela, Tomi . 2016. “A Shadow Rate Model with Time-Varying Lower Bound of Interest Rates.” Bank of Finland Research Discussion Paper 19/2016.

Krippner, Leo. 2013. “Measuring the Stance of Monetary Policy in Zero Lower Bound Environments.” Economics Letters, 118(1), 135–138.

Krippner, Leo. 2014. “A Tractable Framework for Zero-Lower-Bound Gaussian Term Structure Models,” CAMA Working Paper 49/2013, Reserve Bank of New Zealand.

Okina, Kunio, and Shigenori Shiratuka. 2004. “Policy Commitment and Expectation Formation: Japan’s Experience under Zero Interest Rates.” North American Journal of Economics and Finance, 15(1), 75–100.

Nikkei Asian Review. 2016a. “BOJ adopts negative interest rates.” 29 January.

Nikkei Asian Review. 2016b. “Negative-rate JGBs Still Attracting Foreign Buyers.” 14 June.

Pisani-Ferry, Jean, and Guntram Wolf. 2012. “Is LTRO QE in Disguise?” VoxEU.org, 3 May.

Priebsch, Marcel A. 2013. “Computing Arbitrage-Free Yields in Multi-Factor Gaussian Shadow-Rate Term Structure Models.” Finance and Economics Discussion Series, Federal Reserve Board.

Tanizaki, Hisashi. 1996. Nonlinear Filters: Estimation and Applications. Berlin: Springer-Verlag.

Ueno, Yoichi, Naohiko Baba, and Yuji Sakurai. 2006. “The Use of the Black Model of Interest Rates as Options for Monitoring the JGB Market Expectations.” Bank of Japan Working Paper Series 06-E-15.

Wu, Jing Cynthia, and Fan Dora Xia. 2016. “Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound.” Journal of Money, Credit, and Banking, 48(2–3), 253–291.

Appendix A

This appendix shows the derivation of $\widetilde{P_\tau} \equiv E^Q\left[\exp\left(-\widetilde{I_\tau}\right)\right] = E^Q\left[\exp\left(-\alpha_0 - \alpha_1 i_{\frac{1}{4}\tau} - \alpha_2 i_{\frac{3}{4}\tau}\right)\right]$. In the following, the variables $\hat{s}_t \equiv \varphi s_t + (1-\varphi) y_t$ and $\tilde{s}_t \equiv s_t - y_t$ are used. In addition, $\mu_x(\tau)$ and $\sigma_x(\tau)$ represent the expectation and standard deviation of x_τ.

$E^Q\left[\exp\left(-\alpha_0 - \alpha_1 i_{\tau_1} - \alpha_2 i_{\tau_2}\right)\right]$ is computed as follows:

$$
\begin{aligned}
& E^Q\left[\exp\left(-\alpha_0 - \alpha_1 i_{\tau_1} - \alpha_2 i_{\tau_2}\right)\right] \\
= \quad & exp(-\alpha_0) E^Q\left[exp\begin{pmatrix} -\alpha_1\left(\hat{s}_{\tau_1} + (1-\varphi)\tilde{s}_{\tau_1} 1_{\{\tilde{s}_{\tau_1} \geq 0\}}\right) \\ -\alpha_2\left(\hat{s}_{\tau_2} + (1-\varphi)\tilde{s}_{\tau_2} 1_{\{\tilde{s}_{\tau_2} \geq 0\}}\right) \end{pmatrix}\right] \\
= \quad & exp(-\alpha_0) E^Q\left[\begin{matrix} exp\left(-\alpha_1 (1-\varphi)\tilde{s}_{\tau_1} 1_{\{\tilde{s}_{\tau_1} \geq 0\}} - \alpha_2 (1-\varphi)\tilde{s}_{\tau_2} 1_{\{\tilde{s}_{\tau_2} \geq 0\}}\right) \\ \cdot E^Q\left[exp\left(-\alpha_1 \hat{s}_{\tau_1} - \alpha_2 \hat{s}_{\tau_2}\right) \mid \tilde{s}_{\tau_1}, \tilde{s}_{\tau_2}\right] \end{matrix}\right] \\
= \quad & exp(-\alpha_0) E^Q\left[\begin{matrix} exp\left(-\alpha_1 (1-\varphi)\tilde{s}_{\tau_1} 1_{\{\tilde{s}_{\tau_1} \geq 0\}} - \alpha_2 (1-\varphi)\tilde{s}_{\tau_2} 1_{\{\tilde{s}_{\tau_2} \geq 0\}}\right) \\ \cdot \exp\begin{pmatrix} -\alpha_1 \begin{pmatrix} \mu_{\hat{s}}(\tau_1) + \Pi_1(\tau_1,\tau_2)\left(\tilde{s}_{\tau_1} - \mu_{\tilde{s}}(\tau_1)\right) \\ +\Pi_2(\tau_1,\tau_2)\left(\tilde{s}_{\tau_2} - \mu_{\tilde{s}}(\tau_2)\right) \end{pmatrix} \\ -\alpha_2 \begin{pmatrix} \mu_{\hat{s}}(\tau_2) + \Pi_3(\tau_1,\tau_2)\left(\tilde{s}_{\tau_1} - \mu_{\tilde{s}}(\tau_1)\right) \\ +\Pi_4(\tau_1,\tau_2)\left(\tilde{s}_{\tau_2} - \mu_{\tilde{s}}(\tau_2)\right) \end{pmatrix} \end{pmatrix} \\ \times \exp(0.5[\alpha_1\ \alpha_2]\Omega(\tau_1,\tau_2)[\alpha_1\ \alpha_2]') \end{matrix}\right]
\end{aligned}
$$

where $\Pi_1(\mathrm{t,u})$、$\Pi_2(\mathrm{t,u})$、$\Pi_3(\mathrm{t,u})$、$\Pi_4(\mathrm{t,u})$ are given by

$$[\Pi_1(t,u) \quad \Pi_2(t,u)] \equiv \left(\tilde{\Sigma}_{\hat{s}\tilde{s}}(t)\ \tilde{\Sigma}_{\hat{s}\tilde{s}}(t,u)\right)\begin{bmatrix} \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t,u) \\ \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t,u) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(u) \end{bmatrix}^{-1},$$

$$[\Pi_3(t,u) \quad \Pi_4(t,u)] \equiv \left(\tilde{\Sigma}_{\hat{s}\tilde{s}}(u,t)\ \tilde{\Sigma}_{\hat{s}\tilde{s}}(u)\right)\begin{bmatrix} \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t,u) \\ \tilde{\Sigma}_{\tilde{s}\tilde{s}}(t,u) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(u) \end{bmatrix}^{-1}.$$

Here, $\tilde{\Sigma}_{xy}(t)$ and $\tilde{\Sigma}_{xy}(u,t)$ respectively represent the covariance of x_t and y_t and of x_u and y_t. Meanwhile, $\Omega(\tau_1,\tau_2)$ is given by the following:

$$\Omega(\tau_1,\tau_2) \equiv \begin{bmatrix} \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_1) & \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_1,\tau_2) \\ \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_1,\tau_2) & \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_2) \end{bmatrix}$$

$$
\begin{aligned}
&-\begin{bmatrix} \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1) & \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2) \\ \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2,\tau_1) & \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2) \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2) \\ \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2) & \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1) & \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2) \\ \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2,\tau_1) & \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2) \end{bmatrix}'
\end{aligned}
$$

$$
\begin{aligned}
\Omega(\tau_1,\tau_2)[1,1] &= \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_1) \\
&\quad - \frac{\tilde{\Sigma}^2_{\hat{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - 2\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2) + \tilde{\Sigma}^2_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1)}{\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2)^2} \\
\Omega(\tau_1,\tau_2)[1,2] &= \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_1,\tau_2) - \frac{\begin{array}{c}\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - \tilde{\Sigma}^2_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2) \\ +\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1) - \tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2)\end{array}}{\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2)^2} \\
\Omega(\tau_1,\tau_2)[2,1] &= \Omega(\tau_1,\tau_2)[1,2] \\
\Omega(\tau_1,\tau_2)[2,2] &= \tilde{\Sigma}_{\hat{s}\hat{s}}(\tau_2) \\
&\quad - \frac{\tilde{\Sigma}^2_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - 2\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_1,\tau_2)\tilde{\Sigma}_{\hat{s}\tilde{s}}(\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2) + \tilde{\Sigma}^2_{\hat{s}\tilde{s}}(\tau_2)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1)}{\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1)\tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_2) - \tilde{\Sigma}_{\tilde{s}\tilde{s}}(\tau_1,\tau_2)^2}
\end{aligned}
$$

Next, Γ_1, Γ_2, Γ_3, Γ_4 and Γ_5 are defined as follows:

$$
\begin{aligned}
\Gamma_1 &\equiv \begin{array}{l} -\alpha_0 - \alpha_1\big(\mu_{\hat{s}}(\tau_1) - \Pi_1\mu_{\tilde{s}}(\tau_1) - \Pi_2\mu_{\tilde{s}}(\tau_2)\big) \\ \qquad\qquad - \alpha_2\big(\mu_{\hat{s}}(\tau_2) - \Pi_3\mu_{\tilde{s}}(\tau_1) - \Pi_4\mu_{\tilde{s}}(\tau_2)\big) \\ +0.5(\alpha_1^2\Omega_{11} + 2\alpha_1\alpha_2\Omega_{12} + \alpha_2^2\Omega_{22}) \end{array} \\
\Gamma_2 &\equiv -\alpha_1\Pi_1 - \alpha_2\Pi_3 \\
\Gamma_3 &\equiv -\alpha_1\Pi_2 - \alpha_2\Pi_4 \\
\Gamma_4 &\equiv -\alpha_1(1-\varphi) \\
\Gamma_5 &\equiv -\alpha_2(1-\varphi)
\end{aligned}
$$

Then, $E^Q\big[\exp\big(-\alpha_0 - \alpha_1 i_{\tau_1} - \alpha_2 i_{\tau_2}\big)\big]$ can be represented as follows:

$$
\begin{aligned}
&E^Q\big[\exp\big(-\alpha_0 - \alpha_1 i_{\tau_1} - \alpha_2 i_{\tau_2}\big)\big] \\
\equiv\;& \exp(\Gamma_1)E^Q\big[\exp\big(\Gamma_2\tilde{s}_{\tau_1} + \Gamma_3\tilde{s}_{\tau_2} + \Gamma_4\tilde{s}^+_{\tau_1} + \Gamma_5\tilde{s}^+_{\tau_2}\big)\big]
\end{aligned}
$$

The above expectation is divided into four parts. The first part is the case of $\tilde{s}_{\tau_1} \geq 0$ and $\tilde{s}_{\tau_2} \geq 0$:

$$
\begin{aligned}
①\;& \int_{x\geq 0}\int_{y\geq 0} e^{\Gamma_2 x + \Gamma_3 y + \Gamma_4 x + \Gamma_5 y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dx\,dy \\
\equiv\;& \int_{x\geq 0}\int_{y\geq 0} e^{(\Gamma_2+\Gamma_4)x + (\Gamma_3+\Gamma_5)y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dx\,dy \\
=\;& \exp\left(\begin{array}{c} (\Gamma_2+\Gamma_4)\mu_{\tilde{s}}(\tau_1) + (\Gamma_3+\Gamma_5)\mu_{\tilde{s}}(\tau_2) \\ +0.5\left\{\begin{array}{c} (\Gamma_2+\Gamma_4)^2\sigma^2_{\tilde{s}}(\tau_1) + 2(\Gamma_2+\Gamma_4)(\Gamma_3+\Gamma_5) \\ \rho_{\tilde{s}}(\tau_1,\tau_2)\sigma_{\tilde{s}}(\tau_1)\sigma_{\tilde{s}}(\tau_2) + (\Gamma_3+\Gamma_5)^2\sigma^2_{\tilde{s}}(\tau_2) \end{array}\right\} \end{array}\right)
\end{aligned}
$$

$$\times \mathrm{F}^1\begin{pmatrix} \frac{\mu_{\tilde{s}}(\tau_1)}{\sigma_{\tilde{s}}(\tau_1)} + (\Gamma_2+\Gamma_4)\sigma_{\tilde{s}}(\tau_1) + \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_3+\Gamma_5)\sigma_{\tilde{s}}(\tau_2), \\ \frac{\mu_{\tilde{s}}(\tau_2)}{\sigma_{\tilde{s}}(\tau_2)} + \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_2+\Gamma_4)\sigma_{\tilde{s}}(\tau_1) + (\Gamma_3+\Gamma_5)\sigma_{\tilde{s}}(\tau_2), \\ \rho_{\tilde{s}}(\tau_1,\tau_2) \end{pmatrix}$$

The second part is the case of $\tilde{s}_{\tau_1} \geq 0$ and $\tilde{s}_{\tau_2} < 0$:

$$② \int_{x\geq 0}\int_{y<0} e^{\Gamma_2 x+\Gamma_3 y+\Gamma_4 x} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dxdy$$

$$\equiv \int_{x\geq 0}\int_{y<0} e^{(\Gamma_2+\Gamma_4)x+\Gamma_3 y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dxdy$$

$$\exp\begin{pmatrix} (\Gamma_2+\Gamma_4)\mu_{\tilde{s}}(\tau_1) + \Gamma_3\mu_{\tilde{s}}(\tau_2) \\ +0.5\begin{Bmatrix} (\Gamma_2+\Gamma_4)^2\sigma_{\tilde{s}}^2(\tau_1) + 2(\Gamma_2+\Gamma_4)\Gamma_3 \\ \rho_{\tilde{s}}(\tau_1,\tau_2)\sigma_{\tilde{s}}(\tau_1)\sigma_{\tilde{s}}(\tau_2) + {\Gamma_3}^2\sigma_{\tilde{s}}^2(\tau_2) \end{Bmatrix} \end{pmatrix}$$

$$= \times\begin{Bmatrix} \Phi\left(-\frac{\mu_{\tilde{s}}(\tau_2)}{\sigma_{\tilde{s}}(\tau_2)} - \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_2+\Gamma_4)\sigma_{\tilde{s}}(\tau_1) - \Gamma_3\sigma_{\tilde{s}}(\tau_2)\right) \\ -\mathrm{F}^1\begin{pmatrix} -\frac{\mu_{\tilde{s}}(\tau_1)}{\sigma_{\tilde{s}}(\tau_1)} - (\Gamma_2+\Gamma_4)\sigma_{\tilde{s}}(\tau_1) - \rho_{\tilde{s}}(\tau_1,\tau_2)\Gamma_3\sigma_{\tilde{s}}(\tau_2), \\ -\frac{\mu_{\tilde{s}}(\tau_2)}{\sigma_{\tilde{s}}(\tau_2)} - \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_2+\Gamma_4)\sigma_{\tilde{s}}(\tau_1) - \Gamma_3\sigma_{\tilde{s}}(\tau_2), \\ \rho_{\tilde{s}}(\tau_1,\tau_2) \end{pmatrix} \end{Bmatrix}$$

The third part is the case of $\tilde{s}_{\tau_1} < 0$ and $\tilde{s}_{\tau_2} \geq 0$:

$$③ \int_{x<0}\int_{y\geq 0} e^{\Gamma_2 x+\Gamma_3 y+\Gamma_5 y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dxdy$$

$$\equiv \int_{x<0}\int_{y\geq 0} e^{\Gamma_2 x+(\Gamma_3+\Gamma_5)y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)\,dxdy$$

$$\exp\begin{pmatrix} \Gamma_2\mu_{\tilde{s}}(\tau_1) + (\Gamma_3+\Gamma_5)\mu_{\tilde{s}}(\tau_2) \\ +0.5\begin{Bmatrix} {\Gamma_2}^2\sigma_{\tilde{s}}^2(\tau_1) + 2\Gamma_2(\Gamma_3+\Gamma_5) \\ \rho_{\tilde{s}}(\tau_1,\tau_2)\sigma_{\tilde{s}}(\tau_1)\sigma_{\tilde{s}}(\tau_2) + (\Gamma_3+\Gamma_5)^2\sigma_{\tilde{s}}^2(\tau_2) \end{Bmatrix} \end{pmatrix}$$

$$= \times\begin{Bmatrix} \Phi\left(-\frac{\mu_{\tilde{s}}(\tau_1)}{\sigma_{\tilde{s}}(\tau_1)} - \Gamma_2\sigma_{\tilde{s}}(\tau_1) - \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_3+\Gamma_5)\sigma_{\tilde{s}}(\tau_2)\right) \\ -\mathrm{F}^1\begin{pmatrix} -\frac{\mu_{\tilde{s}}(\tau_1)}{\sigma_{\tilde{s}}(\tau_1)} - \Gamma_2\sigma_{\tilde{s}}(\tau_1) - \rho_{\tilde{s}}(\tau_1,\tau_2)(\Gamma_3+\Gamma_5)\sigma_{\tilde{s}}(\tau_2), \\ -\frac{\mu_{\tilde{s}}(\tau_2)}{\sigma_{\tilde{s}}(\tau_2)} - \rho_{\tilde{s}}(\tau_1,\tau_2)\Gamma_2\sigma_{\tilde{s}}(\tau_1) - (\Gamma_3+\Gamma_5)\sigma_{\tilde{s}}(\tau_2), \\ \rho_{\tilde{s}}(\tau_1,\tau_2) \end{pmatrix} \end{Bmatrix}$$

The last part is the case of $\tilde{s}_{\tau_1} < 0$ and $\tilde{s}_{\tau_2} < 0$:

④ $\int_{x<0}\int_{y<0} e^{\Gamma_2 x+\Gamma_3 y} f_{\tilde{s}_{\tau_1},\tilde{s}_{\tau_2}}(x,y)dxdy$

$$= \begin{array}{l} \exp\begin{pmatrix} \Gamma_2\mu_{\tilde{s}}(\tau_1)+\Gamma_3\mu_{\tilde{s}}(\tau_2) \\ +0.5\{{\Gamma_2}^2\sigma_{\tilde{s}}^2(\tau_1)+2\Gamma_2\Gamma_3\rho_{\tilde{s}}(\tau_1,\tau_2)\sigma_{\tilde{s}}(\tau_1)\sigma_{\tilde{s}}(\tau_2)+{\Gamma_3}^2\sigma_{\tilde{s}}^2(\tau_2)\} \end{pmatrix} \\ \times \mathrm{F}^1\begin{pmatrix} -\dfrac{\mu_{\tilde{s}}(\tau_1)}{\sigma_{\tilde{s}}(\tau_1)}-\Gamma_2\sigma_{\tilde{s}}(\tau_1)-\rho_{\tilde{s}}(\tau_1,\tau_2)\Gamma_3\sigma_{\tilde{s}}(\tau_2), \\ -\dfrac{\mu_{\tilde{s}}(\tau_2)}{\sigma_{\tilde{s}}(\tau_2)}-\rho_{\tilde{s}}(\tau_1,\tau_2)\Gamma_2\sigma_{\tilde{s}}(\tau_1)-\Gamma_3\sigma_{\tilde{s}}(\tau_2), \\ \rho_{\tilde{s}}(\tau_1,\tau_2) \end{pmatrix} \end{array}$$

Here, $\mathrm{F}^1\left(\frac{\mu_{\tilde{s}}(t)}{\sigma_{\tilde{s}}(t)},\frac{\mu_{\tilde{s}}(u)}{\sigma_{\tilde{s}}(u)},\rho_{\tilde{s}}(t,u)\right)$ represents the probability from $(-\infty,-\infty)$ to $\left(\frac{\mu_{\tilde{s}}(t)}{\sigma_{\tilde{s}}(t)},\frac{\mu_{\tilde{s}}(u)}{\sigma_{\tilde{s}}(u)}\right)$ of the binomial cumulative distribution with an average of $\vec{0}$ and a variance-covariance matrix given by $\begin{bmatrix} 1 & \rho_{\tilde{s}}(t,u) \\ \rho_{\tilde{s}}(t,u) & 1 \end{bmatrix}$. $\Phi(\cdot)$ represents the standard normal cumulative distribution.

Appendix B

In appendix B, the mechanism behind the limitation of arbitrage between government bonds and reserves is described using a simple model. Specifically, the aim of this appendix is to figure out the reason why arbitrage is between government bond and reserve limited and why the government bond yield can fall below the IOER.

1. Structure of model

There are two type of agent: Bank A in home country (United States) and Bank B in foreign country (Japan). Each bank borrows money and invests in various assets. Specifically, each bank's problem is given by the following sub-sections.

2. Bank A's problem

Bank A maximize the weighted average of net returns form many kinds of asset. The investment in each asset does not only charge balance sheet cost but also lose liquidity respectively. To invest in foreign assets, Bank A use currency swap to raise foreign funds. Specifically, the maximizing problem is given by

$$\max_{W_j, W_l^*} \sum_j R_j \cdot W_j + \sum_l F \cdot R_l^* \cdot W_l^* - L \cdot D - \int_V C(\nu, \epsilon) d\nu$$

subject to

$$\sum_j W_j + X = D$$

$$\sum_l S \cdot W_l^* = X$$

$$V = \sum_j \kappa_j \cdot W_j + \sum_l \kappa_{l^*} \cdot S \cdot W_l^*$$

where R_j and R_l^* are the gross returns from assets dominated in home and foreign currencies, F and S are forward and spot rates of foreign currency, L is the gross rate for borrowing money, $C(V,\epsilon)$ represents the sum of marginal balance sheet cost and the cost for losing liquidity, and κ_j and κ_l^* represents the differences of those cost between each asset. ϵ denotes a state of the economy. Bank A determines allocation of assets, that is W_j for any j and W_l^* for any l. In equilibrium, if Bank A hold asset j or l, the following holds;

$$R_j = L + C(V,\epsilon)\kappa_j$$

$$\frac{F}{S} \cdot R_l^* = L + C(V,\epsilon)\kappa_{l^*}$$

Therefore, if Bank A holds each country's government bond, the following holds;

$$\frac{F}{S} \cdot R_{GB^*} - R_{GB} = C(V,\epsilon)(\kappa_{GB^*} - \kappa_{GB})$$

This equality represents the violation of covered interest parity due to balance sheet cost and liquidity premium if $\kappa_{GB^*} \neq \kappa_{GB}$ and $C(V,\epsilon) > 0$.

Bank B similarly optimizes the weighted average of net returns. The maximizing problem is given by

$$\max_{\widetilde{W_j},\widetilde{W_l}^*} \sum_j \frac{R_j}{F} \cdot \widetilde{W_j} + \sum_l R_l^* \cdot \widetilde{W_l}^* - L^* \cdot \widetilde{D} - \int \tilde{C}(v,\epsilon)dv$$

subject to

$$\sum_j \widetilde{W_j}^* + \tilde{X} = \widetilde{D}$$

$$\sum_l \frac{\widetilde{W_l}}{S} = \tilde{X}$$

$$\tilde{V} = \sum_j \widetilde{\kappa_j} \cdot \frac{\widetilde{W_j}}{S} + \sum_l \widetilde{\kappa_{j^*}} \cdot \widetilde{W_l}^*$$

Now, if Bank A lends to Bank B, the following holds;

$$L^* = R_{AB} = \frac{S}{F} \cdot (L + C(V,\epsilon)\kappa_{BA}).$$

If Bank B holds reserve in a current account in a central bank,

$$R_{R^*} = L^* + \tilde{C}(\tilde{V},\epsilon)\widetilde{\kappa_{R^*}}.$$

Therefore,

$$R_{R^*} = R_{AB} + \tilde{C}(\tilde{V},\epsilon)\widetilde{\kappa_{R^*}}.$$

$$R_{R^*} = R_{GB^*} + \frac{S}{F} C(V,\epsilon)(\kappa_{AB} - \kappa_{GB^*}) + \tilde{C}(\tilde{V},\epsilon)\widetilde{\kappa_{R^*}}.$$

$$R_{R^*} - R_{GB^*} = \frac{S}{F} C(V,\epsilon)(\kappa_{AB} - \kappa_{GB^*}) + \tilde{C}(\tilde{V},\epsilon)\widetilde{\kappa_{R^*}}.$$

Next we define

$$\Delta \equiv R_{GB^*} - \frac{S}{F} R_{GB} = \frac{S}{F} C(V,\epsilon)(\kappa_{GB}^* - \kappa_{GB})$$

Then,

$$R_{R^*} - R_{GB^*} = \Delta \frac{\kappa_{AB} - \kappa_{GB^*}}{\kappa_{GB}^* - \kappa_{GB}} + \tilde{C}(\tilde{V},\epsilon)\widetilde{\kappa_{R^*}}.$$

Here

$$R_{GB}^* - \frac{S}{F} \cdot R_{GB} = \tilde{C}(\tilde{V},\epsilon)(\widetilde{\kappa_{GB^*}} - \widetilde{\kappa_{GB}})$$

holds. Therefore, $\Delta = \tilde{C}(\tilde{V},\epsilon)(\widetilde{\kappa_{GB^*}} - \widetilde{\kappa_{GB}})$. As a result,

$$R_{R^*} - R_{GB^*} = \Delta \frac{\kappa_{AB} - \kappa_{GB^*}}{\kappa_{GB}^* - \kappa_{GB}} + \Delta \frac{\widetilde{\kappa_{R^*}}}{\widetilde{\kappa_{GB^*}} - \widetilde{\kappa_{GB}}}$$

$$R_{R^*} - R_{GB^*} = \Delta \left(\frac{\kappa_{AB} - \kappa_{GB^*}}{\kappa_{GB^*} - \kappa_{GB}} + \frac{\widetilde{\kappa_{R^*}}}{\widetilde{\kappa_{GB^*}} - \widetilde{\kappa_{GB}}} \right)$$

This equation shows that the extent how government bond yields is lower than the interest rate on reserves is strong link to the violation of covered interest rate parity. If U.S. government bonds are more liquid and costs much less than Japanese government bonds, $\widetilde{\kappa_{GB^*}} > \widetilde{\kappa_{GB}}$ and $\Delta > 0$ holds. In addition, if $\kappa_{GB^*} = \widetilde{\kappa_{GB^*}}$ and $\kappa_{GB} = \widetilde{\kappa_{GB}}$ hold, that is, bank A and B are charged in the same manner for holding U.S. government bonds and Japanese government bonds charge, and $\kappa_{AB} > \kappa_{GB^*}$ and $\widetilde{\kappa_{GB^*}} = \widetilde{\kappa_{R^*}}$, then the following relationship holds;

$$R_{R^*} - R_{GB^*} = \Delta\left(\frac{\kappa_{AB}}{\kappa_{GB^*} - \kappa_{GB}}\right)$$

Therefore, if relative costs between holdings of various assets do not change, that is $\frac{\kappa_{AB}}{\kappa_{GB^*}-\kappa_{GB}}$ dose not change, the change of Δ directly influence $R_{R^*} - R_{GB^*}$ or R_{GB^*}. It is straightforward to assume that $C(V,\epsilon)$ and $\tilde{C}(\tilde{V},\epsilon)$ depend on whole market liquidity to describe the current violation of covered interest rate parity due to the divergence on monetary policy. It means that relative costs between holdings of various assets keep steady, but absolute cost does change.

Using the notations in the main text, $R_{R^*} - R_{GB^*}$ can be represented as $y_t - i_t = y_t - \phi_t s_t - (1-\phi_t)y_t = -\phi_t(s_t - y_t)$. In the extended model, after some manipulations, $y_t + \phi_t(s_t - y_t)$ can be regarded as the effective lower bound. In this sense, the effective lower bound is influenced by the violation of covered interest rate parity, that is, the extent of the limit to arbitrage. Hence, considering the influence on home government bond yields from other financial markets including basis swap market, the effective lower bound does not take a constant value. More specifically, even if government bond yields take large negative values which dominate the sum of the various costs including holding and insurance, large basis swap spreads can attract foreign investors who incur cost for extending the size of balance sheet or losing liquidity.

In the modified Black model, the effective lower bound is given by a variable, say, $\hat{y}_t$. Also in this case, $\hat{y}_t$ is influenced by the violation of covered interest rate parity. To the best of my knowledge, $\hat{y}_t$ is estimated as if this is constant in the literature, then no mechanism behind the movement of $\hat{y}_t$ is explored.

Figure 1: Relationship between the short rate and the shadow rate

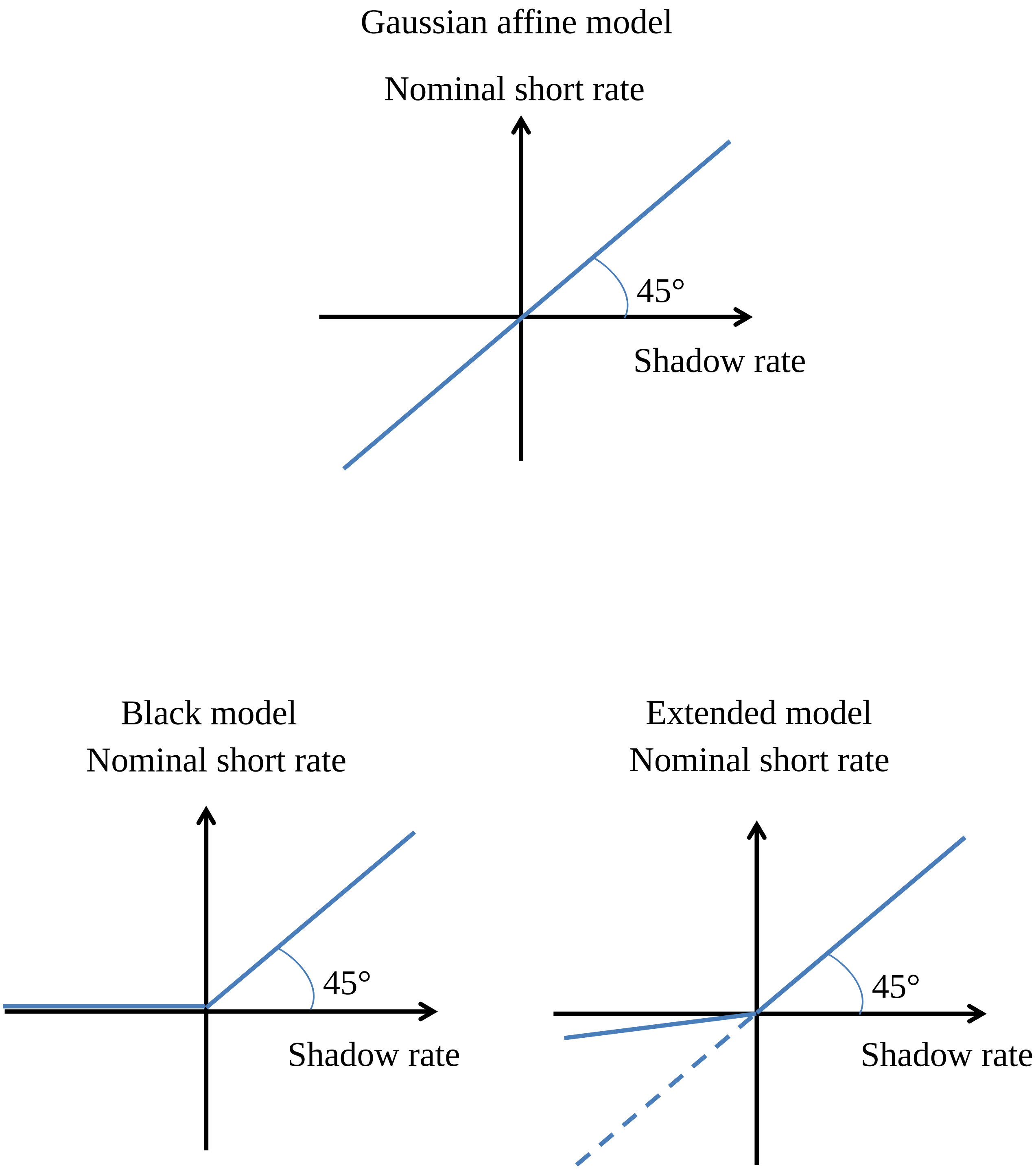
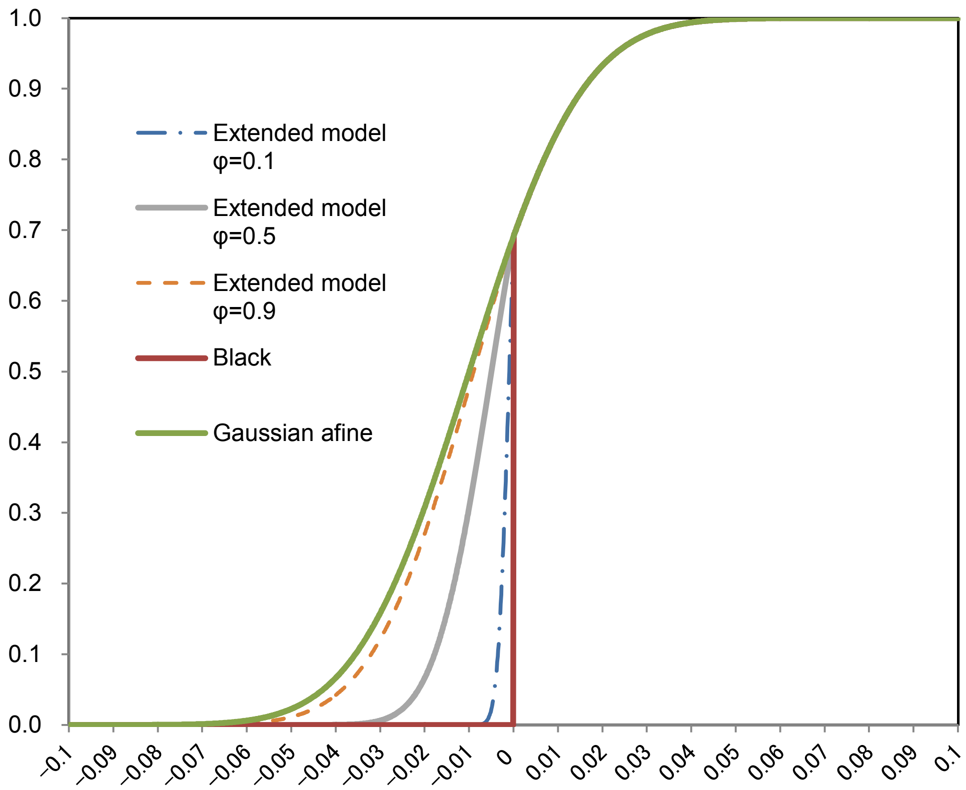
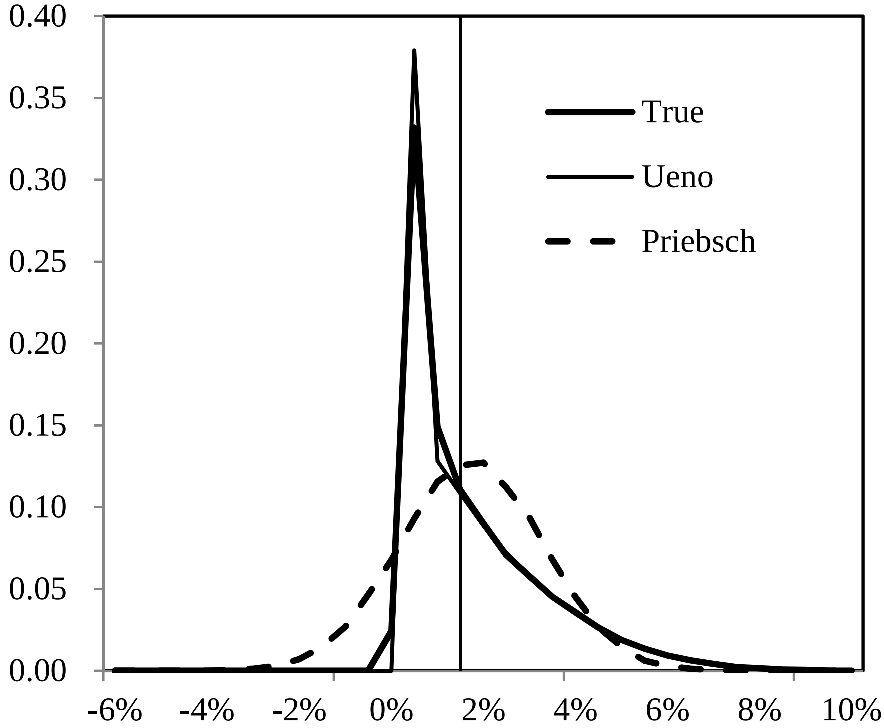


Figure 2: Cumulative probability distribution function of the short rate



Note: one-factor model, $s_t = 0$, $y_t = 0$, $E_t[s_\tau] = -0.01$, $Var_t[s_\tau] = 0.0004$.

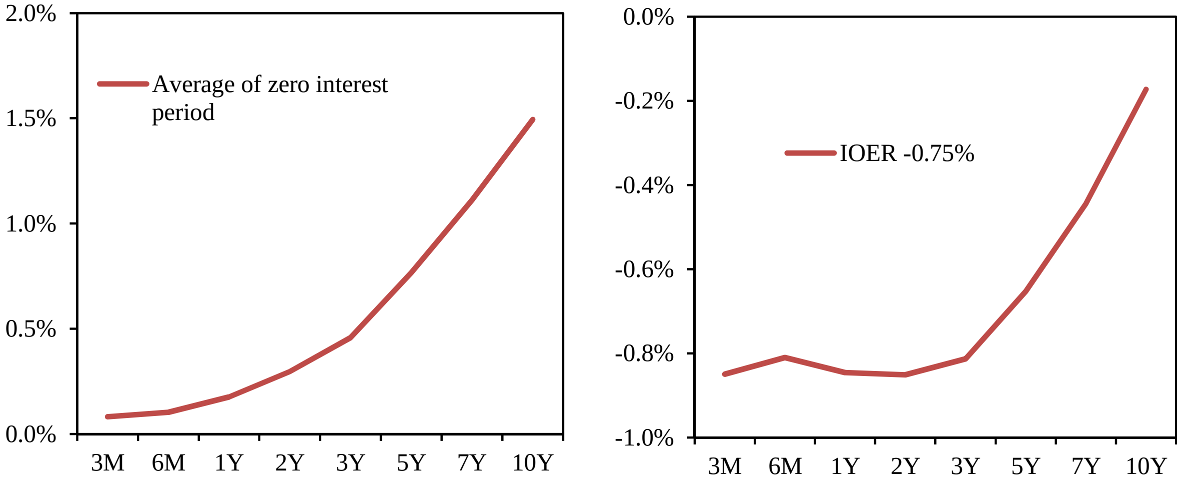
Figure 3: Distribution of I_τ/τ and its approximations



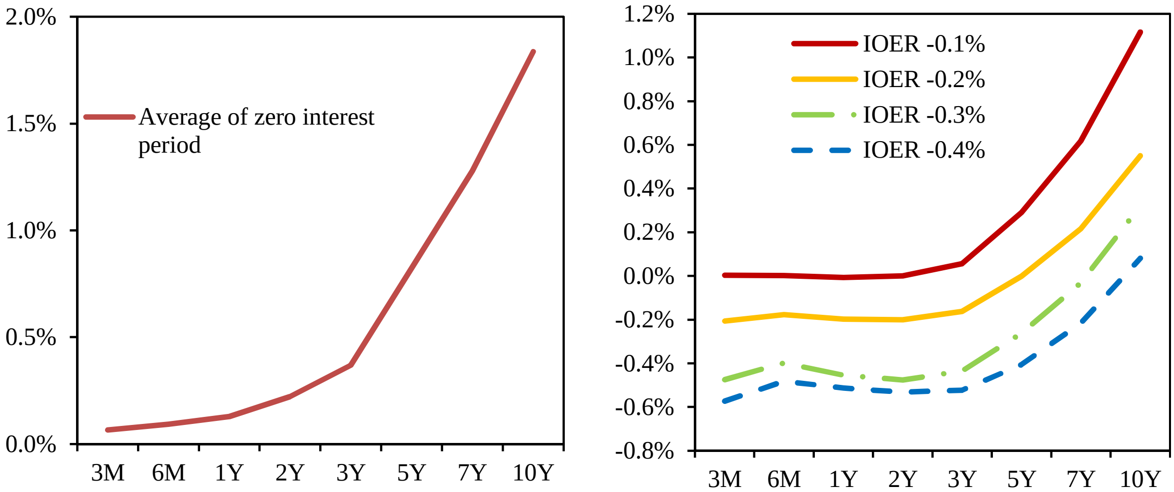
Note: One-factor model, $x_0 = 0$, $y = 0$, $\theta = 0.01$, $\kappa = 0.1$, $\sigma = 0.2$, $\phi = 0$. The maturity is 10 year. 100,000 paths are generated.

Figure 4: Typical yield curves

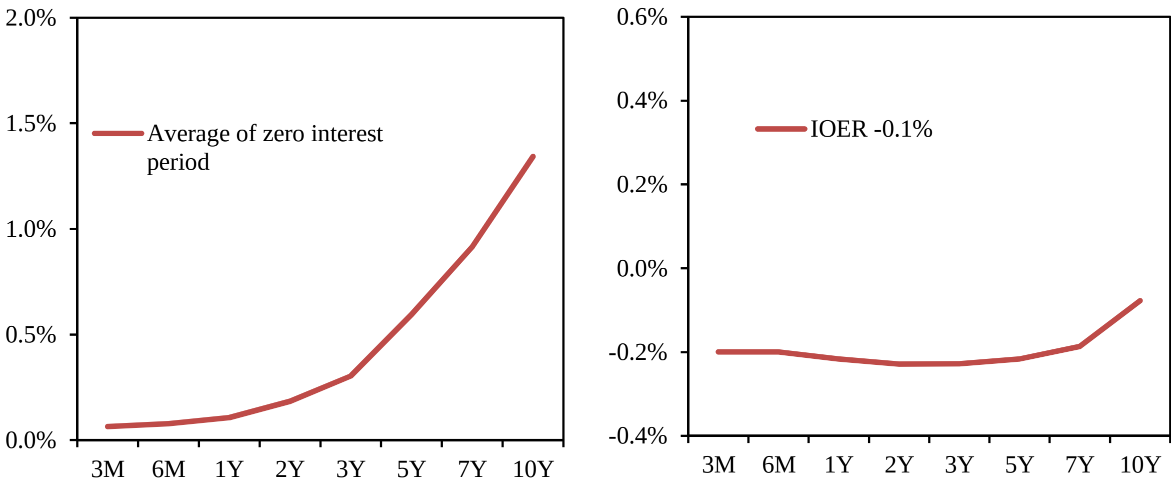
Switzerland



Germany



Japan



Notes: "Average of zero interest period" refers to the average when the 3-month rate was non-negative and less than 0.25%.
The panels on the right show the averages of yields while the IOER is equal to a particular level.
Source: Bloomberg

Figure 5: Term structure of volatility: Switzerland

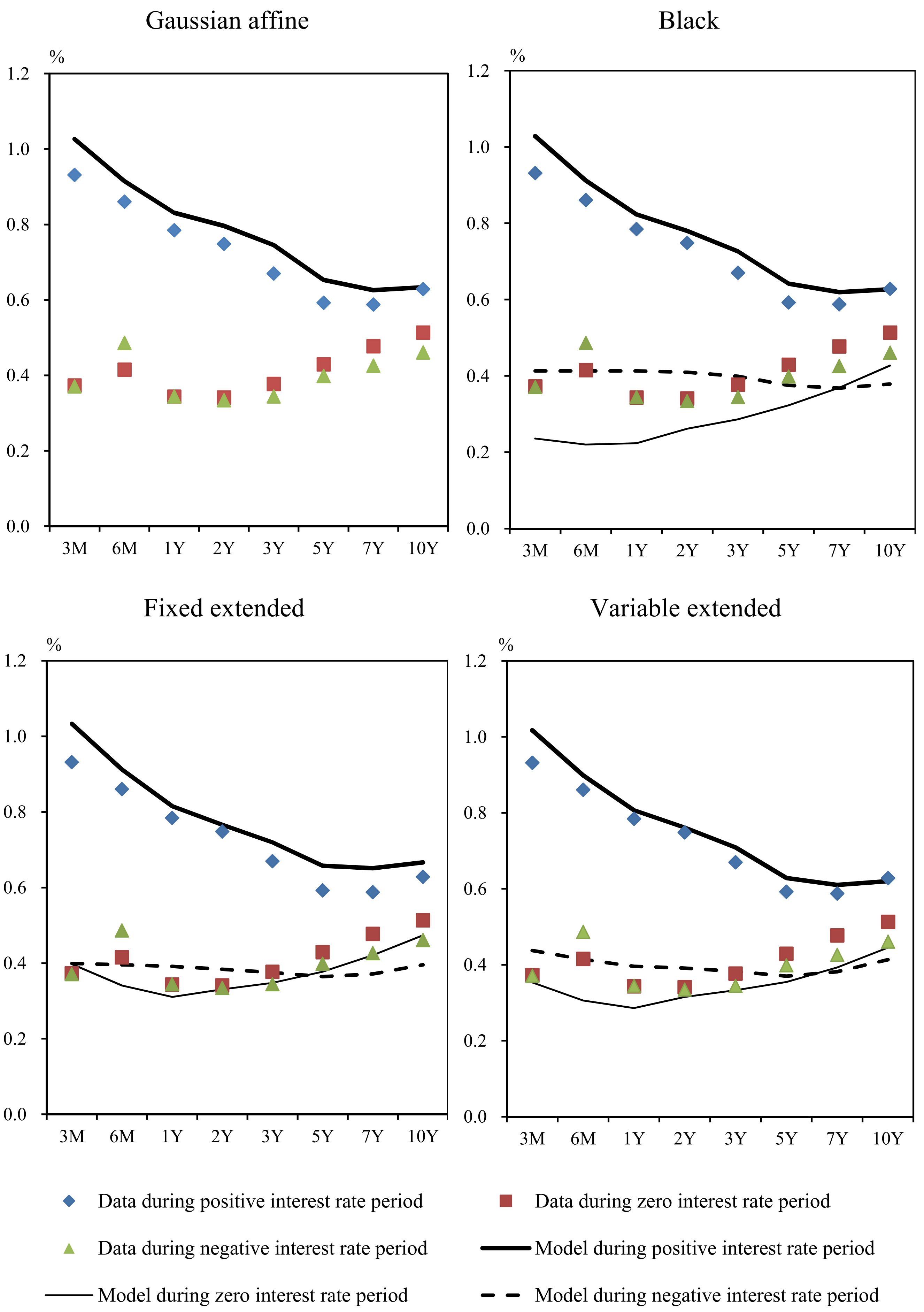


Figure 6: Term structure of volatility: Germany

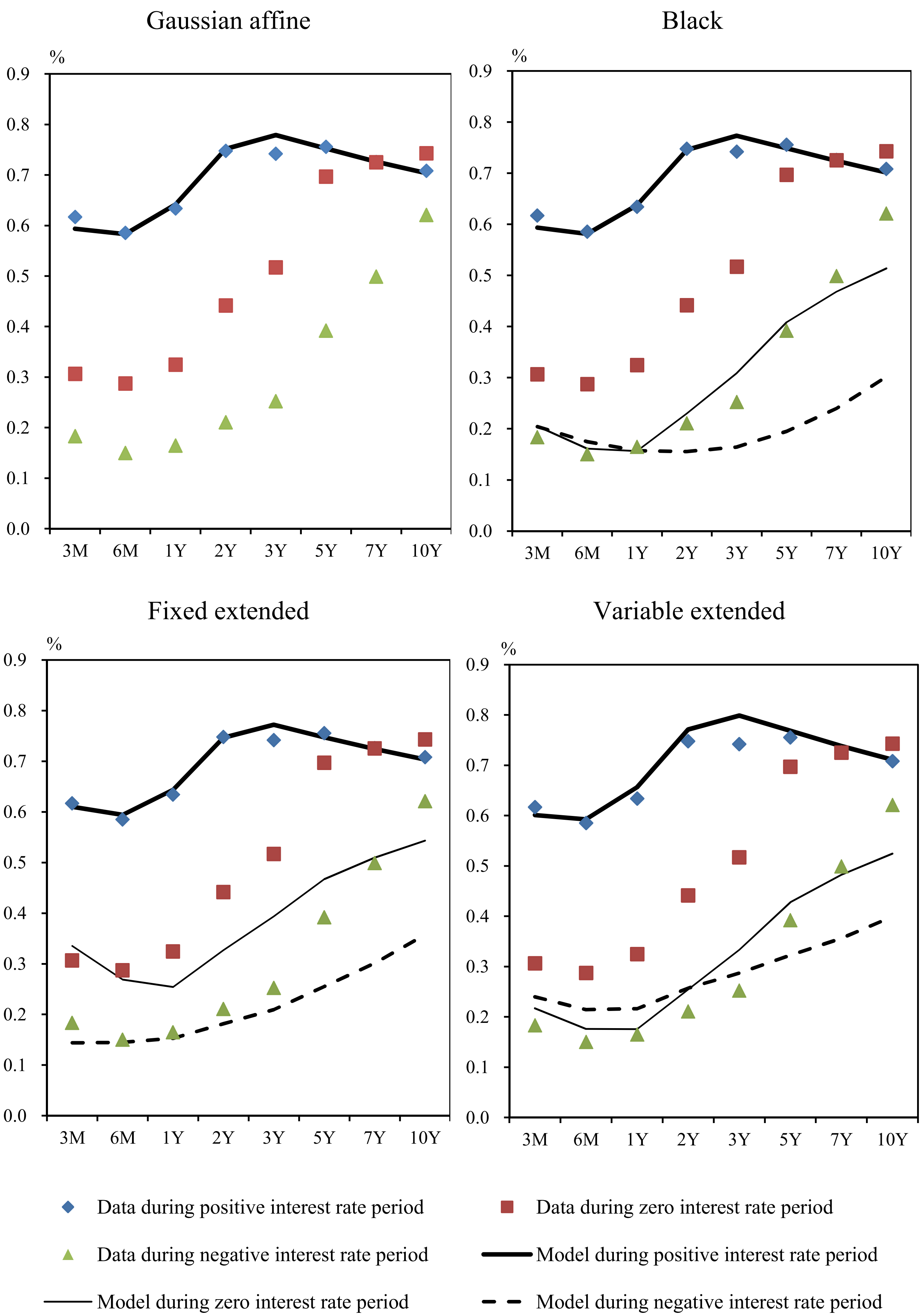


Figure 7: Term structure of volatility: Japan

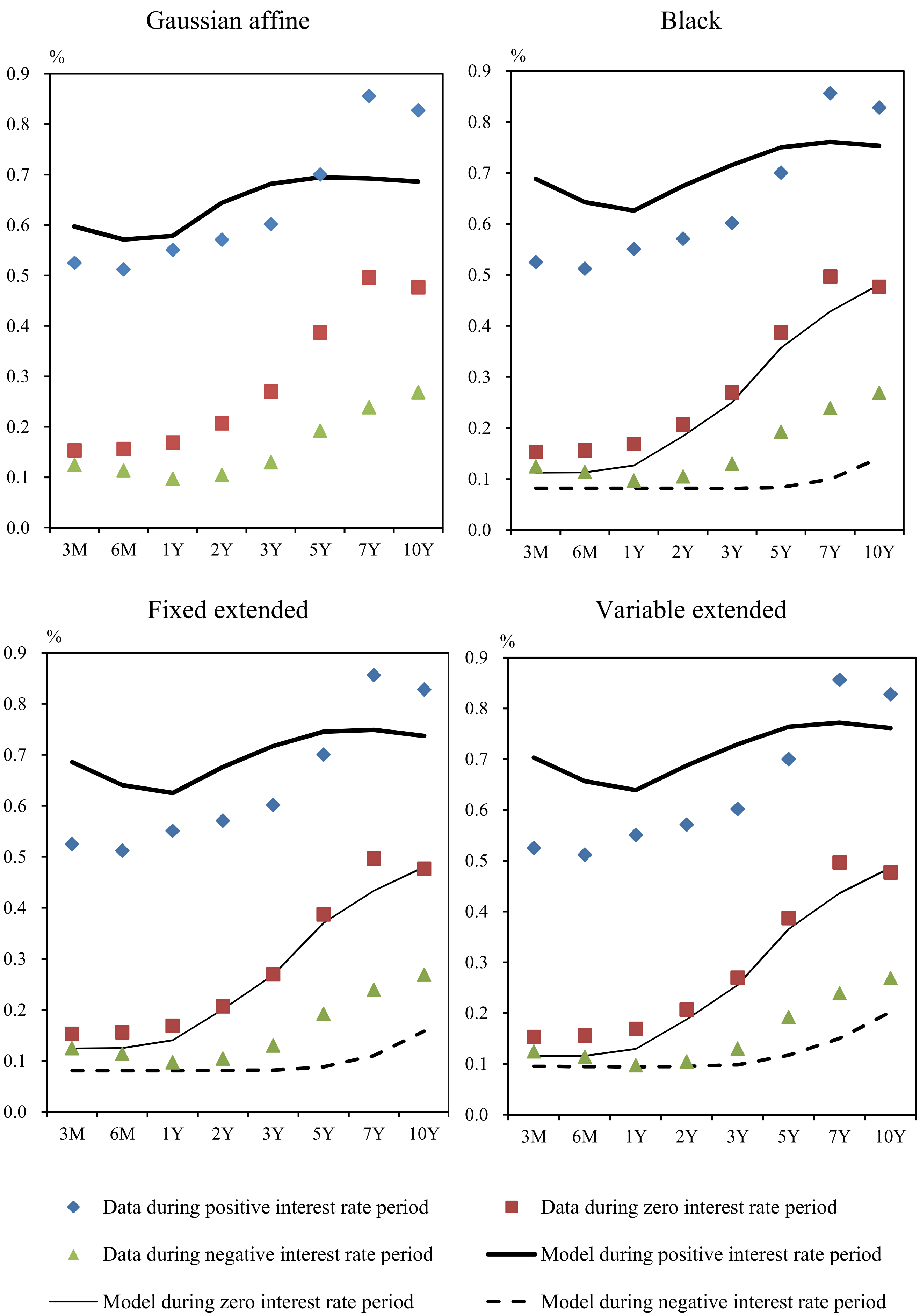
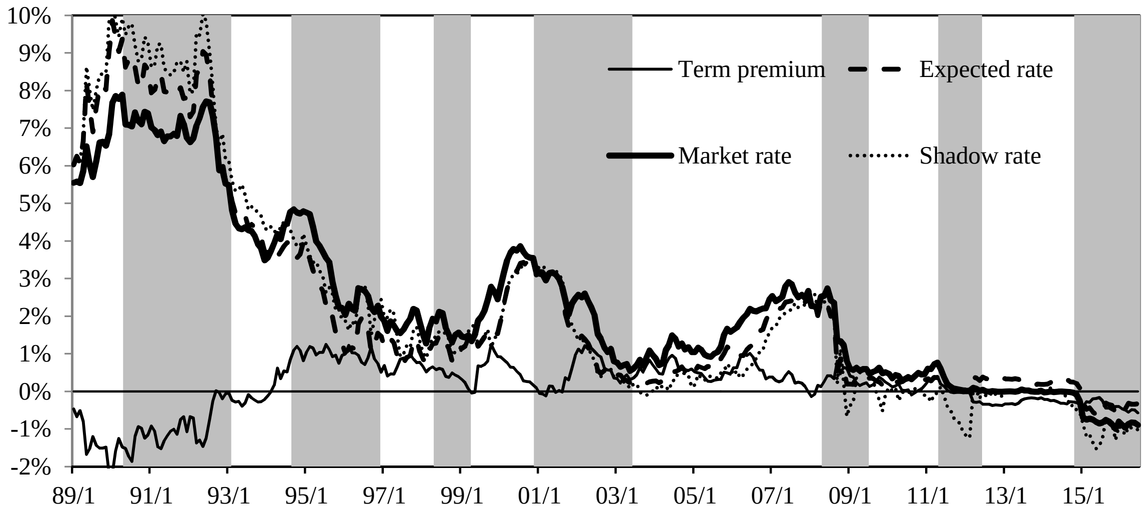


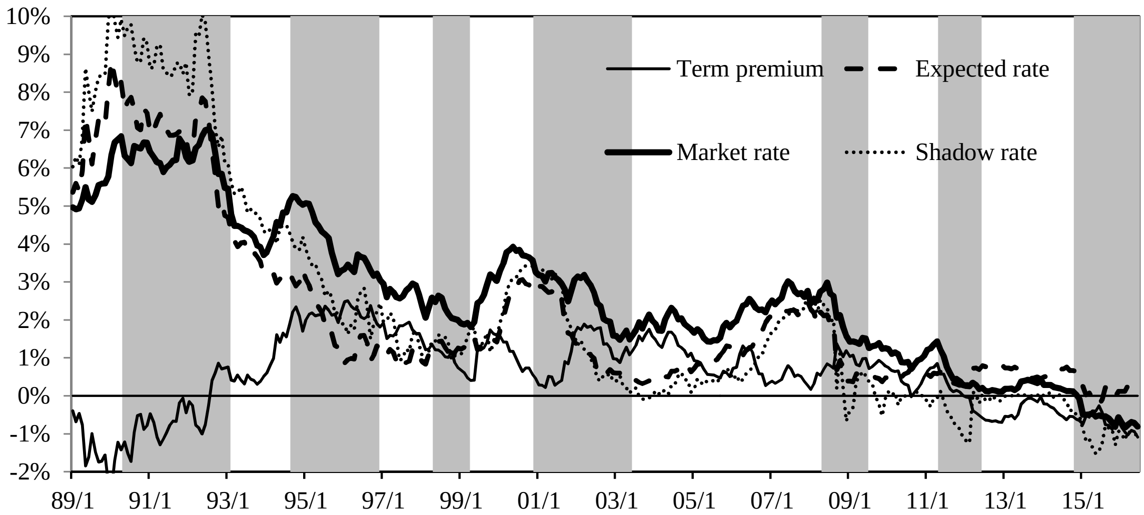
Figure 8: Shadow rate, expected interest rate, and yield term premia:

Switzerland

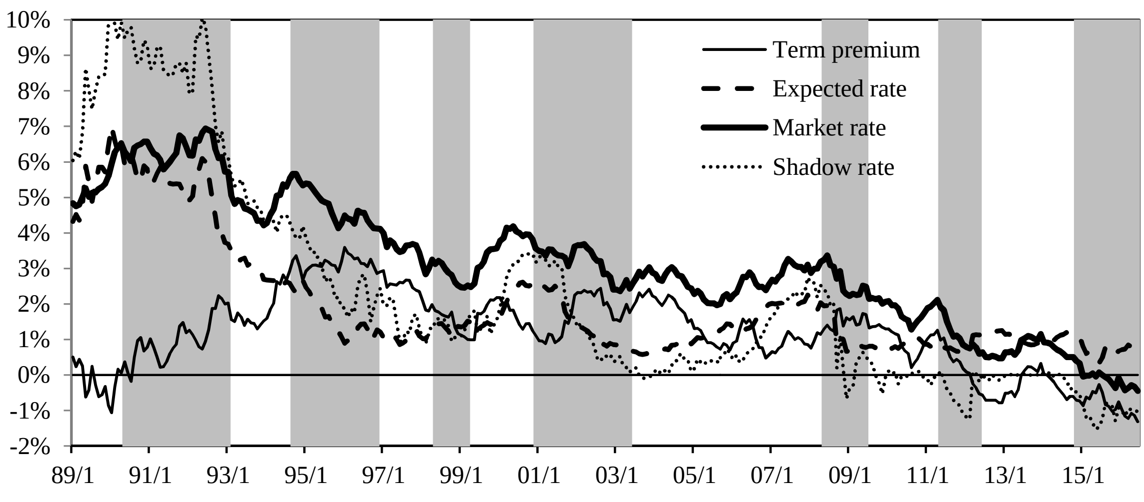
2Y



5Y



10Y

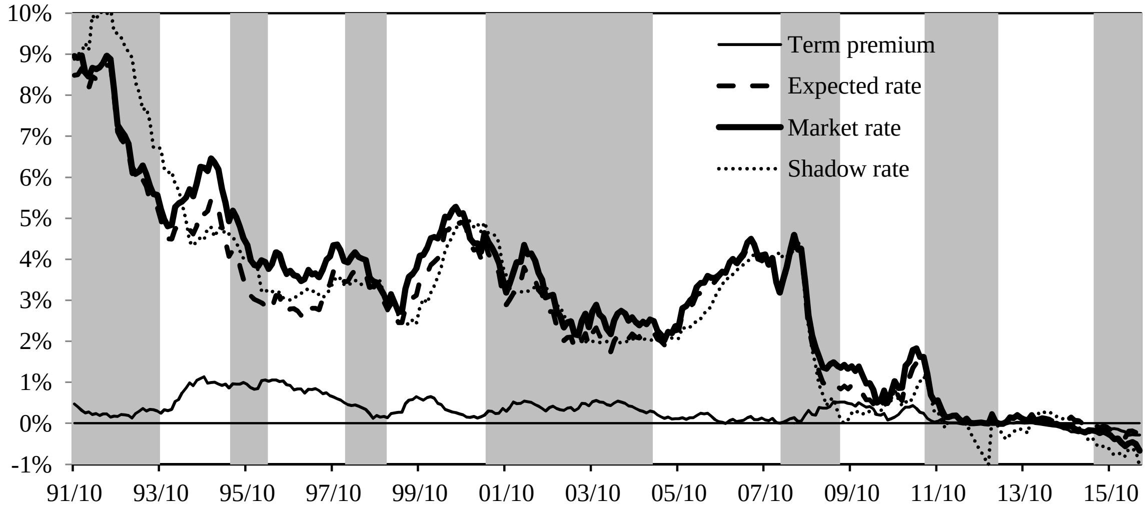


Note: Shade areas indicate recesion periods

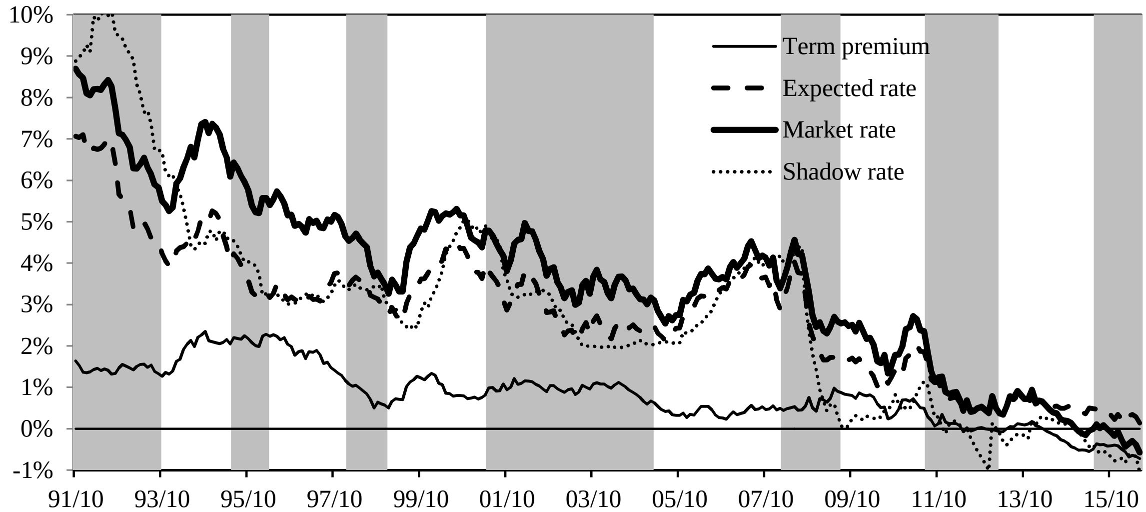
Figure 9: Shadow rate, expected interest rate, and yield term premia:

Germany

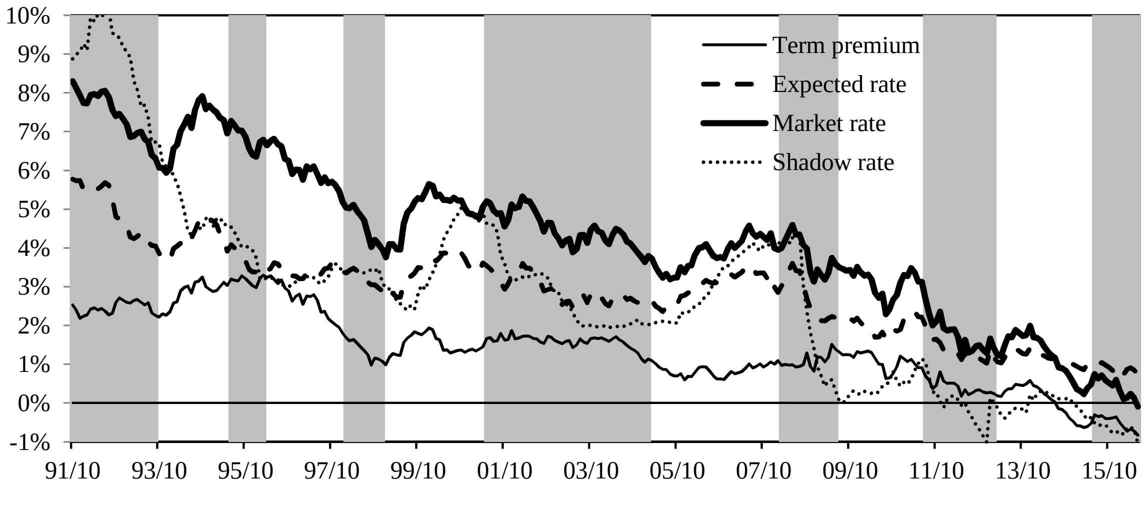
2Y



5Y



10Y

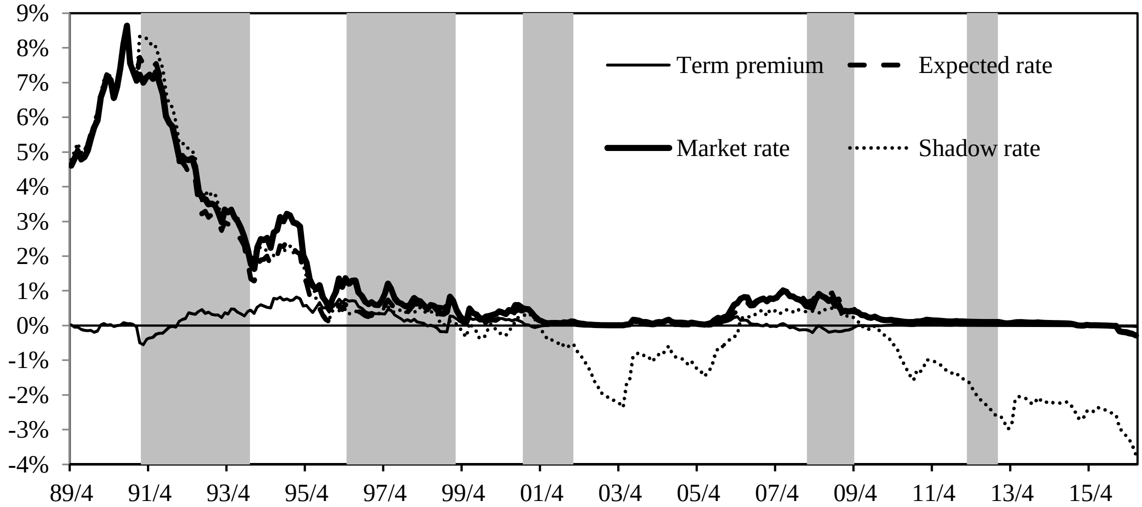


Note: Shade areas indicate recesion periods

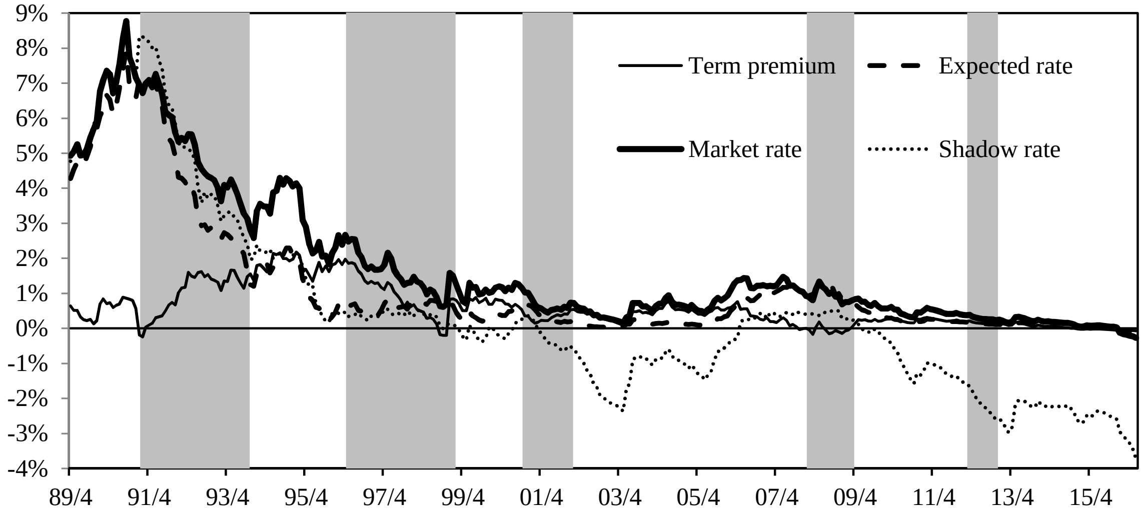
Figure 10: Shadow rate, expected interest rate, and yield term premia:

Japan

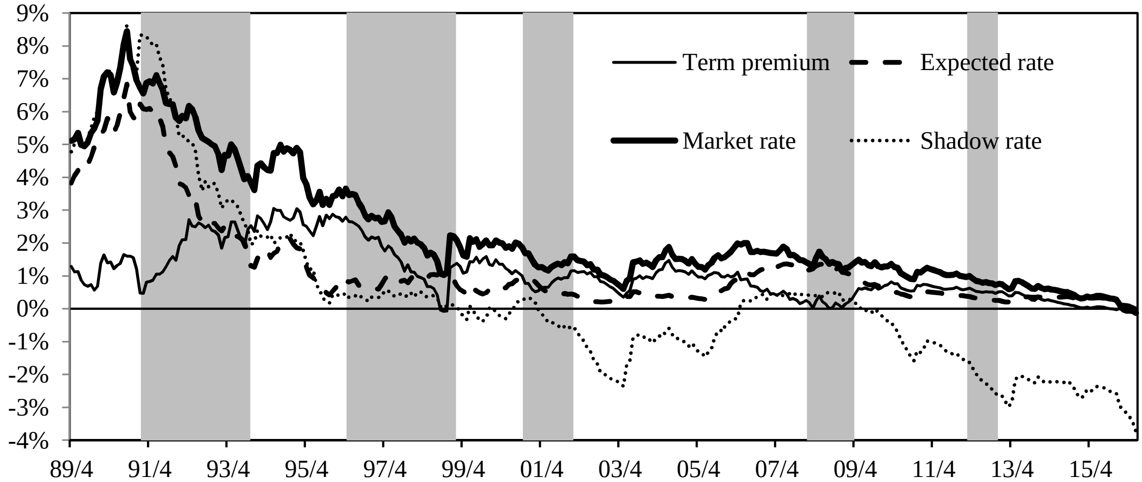
2Y



5Y



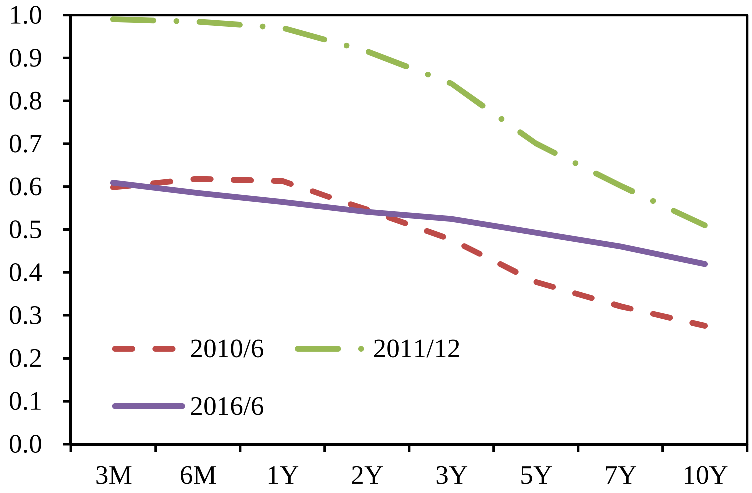
10Y



Note: Shade areas indicate recesion periods

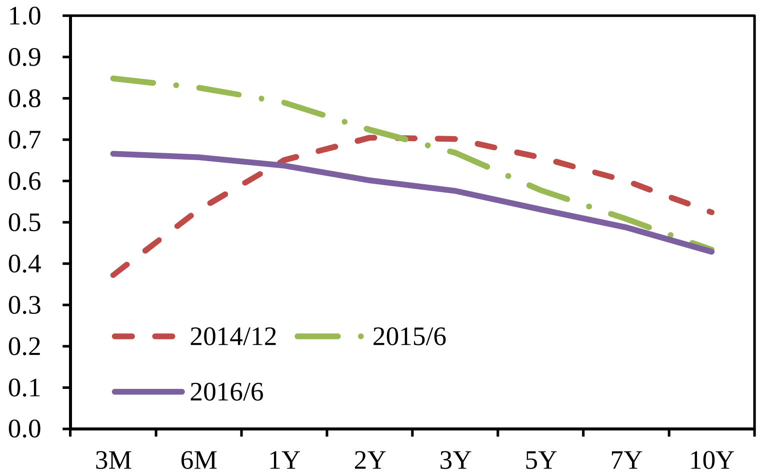
Figure 11 : Sensitivity of yield curves to the IOER

Switzerland



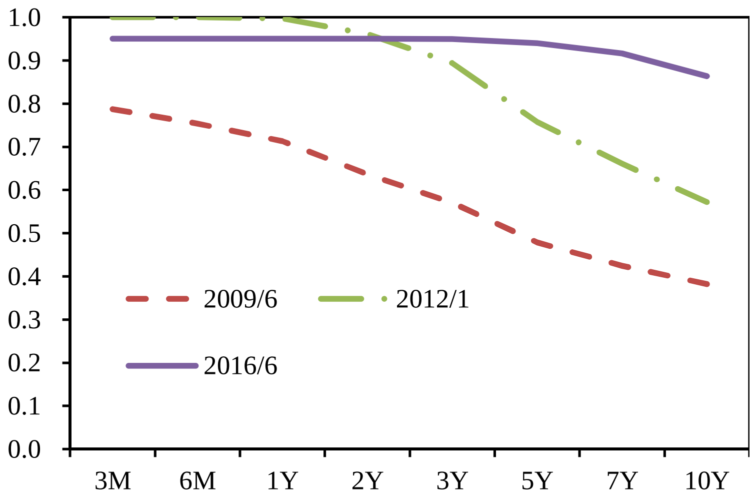
	$s(x_t)$	φ_t
2010/6	0.0%	0.00
2011/12	-1.1%	0.00
2016/6	-1.3%	0.34

Germany



	$s(x_t)$	φ_t
2014/12	0.0%	0.03
2015/6	-0.5%	0.12
2016/6	-1.0%	0.33

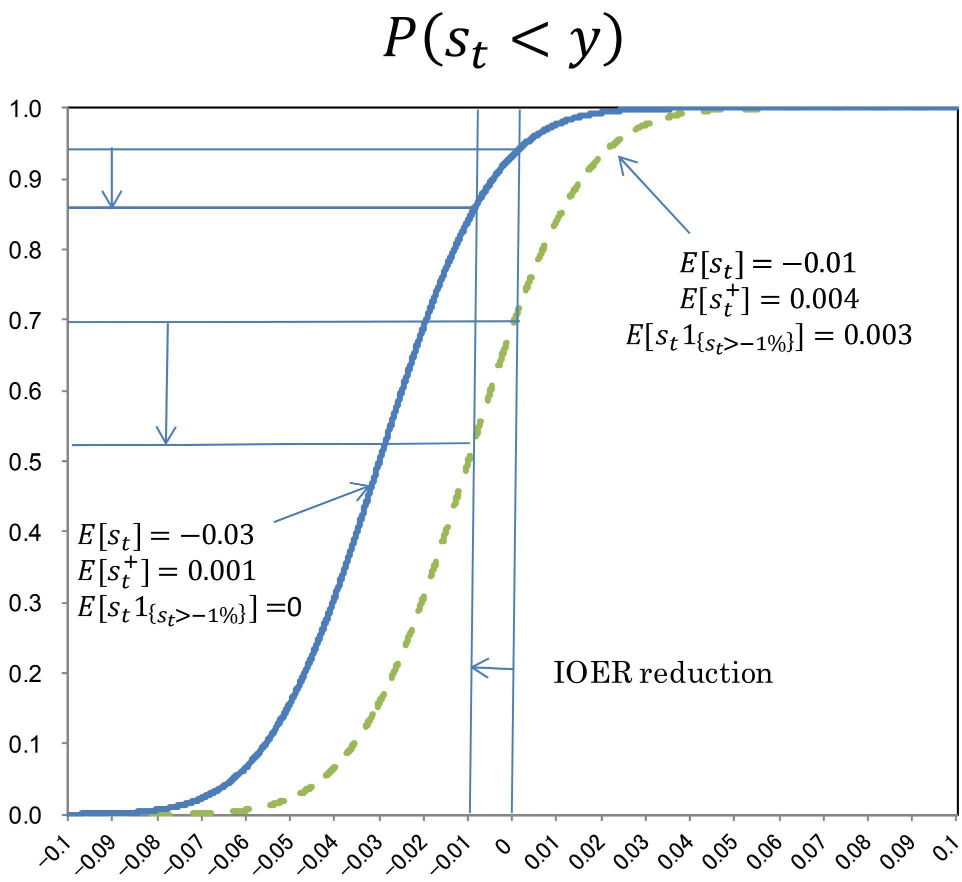
Japan



	$s(x_t)$	φ_t
2009/6	0.0%	0.00
2012/1	-1.5%	0.02
2016/6	-3.7%	0.05

Figure 12：Sensitivity of yield curves to the IOER

$$E[i_t] = \mathrm{E}\left[s_t 1_{\{s_t \geq y\}}\right] + y \times P(s_t < y)$$

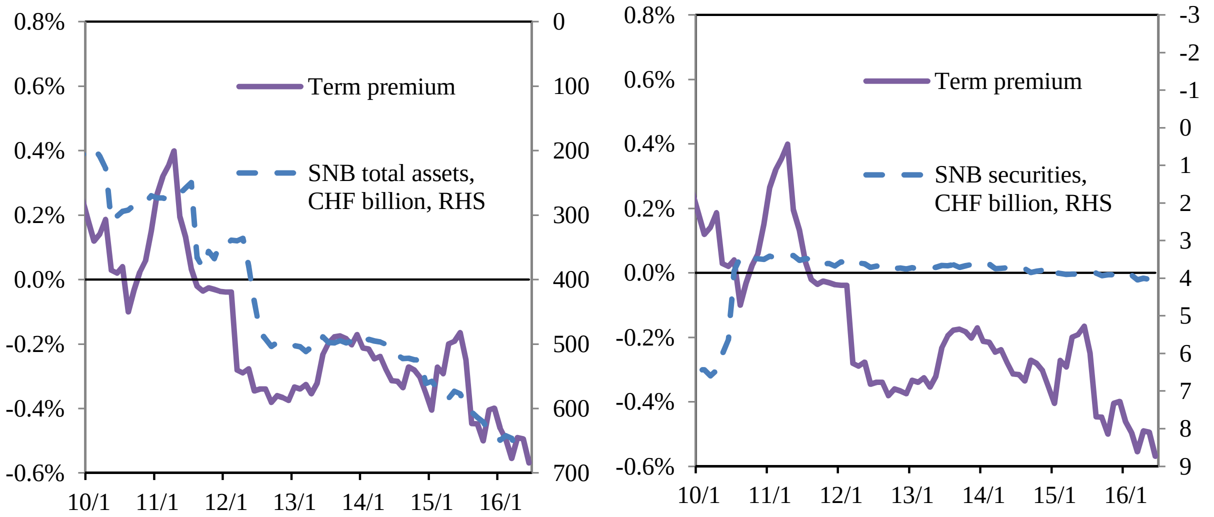


$E[i_t]$	$y = 0\%$	$y = -1\%$	Diff.
$\mathrm{E}[s_t] = -1\%$	0.40%	-0.20%	-0.60%P
$\mathrm{E}[s_t] = -3\%$	0.10%	-0.85%	-0.95%P

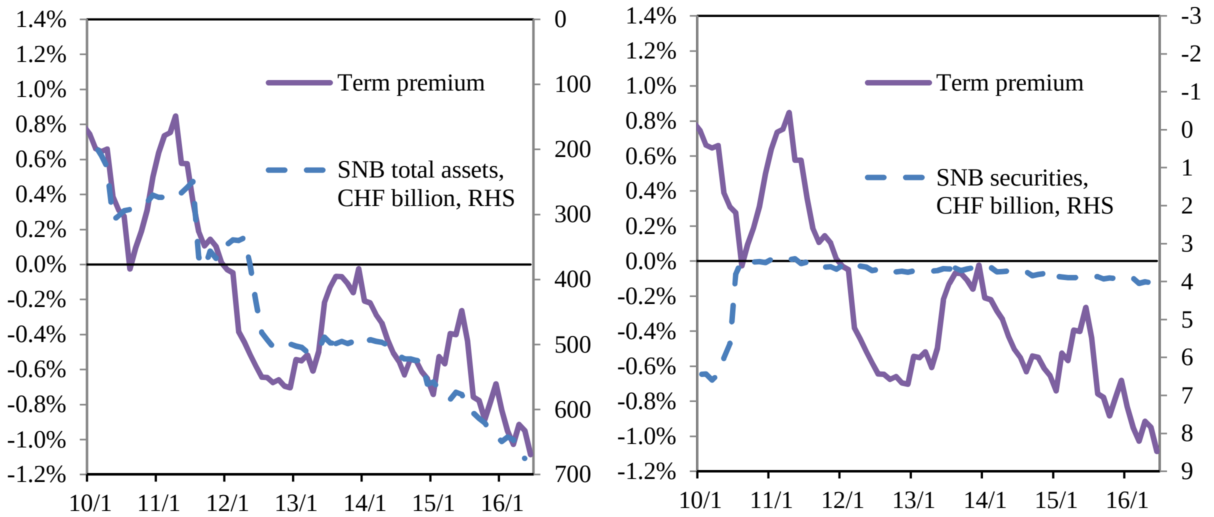
Figure 13: Relationship between quantitative easing and yield term premia

Switzerland

2Y



5Y



10Y

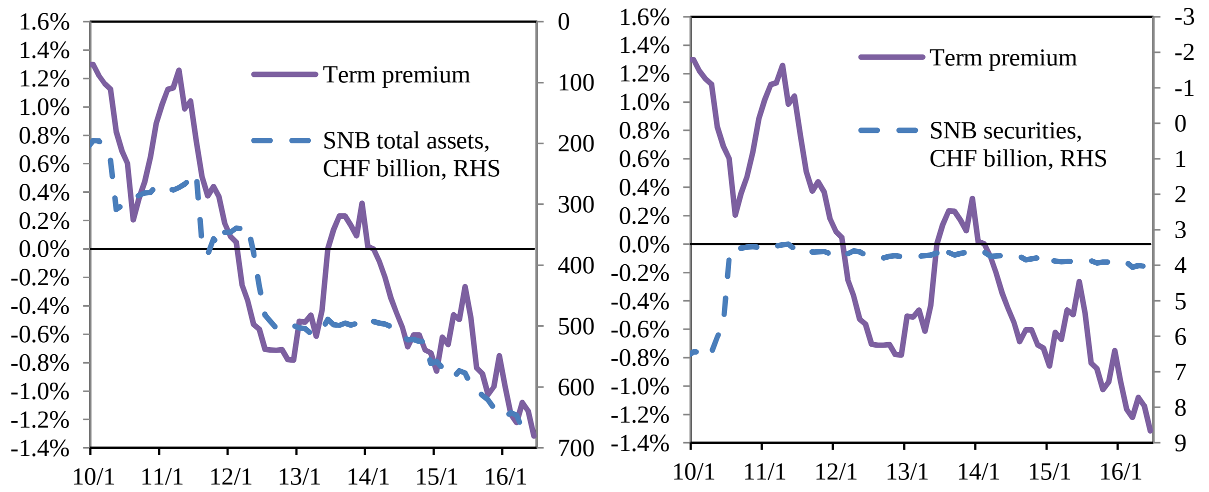
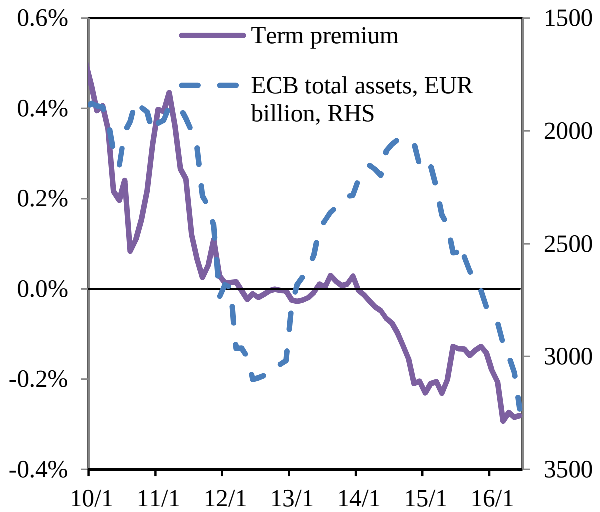
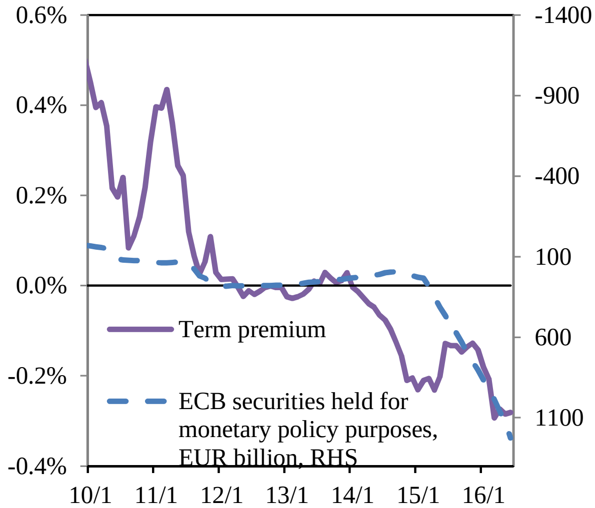


Figure 14: Relationship between quantitative easing and yield term premia

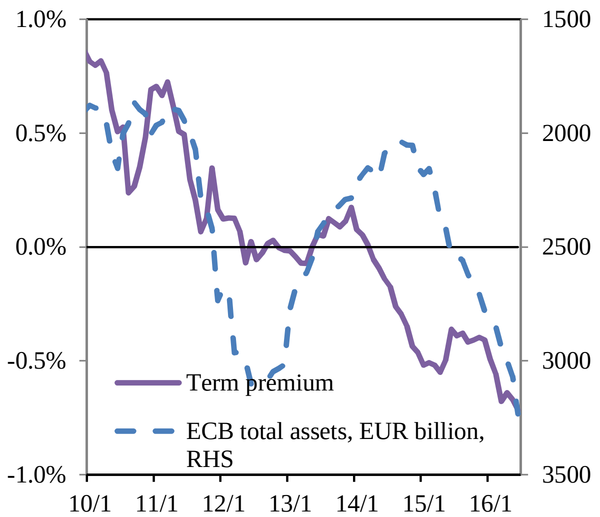
Germany

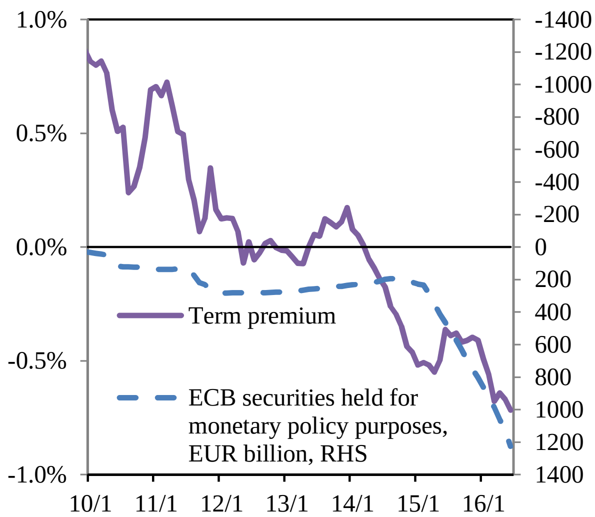
2Y



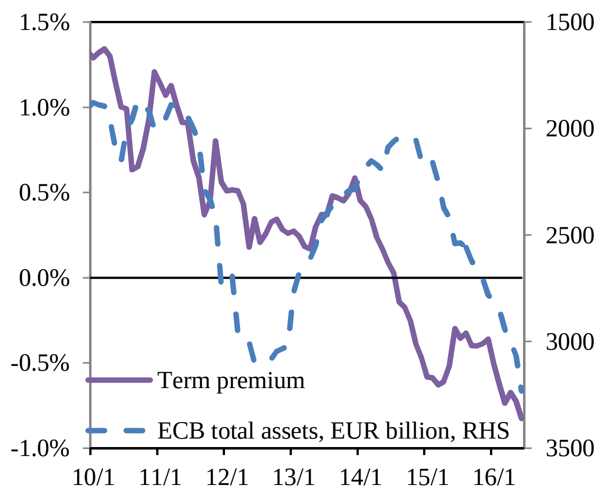


5Y





10Y



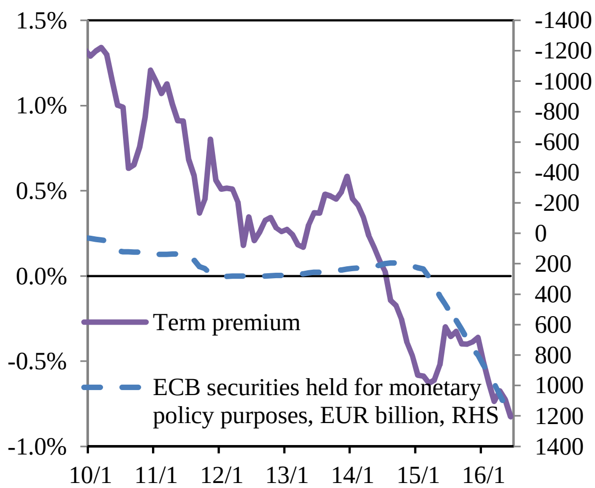
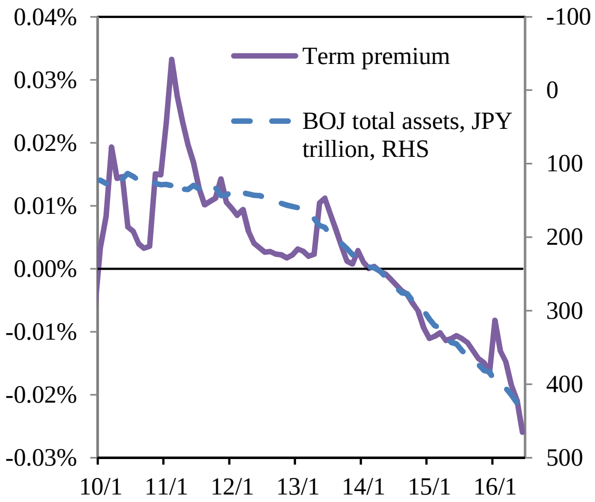
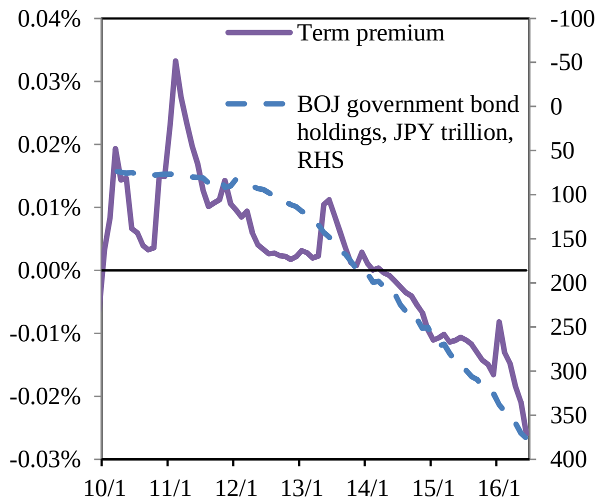


Figure 15: Relationship between quantitative easing and yield term premia

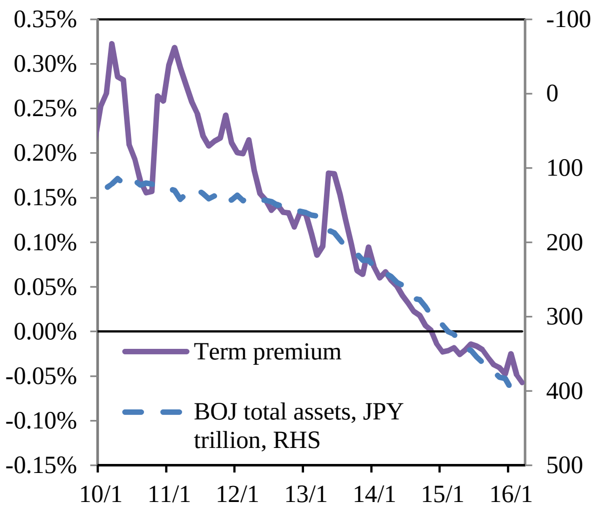
Japan

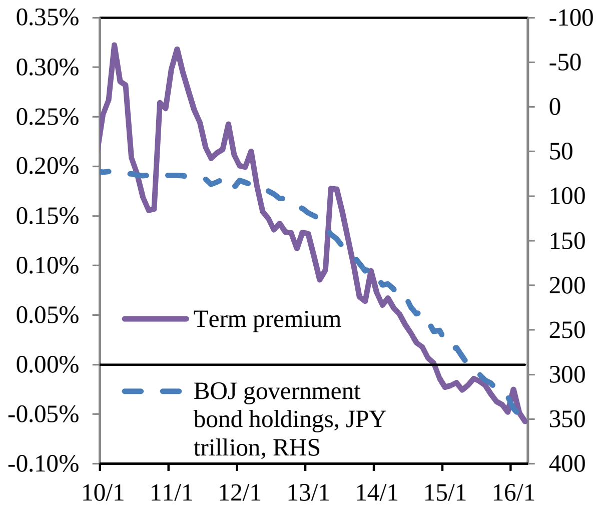
2Y



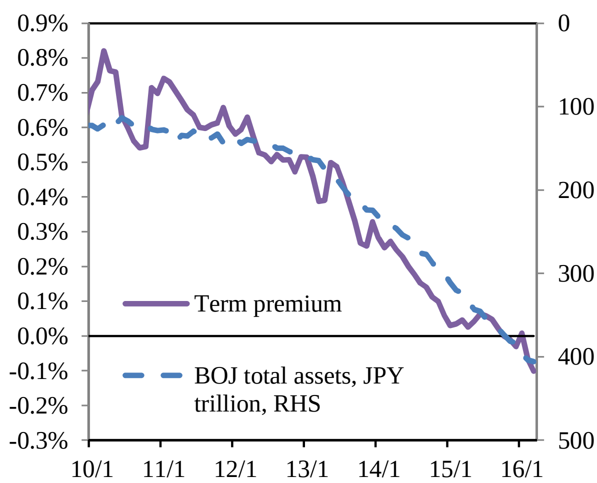


5Y





10Y



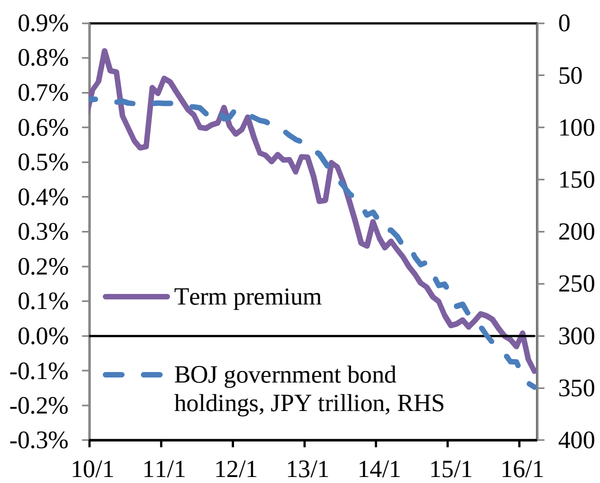


Figure 16: Decomposition of yield curves: Switzerland

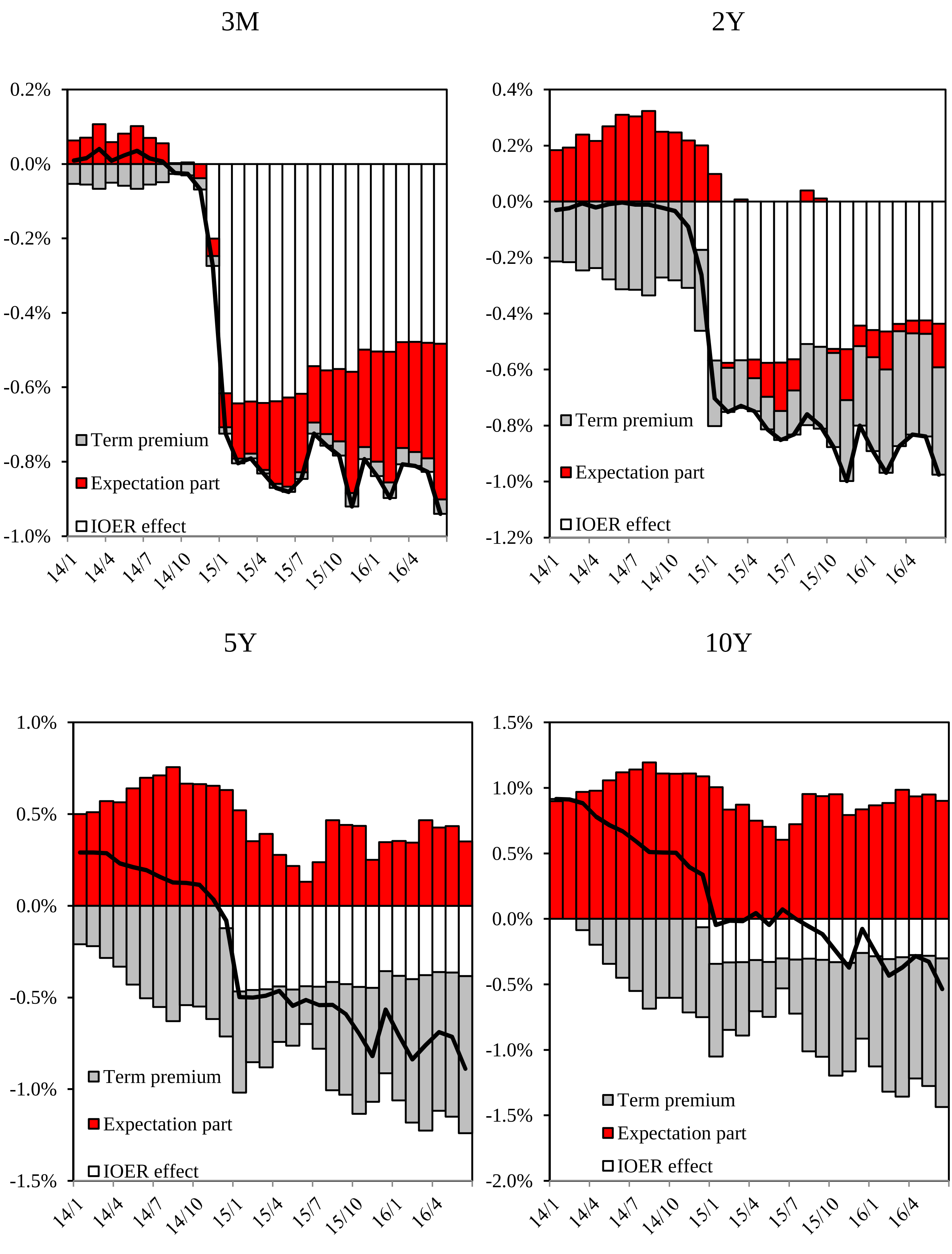


Figure 17: Decomposition of yield curves: Germany

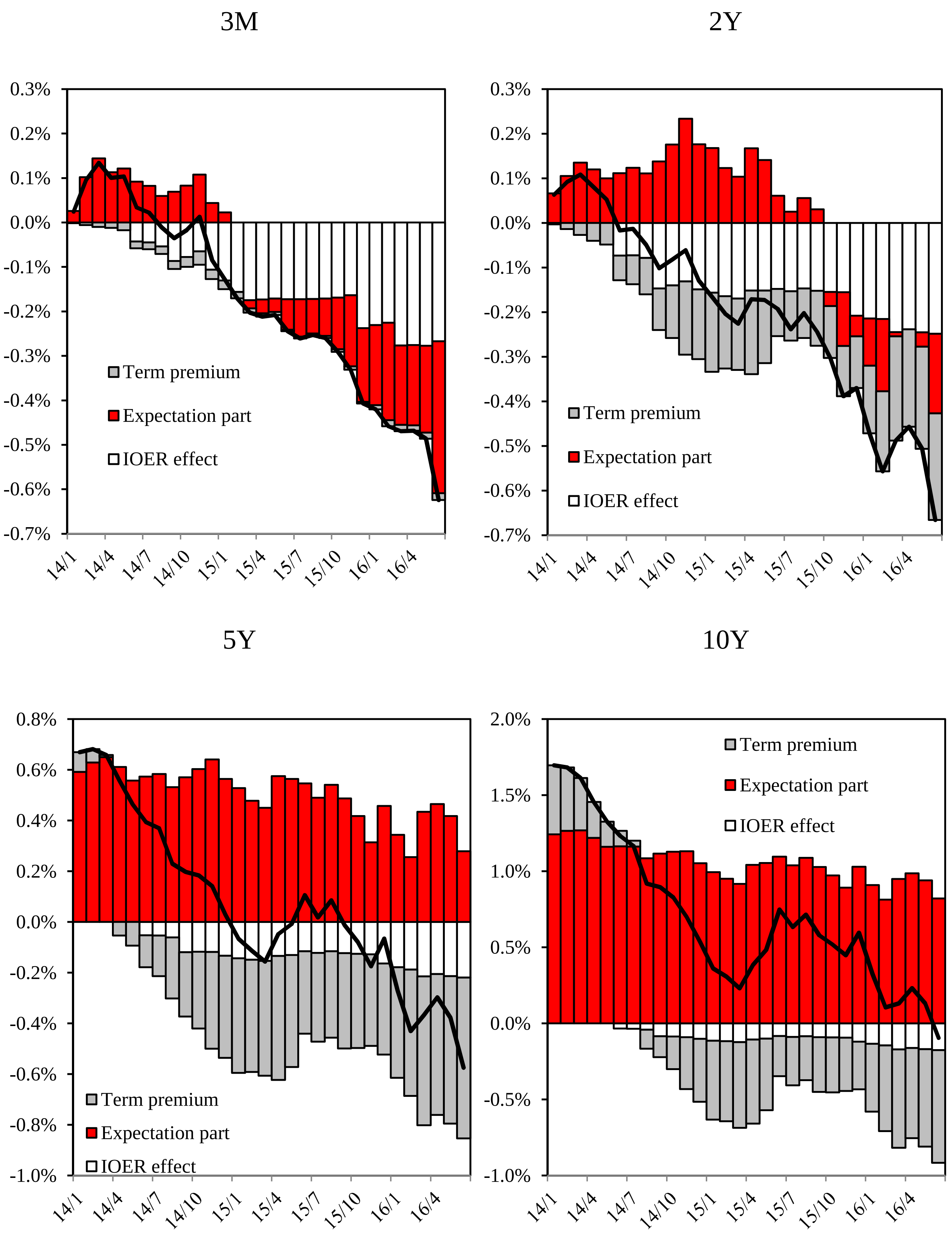


Figure 18: Decomposition of yield curves: Japan

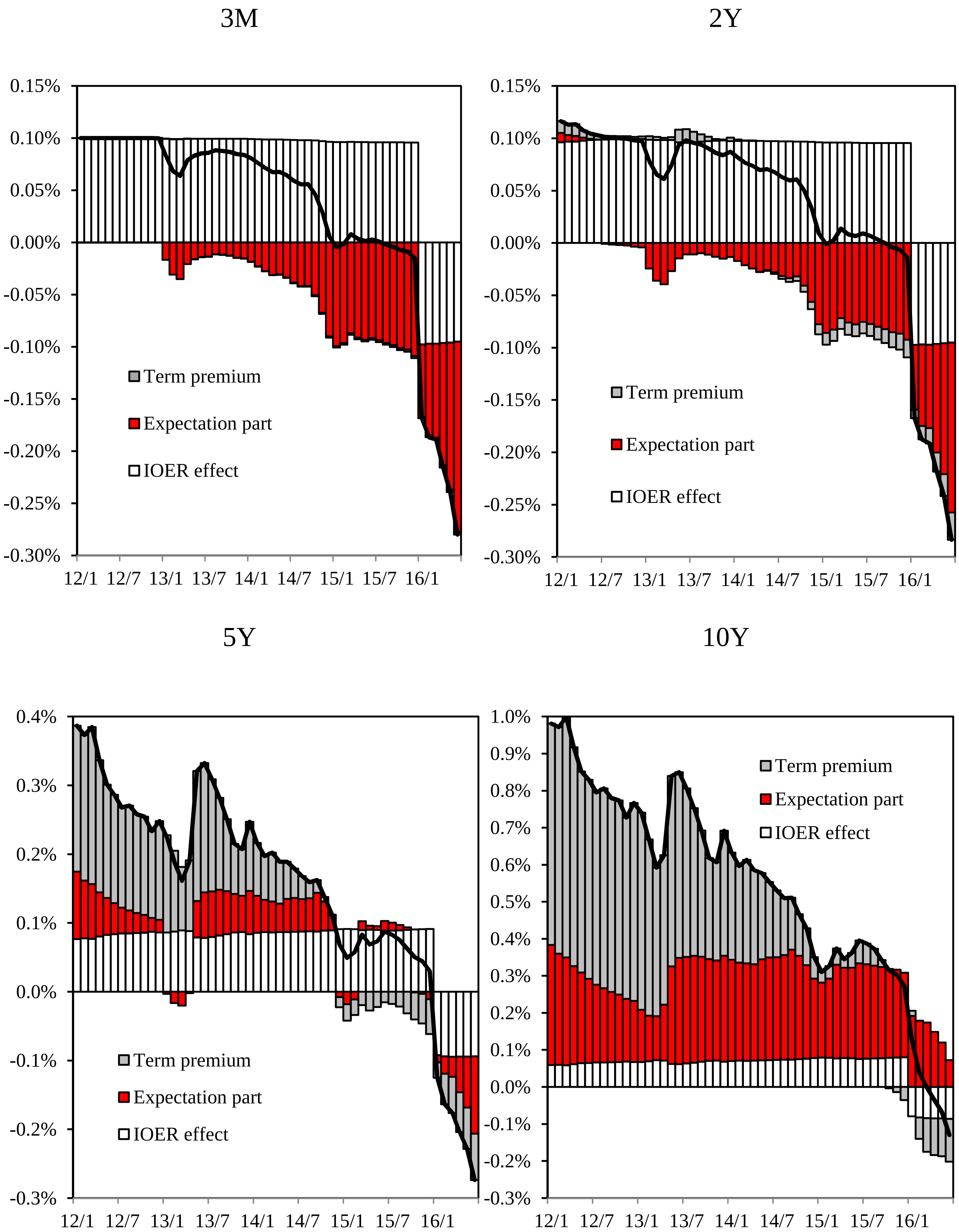
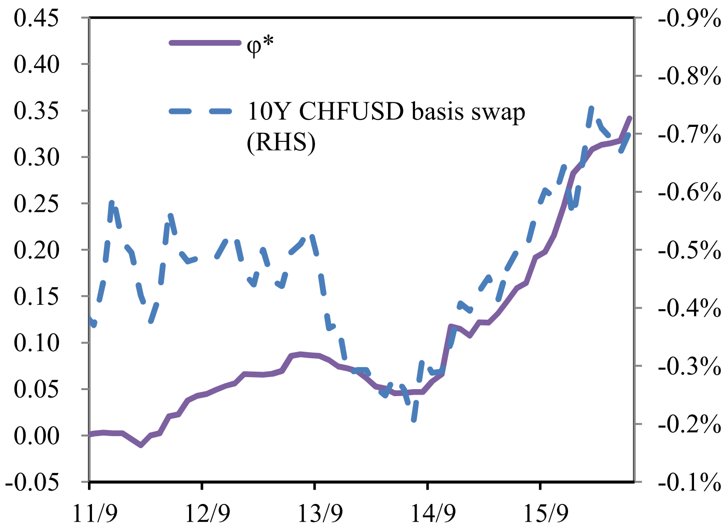
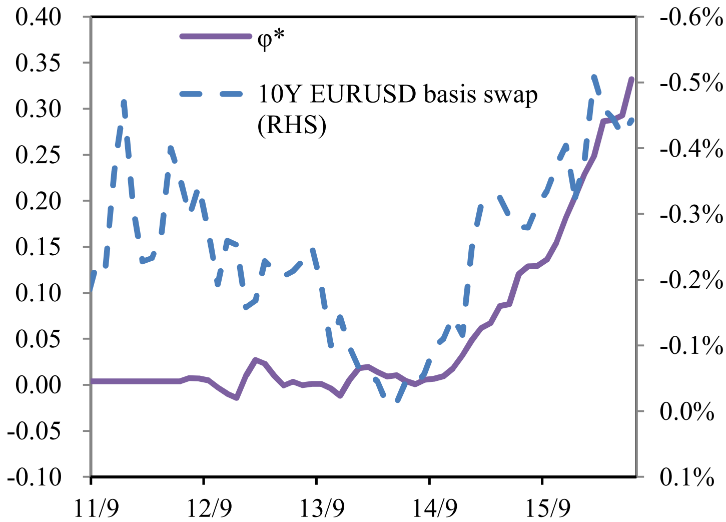


Figure 19: Basis swap and the power of arbitrage

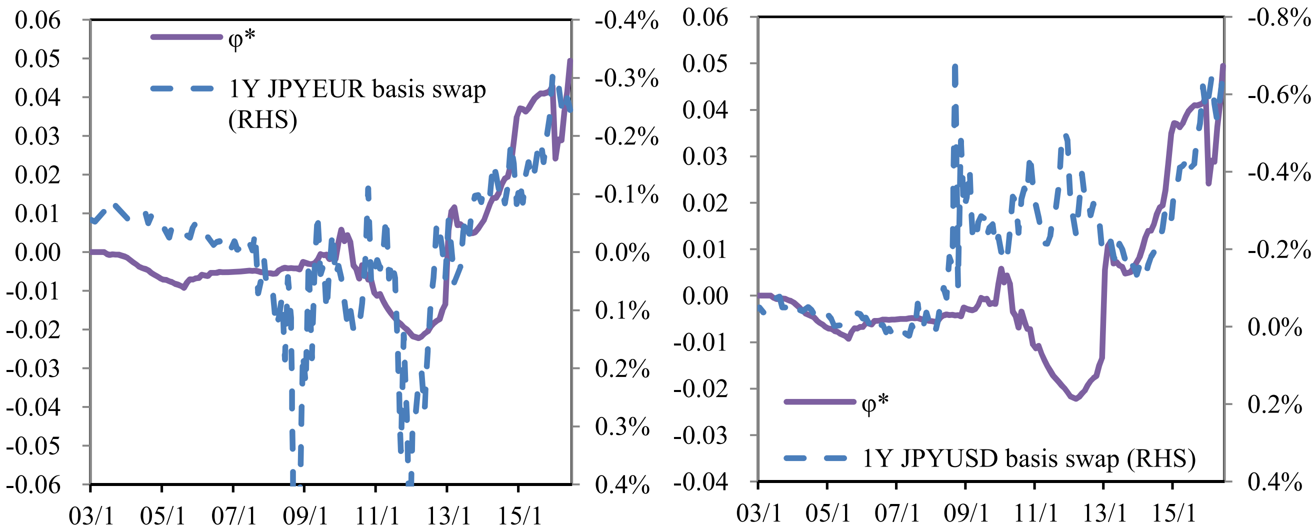
Switzerland



Germany



Japan



Source: Bloomberg

Table 1: Exact and approximated bond prices

Shadow rate		Maturity			
		1Y	5Y	10Y	30Y
1%	Exat	0.98829	0.92449	0.84104	0.58363
	Priebsch (2013)	0.98829	0.92456	0.84178	0.59507
	Rate of deviation	0.00050%	0.00723%	0.08848%	1.95929%
	Difference bps	0.050	0.145	0.884	6.468
	This paper	0.98829	0.92449	0.84101	0.58294
	Rate of deviation	0.00049%	-0.00011%	-0.00369%	-0.11872%
	Difference bps	0.049	-0.002	-0.037	-0.396
0%	Exact	0.99463	0.94622	0.87124	0.61258
	Priebsch (2013)	0.99463	0.94628	0.87192	0.62391
	Rate of deviation	-0.00050%	0.00612%	0.07800%	1.84886%
	Difference bps	-0.050	0.122	0.780	6.107
	This paper	0.99462	0.94622	0.87122	0.61189
	Rate of deviation	-0.00051%	0.00007%	-0.00234%	-0.11223%
	Difference bps	-0.051	0.001	-0.023	-0.374

Note: One-factor model, $y = 0$, $\theta = 0.01, \kappa = 0.1, \sigma = 0.2, \phi = 0$.

Table 2 : Estimated parameters: Switzerland

	(Row,column)	Gaussian affine	Black	Fixed extended	Variable extended
κ_x^Q	(1,1)	0.01328	0.01427	0.01364	0.01345
	(2,2)	0.82527	0.84533	0.87102	0.85500
	(3,3)	0.82527	0.84533	0.87102	0.85500
	(2,2)	1.00000	1.00000	1.00000	1.00000
σ_x	(1,1)	0.00884	0.00905	0.00945	0.00889
	(2,1)	-0.00470	-0.00472	-0.00476	-0.00470
	(3,1)	0.01448	0.01481	0.01469	0.01463
	(2,2)	0.01146	0.01167	0.01164	0.01163
	(3,2)	0.00729	0.00706	0.00736	0.00736
	(3,3)	0.02020	0.02068	0.02052	0.02043
$\kappa_x^Q \theta_x^Q$	(1,1)	0.00238	0.00235	0.00226	0.00233
$U = \left(I - e^{-\kappa_x^P/12}\right)\theta_x^P$	(1,1)	0.00009	0.00009	0.00009	0.00009
	(2,1)	0.00024	0.00009	0.00008	0.00008
	(3,1)	0.00017	0.00018	0.00018	0.00018
κ_x^P	(1,1)	0.00023	0.00023	0.00023	0.00023
	(2,1)	0.04567	0.04652	0.04599	0.04622
	(3,1)	-0.45438	-0.43784	-0.45817	-0.45813
	(1,2)	-0.08351	-0.08515	-0.08613	-0.08412
	(2,2)	0.28201	0.26066	0.26038	0.26745
	(3,2)	0.72248	0.73633	0.76155	0.72918
	(1,3)	0.49239	0.50009	0.50301	0.49690
	(2,3)	0.13674	0.13755	0.13836	0.13713
	(3,3)	2.42883	2.42427	2.56704	2.43649
σ_e	(1,1)	0.00085	0.00086	0.00085	0.00084
σ_y	(1,1)	0.00456	0.00456	0.00456	0.00456
φ	(1,1)	1.00000	0.00000	0.05547	-0.00001
σ_φ	(1,1)	0.00000	0.00000	0.00000	0.05294
Average of log likelihood		41.46989	41.54190	41.84379	41.93576
P-value (vs. fixed extended)		0.00000	0.00000	—	—
BIC		-27288.01	-27335.53	-27531.36	-27588.64
T		330	330	330	330

Table 3 ： Estimated parameters: Germany

	(Row,column)	Gaussian affine	Black	Fixed extended	Variable extended
κ_x^Q	(1,1)	0.06034	0.06136	0.05828	0.05838
	(2,2)	0.67333	0.68030	0.67638	0.67487
	(3,3)	0.67333	0.68030	0.67638	0.67487
	(2,2)	1.00000	1.00000	1.00000	1.00000
σ_x	(1,1)	0.01354	0.01369	0.01357	0.01363
	(2,1)	-0.01259	-0.01268	-0.01266	-0.01274
	(3,1)	0.01044	0.00997	0.01049	0.01047
	(2,2)	0.00660	0.00668	0.00682	0.00670
	(3,2)	0.00432	0.00436	0.00438	0.00440
	(3,3)	0.01671	0.01699	0.01671	0.01737
$\kappa_x^Q \theta_x^Q$	(1,1)	0.00419	0.00413	0.00404	0.00404
$U = \left(I - e^{-\kappa_x^P/12}\right)\theta_x^P$	(1,1)	-0.00022	-0.00022	-0.00022	-0.00022
	(2,1)	0.00037	0.00037	0.00036	0.00036
	(3,1)	-0.00092	-0.00092	-0.00087	-0.00087
κ_x^P	(1,1)	-0.12336	-0.11853	-0.12277	-0.11970
	(2,1)	0.09179	0.09330	0.09496	0.09549
	(3,1)	-0.48265	-0.48697	-0.48315	-0.48579
	(1,2)	-0.24473	-0.24791	-0.24595	-0.24720
	(2,2)	0.58976	0.58801	0.56837	0.57091
	(3,2)	-0.35814	-0.35893	-0.36137	-0.36338
	(1,3)	0.28592	0.28802	0.28757	0.28705
	(2,3)	0.55794	0.56737	0.56986	0.56903
	(3,3)	1.06006	1.07467	1.06730	1.06773
σ_e	(1,1)	0.00068	0.00069	0.00067	0.00067
σ_y	(1,1)	0.00141	0.00141	0.00141	0.00141
φ	(1,1)	1.00000	0.00000	0.09992	0.00390
σ_φ	(1,1)	0.00000	0.00000	0.00000	0.05112
Average of log likelihood		42.88396	42.79203	43.21875	43.31568
P-value (vs. fixed extended)		0.00000	0.00000	—	—
BIC		-25392.05	-25337.45	-25587.54	-25641.74
T		297	297	297	297

Table 4： Estimated parameters: Japan

	(Row,column)	Gaussian affine	Black	Fixed extended	Variable extended
κ_x^Q	(1,1)	0.07662	0.07330	0.07461	0.07424
	(2,2)	0.53454	0.54998	0.53552	0.54772
	(3,3)	0.53454	0.54998	0.53552	0.54772
	(2,2)	1.00000	1.00000	1.00000	1.00000
σ_x	(1,1)	0.01719	0.01873	0.01869	0.01891
	(2,1)	-0.01456	-0.01421	-0.01437	-0.01412
	(3,1)	0.01052	0.01019	0.01019	0.01009
	(2,2)	0.00601	0.00633	0.00643	0.00635
	(3,2)	0.00193	0.00203	0.00205	0.00204
	(3,3)	0.01105	0.01156	0.01172	0.01168
$\kappa_x^Q \theta_x^Q$	(1,1)	0.00341	0.00269	0.00269	0.00263
$U = \left(I - e^{-\kappa_x^P/12}\right)\theta_x^P$	(1,1)	0.00080	0.00078	0.00078	0.00079
	(2,1)	-0.00017	-0.00016	-0.00016	-0.00016
	(3,1)	0.00112	0.00108	0.00108	0.00108
κ_x^P	(1,1)	-0.26649	-0.28366	-0.29479	-0.29371
	(2,1)	0.24541	0.24352	0.24165	0.24331
	(3,1)	-0.28897	-0.28776	-0.29310	-0.29068
	(1,2)	-0.14998	-0.14973	-0.15079	-0.15113
	(2,2)	0.56251	0.59253	0.59719	0.60026
	(3,2)	0.71080	0.70604	0.70223	0.70773
	(1,3)	1.32323	1.31862	1.30726	1.33745
	(2,3)	-0.00436	-0.00441	-0.00448	-0.00440
	(3,3)	2.73702	2.74302	2.70342	2.74048
σ_e	(1,1)	0.00096	0.00092	0.00090	0.00091
σ_y	(1,1)	0.00082	0.00082	0.00082	0.00082
φ	(1,1)	1.00000	0.00000	0.01533	0.00000
σ_φ	(1,1)	0.00000	0.00000	0.00000	0.01639
Average of log likelihood		41.41125	42.40204	42.64893	42.77034
P-value (vs. fixed extended)		0.00000	0.00000	—	—
BIC		-27000.93	-27648.91	-27806.96	-27882.95
T		327	327	327	327

Table 5: Fitting results (RMSEs) : Switzrland

All ovservations

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	bps Average
Gaussian affine	7.86	7.43	11.33	5.77	6.88	6.27	4.49	5.75	6.97
Black	8.17	7.74	11.52	6.68	7.63	6.65	4.80	5.87	7.38
Fixed extended	7.67	7.71	11.08	6.01	7.33	6.79	5.04	6.30	7.24
Variable extended	7.57	7.34	11.14	6.00	7.16	6.35	4.53	5.80	6.99

Negative interest rate period (3-month rate is negative)

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	bps Average
Gaussian affine	2.78	2.85	2.14	1.49	2.09	2.24	2.01	2.22	2.23
Black	11.37	8.78	11.35	12.46	11.85	9.54	6.50	4.87	9.59
Fixed extended	4.82	5.48	5.01	5.90	6.76	6.51	4.56	3.71	5.34
Variable extended	3.15	3.18	2.57	2.84	3.65	3.70	2.90	2.76	3.10

Zero interest rate period (3-month rate is non-negative and less than 0.25%)

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	bps Average
Gaussian affine	3.78	2.76	6.42	4.63	5.40	7.50	3.46	5.61	4.95
Black	3.45	3.92	5.41	4.31	5.76	7.35	4.37	5.11	4.96
Fixed extended	2.85	3.31	5.54	4.42	6.04	8.05	4.07	5.48	4.97
Variable extended	3.00	3.38	5.56	4.36	5.82	7.74	4.14	5.28	4.91

Positive interest rate period (3-month rate is greater than 0.25%)

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	bps Average
Gaussian affine	9.07	8.68	12.98	6.33	7.56	6.06	4.93	6.02	7.70
Black	8.91	8.56	12.94	6.58	7.69	6.06	4.76	6.18	7.71
Fixed extended	8.88	8.85	12.73	6.47	7.75	6.35	5.36	6.73	7.89
Variable extended	8.81	8.47	12.87	6.65	7.78	6.02	4.77	6.16	7.69

Table 6: Fitting results (RMSEs): Germany

All ovservations

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	5.19	5.59	8.79	5.72	6.66	6.82	6.87	6.20	6.48
Black	5.43	5.69	8.76	5.79	6.21	6.01	5.99	5.01	6.11
Fixed extended	4.37	5.33	8.36	4.92	5.71	6.14	6.81	5.55	5.90
Variable extended	4.78	5.50	8.51	4.57	5.35	5.42	6.03	5.14	5.66

Negative interest rate period (3-month rate is negative)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	2.81	3.14	1.59	1.92	1.31	3.48	3.84	5.12	2.90
Black	12.30	7.18	10.05	11.43	10.78	7.23	5.14	4.62	8.59
Fixed extended	5.28	3.72	3.31	3.33	3.41	2.92	2.82	3.47	3.53
Variable extended	4.36	3.23	2.24	1.78	2.04	2.25	2.67	3.42	2.75

Zero interest rate period (3-month rate is non-negative and less than 0.25%)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	4.38	2.23	7.04	3.46	5.62	6.23	3.85	6.89	4.96
Black	2.75	3.42	5.59	4.66	5.43	3.77	3.73	4.07	4.18
Fixed extended	2.34	2.94	5.83	3.42	5.35	5.21	4.35	5.07	4.31
Variable extended	2.53	3.09	5.29	3.12	4.91	4.10	3.81	4.48	3.92

Positive interest rate period (3-month rate is greater than 0.25%)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	5.49	6.14	9.44	6.24	7.11	7.15	7.46	6.17	6.90
Black	4.60	5.82	9.05	5.10	5.69	6.18	6.35	5.19	6.00
Fixed extended	4.53	5.76	9.04	5.25	5.94	6.51	7.39	5.78	6.28
Variable extended	5.09	5.97	9.29	4.95	5.64	5.82	6.54	5.37	6.08

Table 7 : Fitting results (RMSEs): Japan

All ovservations

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	3.99	3.15	7.50	6.49	6.50	7.48	12.63	10.74	7.31
Black	4.95	4.50	8.13	6.95	7.33	7.51	12.39	9.66	7.68
Fixed extended	4.22	3.72	7.71	6.73	7.04	7.19	12.25	9.66	7.32
Variable extended	3.96	3.46	7.62	6.26	6.83	7.08	12.13	9.57	7.11

Negative interest rate period (3-month rate is negative)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	0.56	1.10	2.08	1.08	1.53	3.20	1.11	4.57	1.90
Black	12.16	11.97	12.04	11.66	10.79	9.24	8.27	6.87	10.37
Fixed extended	7.01	6.76	6.54	6.18	5.63	5.03	5.14	5.15	5.93
Variable extended	3.91	3.62	2.97	2.78	2.72	2.64	3.44	4.59	3.33

Zero interest rate period (3-month rate is non-negative and less than 0.25%)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	3.08	2.19	4.16	3.47	3.31	5.66	5.61	6.86	4.29
Black	2.77	3.02	3.34	3.28	4.03	4.59	4.98	5.88	3.99
Fixed extended	2.64	3.14	3.49	3.15	3.77	4.60	5.06	5.75	3.95
Variable extended	2.28	2.71	3.14	3.13	4.00	4.53	4.95	5.95	3.84

Positive interest rate period (3-month rate is greater than 0.25%)

bps

	3M	6M	1Y	2Y	3Y	5Y	7Y	10Y	Average
Gaussian affine	5.01	4.09	10.30	9.00	9.07	9.41	17.96	14.26	9.89
Black	5.11	3.97	10.77	8.81	9.30	9.57	17.53	12.83	9.74
Fixed extended	5.06	3.72	10.66	9.24	9.54	9.44	17.45	13.00	9.76
Variable extended	5.22	4.11	10.87	8.70	9.26	9.41	17.34	12.78	9.71