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Ryo Kato* and Takayuki Tsuruga**

Abstract

This paper develops a dynamic general equilibrium model that explicitly includes a banking sector with a maturity mismatch. We demonstrate that, despite the perfect competition in the banking sector, rational banks take on excessive risks systemically, resulting in overleverage and inefficiently high crisis probabilities. The model accounts for the banks' rational over-optimism regarding future capital prices which arises from pecuniary externalities on their own solvency. Using the model as an example, we introduce MSR (marginal systemic risk) as a general measure to assess the macroeconomic exposure to systemic risks.

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1 Introduction

Theories of banking and financial intermediation have come a long way, elaborating on the benefits and the inherent fragility of banking systems. In the meantime, macroeconomists have made a variety of attempts to incorporate realistic financial intermediaries — banks in particular— in dynamic general equilibrium models. A juxtaposition of the standard models of banking (Diamond and Dybvig 1983, Allen and Gale 1998, 2004) with a few dynamic macroeconomic models (Bernanke and Gertler 1989, Carlstrom and Fuerst 1997, Kiyotaki and Moore 1997) can clearly point to a yet-to-be filled gap between the micro-theory of banking and macroeconomics. The gap may be summarized as follows: Many macroeconomic models have successfully incorporated the complicated lender-borrower relationship and analyzed its consequences and implications for the macroeconomy, while, conceivably, few macroeconomic models have crystallized the roles and perils of banking systems in the light of a number of real-world experiences of a financial crisis. This potential deficiency in macroeconomics manifests itself when the banks typically modeled in micro-theories of banking as well as in the real world are compared to the over-simplified "banks" in macroeconomic models. Banks in the real world, as reasonably modeled by micro-theories, provide unique services for their customers. Banks typically raise funds via short-term liabilities and invest them, in part, in illiquid assets.[1] This notable business line undertaken by banks is widely acknowledged as a maturity mismatch or maturity transformation. On top of this, as noted by Allen and Gale (2007), micro-theories of banking have broadly emphasized other special elements in banking business, such as provision of liquidity insurance for depositors against liquidity (preference) shocks and inherent exposure to crisis risks. In contrast, however, these special elements in banks are given short shrift in macroeconomics.

This paper develops a dynamic general equilibrium model with a realistic banking sector to address basic yet unsolved questions: Does a banking sector with maturity mismatch affect macroeconomic fluctuations? If yes, how could it improve or undermine economic welfare? We integrate a model of banking developed by Diamond and Rajan (2001a, 2009) into an overlapping generations (OLG) model, where a liquidity shortage in banks with maturity mismatch precipitates inefficient financial crises that can result in a devastating decline in macroeconomic activity. As opposed to many existing models with "bubbles," for example, where a bubble bursts at a given (exogenous) probability, our model shows that, while not including asset price bubbles, the probability of a crisis varies depending on the leverage that banks determine based on their rational decisions. More broadly, we argue that, regardless of bubbles taking place or not, the banking sector in a laissez-faire economy generally cannot achieve the constrained optimum (a second-best allocation) and tends to be overleveraged as a result of perfect competition.

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[1]While we model banks that make loans funded from demand deposits, we do not necessarily limit our attention to commercial banks. Rather, the bank in our paper can broadly refer to financial intermediaries that raise funds via demandable claims such as repo and transform maturity in their asset.

We find as the main result, that a perfectly competitive banking sector tends to take on excessive systemic risks and gives rise to an inefficiently high crisis probability. The result may appear to go against the first theorem of welfare economics. To better understand the point, we introduce a new measure, marginal systemic risk (MSR) to assess the distortion as a fairly general concept. MSR is defined as the deviation from the socially optimal marginal increases in crisis probability vis-à-vis any target risk-taking variable. If an economy achieves the social optimum, MSR is zero in this economy, whereas an economy that takes excessive risks implies a positive MSR. In our model, MSR is defined against bank leverage and is proved to be positive, which arises from the overleveraged banking system. More specifically, MSR in our model depends on the marginal changes in the shadow values of banks' solvency constraints as lower solvency gives rise to a higher crisis probability. In general, the value of bank assets typically depends on asset prices. In our model, the banks' solvency constraint depends on expected capital prices. Because the constraint itself depends on capital prices, price-taking banks do not internalize the marginal changes in the shadow values of the constraint against their own decision on leverages. The pecuniary externality regarding the solvency constraint creates the positive MSR and the overleveraged banking system in our model.

More generally, the non-zero MSR in our model can be interpreted in line with the first welfare theorem: Maturity-mismatching banks normally provide non state-contingent debt, typically represented by demand deposits, rather than fully state-contingent bonds (i.e., Arrow securities), where otherwise bank runs need not be considered. Diamond and Rajan (2001a, 2001b) highlight the advantage of the banking business by demonstrating that such issuance of non state-contingent short-term debt streamlines financial intermediation and promotes liquidity creation. While their argument that banking business can improve welfare applies fairly generally, the financial intermediation by banks cannot fully compensate for the market incompleteness. Given the incompleteness, there is no guarantee that a competitive equilibrium can achieve the (constrained) social optimum (i.e., the second-best allocation). With an incomplete financial market, pecuniary externality can arise in a variety of forms. Because of such externalities, price-taking banks cannot strike the right balance between the cost and the benefit of a marginal increase in their own leverage. The inefficient allocations in our model appear to contrast with Allen and Gale (1998), in which the banking sector achieves the second-best allocation. Their model includes a single market where the banking sector, as the result of the perfect competition, internalizes all the effects through price changes. In contrast, our model includes more than one market (i.e., the capital and labor markets) in which banks take prices as given. The price-taking banks, given the incompleteness of the financial markets, give rise to a positive MSR, or equivalently, over-risk-taking in a manner similar to that studied by a number of recent works on pecuniary externalities (Lorenzoni 2008, Jeane and Korinek 2011, Korinek 2010, Bianchi 2010).

Because of the incomplete financial markets and the resulting pecuniary externalities, non-

zero MSR (or an excessive risk-taking, equivalently) arises not surprisingly. In the aforementioned models with pecuniary externalities, decentralized agents take the capital price (or other asset prices) as given, and such price-taking behavior distorts the efficient allocations as opposed to the cases presumed by the first welfare theorem. While our model shares positive MSRs with early works on pecuniary externalities that result in over-credit (e.g., Lorenzoni 2008), our perspective on financial crises varies remarkably from early studies. A notable difference is that our model explicitly includes the maturity-mismatching banks and financial crises are precipitated by insolvencies of the banking system triggered by a liquidity shortage. Furthermore, it could be noted that the market incompleteness in our model arises from the maturity mismatch in the banking sector itself rather than from any other extra friction or constraint (e.g., asymmetric information or a borrowing constraint).

The remaining question is why the competitive banking sector gives rise to a strictly positive MSR, or overleverage, rather than a negative MSR (i.e., underleverage). Suppose that a bank chooses to lever up to a higher level. While the bank can attract more creditors/depositors by increasing the leverage (i.e., by offering a higher return on their debt), it would be exposed to a higher risk of insolvency. Therefore, each bank needs to recognize their solvency on the brink of a default, to strike the right balance between the cost and the benefit of increasing their size of leverage. To this end, banks try to precisely assess the values of their own assets to estimate the marginal cost of increasing the leverage, taking the price of their illiquid assets as given. Notably, when the banks calculate the value of the illiquid assets, the pecuniary externalities are not factored in and this disregarded pecuniary externalities prompt the banks to be overleveraged. In general, highly leveraged banks hold large assets in an attempt to enhance their solvency. By holding more assets, they try to pursue higher returns to weather high liquidity shocks with high profitability. In general equilibrium, however, the banking sector as a whole overinvests owing to the externalities and this overinvestment in illiquid assets pares down the price of the illiquid assets. The decreased value of the illiquid assets undermines the banks' solvency more quickly on the brink of a systemic financial crisis compared to the case of a run on an individual bank. When individual banks determine their size of leverage, each of them takes future capital prices as given and perfect competition among banks does not allow them to offer a lower deposit rate even though the overleverage leads to an inefficiently high crisis probability. Put another way, the pecuniary externality ill-incentivizes banks to take on excessive risks *systemically*. In retrospect, the banks' decisions on their leverage always appear to be over-optimistic, accompanied by their over-optimistic outlook of future capital prices. Broadly speaking, banks correctly recognize their private cost and benefit of a high leverage while, from a viewpoint of the social planner, they always underestimate the marginal cost of raising leverage. This "rational over-optimism" entrenched in banking systems can pave the way to better understand banks' roles and incentives that have been observed repeatedly in preceding financial crises.

The rest of the paper proceeds as follows. Section 2 illustrates the macroeconomic model with maturity-mismatching banks. Using the model, we characterize the banks' optimal leverage and the competitive market equilibrium. In Section 3, we compare the laissez-faire equilibrium characterized in Section 2 with the allocation achieved by the social planning bankers. Meanwhile, Section 3 introduces MSR to elucidate why a competitive banking sector tends to be overleveraged. Section 4 discusses numerically the probability of financial crisis, the size of distortions, and the issue of crisis prevention along with the comparisons with a few empirical studies. Section 5 discusses relations with the existing models. Section 6 concludes.

2 A Macroeoconomy with Banks

2.1 Agents, Endowment, Preferences, and Technology

We consider an infinite-horizon OLG model incorporating banks with maturity mismatch. Each generation of agents consists of households, entrepreneurs, and bankers. Each period, generation t is born at the beginning of period t and lives for two periods t and $t+1$. Each agent is identical and constant in the population. Furthermore, an initial old generation lives for one period and the subsequent generations live for two periods.

Households are risk averse and subject to liquidity shock, which affects their preference of consumption over the two periods. The liquidity shock is an aggregate shock and the only source of the uncertainty in the model. The households aim to smooth out their consumption intertemporally. Following models in the theories of banking (e.g., Diamond and Rajan 2009), households are endowed with a unit of consumption goods at birth and do not consume the initially endowed consumption goods at the beginning of period t. The households deposit all initial endowments at banks operating in the same generation. They receive wages w_t in the competitive labor market by supplying one unit of labor in both periods, t and $t+1$.

Entrepreneurs are risk neutral and have access to capital producing technology. They launch long-term investment projects at the beginning of period t, by borrowing households' endowment via the bankers in the same generation. The investment project needs one period for gestation, and capital goods are produced in period $t+1$. We call this capital producing technology a "project." They sell the capital good in the competitive market for the capital good price q_{t+1}.

Banks raise funds from households and lend them to entrepreneurs at the beginning of period t.[2] In principle, we follow Diamond and Rajan (2001a) to model banks. Banks are risk neutral and competitive in raising and lending funds in the markets. They issue demand deposits (short-term debt) and commit to repaying households. As a nature of demand deposits, banks can provide insurance against the depositor's liquidity shocks. However, when households demand repayment

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[2]We assume intra-generational banking, which effectively means that all bankers die at the end of the second period.

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before the completion of the entrepreneur's projects, banks must liquidate premature projects to meet the demand for repayment. This maturity mismatch, represented by the combination of the long-term assets and the short-term liability, leaves banks exposed to risks of a default because, depending on the amount of withdrawals in the interim period, the banks' solvency is endangered.

The technology of consumption good production Y_t is represented by a standard constant-returns-to-scale Cobb-Douglas production function:

$$Y_t = F(K_t, H_t) = K_t^\alpha H_t^{1-\alpha},$$

where K_t and H_t denotes the capital stock and hours worked, respectively. Demands for labor and capital satisfy

$$w_t \quad = \quad F_{H,t} = (1-\alpha)\left(\frac{K_t}{H_t}\right)^\alpha \tag{1}$$

$$q_t \quad = \quad F_{K,t} = \alpha\left(\frac{K_t}{H_t}\right)^{\alpha-1}. \tag{2}$$

In what follows, we describe each agent's decisions (consumptions, withdrawal, and liquidation of entrepreneur's projects) *after* the liquidity shock is realized. Then, we move on to the bank's decision on the level of leverage *before* the realization of the liquidity shock. Table 1 summarizes the sequence of events in each generation.

2.2 Households

Under the perfectly competitive banking sector, each household accepts the banks' offer on deposit face value D_t at the beginning of period t, and observes the liquidity shock θ_t in the middle of period t. The liquidity shock is common across all households and has the probability density function $f(\theta_t)$ with a support of $[0,1]$. This shock represents households' preference for consumption when young and signals the need for liquidity in period t.[3]

After these events, households make their decisions for consumption smoothing. In particular,

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[3] Although in fact all households are subject to the same aggregate shock, we assume that an infinitesimally small number of households are believed to face a different θ_t from other households. This assumption ensures the existence of a Nash equilibrium, in which all households run to the banks when households believe that the banks are insolvent under the observed θ_t.

households then choose withdrawal g_t to maximize

$$
\begin{aligned}
U\left(C_{1,t}, C_{2,t+1}\right) &= \theta_t \log C_{1,t} + (1-\theta_t) \log C_{2,t+1}, \\
\text{s.t. } C_{1,t} &= \begin{cases} w_t + g_t \text{ with probability } 1-\pi_t \\ w_t + X \text{ with probability } \pi_t \end{cases} \\
C_{2,t+1} &= \begin{cases} w_{t+1} + R_t\left(D_t - g_t\right) \text{ with probability } 1-\pi_t \\ w_{t+1} \text{ with probability } \pi_t, \end{cases}
\end{aligned}
\tag{3}
$$

where $C_{1,t}$ and $C_{2,t+1}$ denote the consumption of households born in period t when young and old, respectively. Each household supplies a unit of labor every period and receives wage income w_t in period t and w_{t+1} in period $t+1$. Here R_t denotes the one-period gross interest rate from period t to $t+1$. In addition, π_t is the *ex post* probability of a financial crisis. With the probability $1 - \pi_t$, households withdraw g_t in period t and all the rest of deposits in period $t+1$.[4] With the probability π_t, a financial crisis arises and households' withdrawals amount to the liquidation value of premature projects, X (< 1), in period t and consumption is not smoothed out over the two periods.

When the households can smooth out their consumption, the intertemporal first-order condition for consumption is satisfied:

$$
\frac{\theta_t}{1-\theta_t}\left(\frac{C_{1,t}}{C_{2,t+1}}\right)^{-1} = R_t,
\tag{4}
$$

while the intertemporal budget constraint holds with equality $C_{1,t} + C_{2,t+1}/R_t = w_t + D_t + w_{t+1}/R_t \equiv m_t$, where m_t is the lifetime income for households. Given the Euler equation (4), the withdrawals in the absence of crisis can be written as

$$
g_t = \theta_t\left(\frac{w_{t+1}}{R_t} + D_t\right) - (1-\theta_t) w_t.
\tag{5}
$$

The withdrawal function implies that large θ_t and D_t are likely to precipitate a financial crisis.

2.3 Entrepreneurs

Entrepreneurs are risk neural and maximize their lifetime utility represented by $E\left(C_{1,t}^e + C_{2,t+1}^e\right)$, where $C_{1,t}^e$ and $C_{2,t+1}^e$ denote entrepreneurs' consumption when young and old. They use a unit of consumption goods financed from banks for their capital good production, and this production technology takes one period for gestation before its completion. In period $t+1$, the project yields a random capital good output $\tilde{\omega}$, which takes a value distributed over $[\omega_L, \omega_H]$ with the probability

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[4]In the maximization problem of households, we assume that wage income in period t is low relative to the initial endowment, ensuring a non-negative withdrawal g_t in the equilibrium. Furthermore, we assume the perishability of consumption goods, so any positive gross rate of interest rate can be an equilibrium interest rate.

density function $h(\omega)$.[5] If this project is prematurely liquidated in period t, the transformation from the consumption good into capital is incomplete. As a result, the output is reduced to X units of consumption goods and is repaid fully to banks in period t. When the project is completed in period $t+1$, however, entrepreneurs can sell their output in the capital good market for the price of q_{t+1}. After the repayment to banks, they are left with $1 - \gamma$ of the share of their profit and enjoy their own consumption based on their linear utility. We assume that entrepreneurs are endowed with $\underline{I}$ units of capital goods in period $t+1$.[6] They sell this endowment capital together with the newly created capital made from the consumption goods transferred from the households at the beginning of $t+1$.

2.4 Banks

Banks are also risk neutral and maximize their lifetime utility $E\left(C_{1,t}^b + C_{2,t+1}^b\right)$, where $C_{1,t}^b$ and $C_{2,t+1}^b$ denote consumption of banks when young and old. We borrow the microfoundation of the banking business from Diamond and Rajan (2001a, 2001b): banks issue demand deposits (short-term debt) as a commitment device to compensate for the lack of transferability of their collection skill and to promote liquidity creation. As discussed in a number of preceding works on banking (such as Diamond and Dybvig, 1983 and Allen and Gale, 1998), the banks determine D_t before observing the liquidity shock, which is realized in the middle of period t. We stress that D_t has a one-to-one relationship with bank leverage and, hence, we refer to D_t as leverage hereafter. We will discuss this issue later in this section and in Section 4 in terms of numerical interpretations.

Each bank has no initial endowment at birth but has a special skill to acquire knowledge about entrepreneurs' business. This skill enables a bank to act as the relationship lender that can develop the alternative, but less efficient, use of an entrepreneur's project. As discussed in Diamond and Rajan (2001a, 2001b, 2009), this knowledge enables the bank to acquire a fraction γ of the realized project's outcome in period $t+1$.[7]

We also follow the preceding works on the assumption for the distribution of entrepreneurs to which each bank makes loans. Each bank attracts many entrepreneurs through the competitive offer on the loan, resulting in the identical portfolio shared by all the symmetric banks. This setup effectively leads to a convenient outcome in the model: each bank and the aggregate economy face an identical distribution of entrepreneurs. In period t, the banks receive signals ω that perfectly predict the realized value of $\tilde{\omega}$ in period $t+1$. With this information ω and the households' liquidity demands observed in period t, each bank chooses one of the options: (i) to liquidate projects in

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[5] Following the literature, we take the assumption that there is no aggregate uncertainty in the project outcome.

[6] For simplicity, we assume a 100 percent depreciation rate in the law of motion for capital. The introduction of the endowment of capital goods here guarantees a finite capital price in the case of a financial crisis in which all projects are scrapped due to full liquidation.

[7] We also assume that once a bank has made a loan to entrepreneurs, no other lenders can replicate the collection skill.

period t, obtaining X of consumption goods per project; or (ii) to collect a fraction $\gamma q_{t+1}\omega$ from a completed project in period $t+1$. The bank liquidates the project if the outcome of a project falls short of $\omega\left(R_t/q_{t+1}\right)$, defined as a function of R_t/q_{t+1}:[8]

$$\omega\left(\frac{R_t}{q_{t+1}}\right) = \frac{X}{\gamma}\frac{R_t}{q_{t+1}}. \tag{6}$$

Otherwise, the bank continues the project, and then receives repayment of $\gamma q_{t+1}\omega$, — and entrepreneurs consume the remaining fraction of outcome, $(1-\gamma)\,q_{t+1}\omega$, — per project. After repaying the full amount of the households' withdrawals, the banks consume their own capital, denoted as $C_{1,t}^b + C_{2,t+1}^b$.

Let the bank's asset be $A\left(R_t/q_{t+1}\right)$. The bank's asset at the beginning of period t (i.e., prior to the withdrawals) can be expressed as

$$\begin{aligned} A\left(\frac{R_t}{q_{t+1}}\right) &= \int_{\omega_L}^{\omega_{t+1}^*} Xh\left(\omega\right)d\omega + \frac{\gamma q_{t+1}}{R_t}\int_{\omega_{t+1}^*}^{\omega_H} \omega h\left(\omega\right)d\omega \\ &= L\left(\frac{R_t}{q_{t+1}}\right) + \frac{\gamma q_{t+1}}{R_t}I\left(\frac{R_t}{q_{t+1}}\right). \end{aligned} \tag{7}$$

The bank asset denoted in (7) can be decomposed into two components: the values of the prematurely liquidated projects denoted as $L_t = L\left(R_t/q_{t+1}\right) \equiv \int_{\omega_L}^{\omega_{t+1}^*} Xh\left(\omega\right)d\omega$, which is used to meet the liquidity demand (i.e., withdrawals) from the households, and the banks' share of the investment output (measured in the present value of consumption goods) denoted as $\gamma q_{t+1}I_t/R_t$, where I_t is given by $I_t = I\left(R_t/q_{t+1}\right) = \int_{\omega_{t+1}^*}^{\omega_H} \omega h\left(\omega\right)d\omega$.

The banks are subject to the solvency constraint $D_t \leq A\left(R_t/q_{t+1}\right)$. Because banks' assets decrease when they are discounted by a high R_t and advance on high expected capital prices q_{t+1}, it can be easily shown that $A\left(\cdot\right)$ monotonically decreases with R_t/q_{t+1}. We can then define the relative price R_t^*/q_{t+1}^* which satisfies the solvency constraint with equality

$$D_t = A\left(R_t^*/q_{t+1}^*\right). \tag{8}$$

We refer to R_t^* and q_{t+1}^* as the *threshold* interest rate and capital price, respectively. Hereafter, we denote a variable with an asterisk as the variable on the threshold. Similarly, we denote a function with an asterisk as the function evaluated at the threshold variable. For the purpose of subsequent discussion, we note that given $A\left(R_t/q_{t+1}\right)$, the *ex post* bank capital and bank leverage are denoted as $A_t - D_t$ and $D_t/(A_t - D_t)$, respectively, both of which are determined uniquely once D_t is chosen.

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[8]Equation (6) can be reinterpreted as follows: $\gamma q_{t+1}\omega\left(R_t/q_{t+1}\right)/X$ corresponds to the marginal rate of transformation (MRT) between the period t consumption goods (i.e., liquidation) and the period $t+1$ consumption goods (i.e., continuation of projects). The MRT is here compared with the marginal rate of substitution of the households that is observed as the interest rate, R_t.

2.5 Market Clearing Conditions

Four markets need to clear in the competitive equilibrium: (i) Liquidity; (ii) consumption goods; (iii) capital goods; and (iv) labor. The liquidity market clearing condition is given by

$$L\left(\frac{R_t}{q_{t+1}}\right) = \theta_t\left(\frac{w_{t+1}}{R_t} + D_t\right) - (1 - \theta_t)\,w_t. \tag{9}$$

Next, the market clearing condition for consumption goods is

$$Y_t + L\left(\frac{R_t}{q_{t+1}}\right) = C_{1,t} + C_{2,t} + C_{2,t}^e + C_{2,t}^b. \tag{10}$$

The left-hand side of (10) includes the supply of goods from the liquidity market. This is because consumption of households in period t consists of wage income and withdrawal. On the right-hand side of (10), $C_{2,t}$, $C_{2,t}^e$, and $C_{2,t}^b$ denote consumption when generation $t-1$ is the old.

The capital good market clearing condition is

$$K_{t+1} = \begin{cases} \underline{I} + I\left(R_t/q_{t+1}\right) & \text{with probability } 1 - \pi_t \\ \underline{I} & \text{with probability } \pi_t. \end{cases} \tag{11}$$

Here the equation suggests that the capital good supply sharply declines, conditional on a crisis. Throughout the paper, we use $\underline{w}$ and $\underline{F}_H$ to denote the wage rate and the marginal product of labor evaluated at $K_{t+1} = \underline{I}$.

Finally, both young and old generations supply a unit of labor each period. Therefore, H_t equals two for all t.

2.6 Optimal Bank Leverage

We now consider the banks' optimal choice for the leverage. The banks choose the size of their leverage before the realization of the liquidity shock. We focus on the laissez-faire banks in this subsection, and will discuss the social planning banks in Section 3.

The banks are competitive in issuing demand deposits. and we assume that households' endowments are scarce in comparison to entrepreneurs' projects. As a result of perfect competition, the banks make a competitive offer of deposits for households, aiming at maximizing the household welfare (Allen and Gale 1998, 2007), while in fact they are maximizing their own profit (linear utility). Maximizing the household utility via the deposit offers means that banks internalize the liquidity market clearing condition in determining the offer. Through this internalization, the banks take into account possible changes in the crisis probability π_t, which appears in (3) and (11). On the other hand, outside the liquidity market, they take the capital prices and wages as given.

To understand how the banks' choice on D_t affects π_t, we define the threshold level of the

liquidity shock θ_t^* that clears the liquidity market at the threshold interest rate and capital prices, R_t^* and q_{t+1}^*. The market clearing condition implies that θ_t^* is given by

$$\theta_t^* = \frac{L\left(R_t^*/q_{t+1}^*\right) + w_t}{w_t + D_t + w_{t+1}^*/R_t^*}. \tag{12}$$

The above equation means that given D_t, changes in R_t^* and q_{t+1}^* always give rise to changes in θ_t^* for the liquidity market to clear. By the solvency constraint (8), any level of D_t, once chosen, determines the threshold relative price, R_t^*/q_{t+1}^*. Hence, θ_t^* can be interpreted as the liquidity shock on the brink of a financial crisis. Namely, when θ_t is strictly greater than θ_t^*, the banks turn out to be insolvent and a crisis is precipitated. Thus, the crisis probability π_t has a one-to-one relationship to θ_t^* via the probability density function $f(\theta_t)$:

$$\pi_t = \int_{\theta_t^*}^1 f(\theta_t)\, d\theta_t. \tag{13}$$

In principle, the banks' choice of the leverage specifies R_t^*/q_{t+1}^* and this threshold relative price determines the threshold level of the liquidity shock θ_t^*, completing the link between the bank leverage and the crisis probability. We are now ready to set up the optimization problem for the banks to determine their size of the leverage. In the problem, as discussed, banks take into account the endogenously changing θ_t^*.

Problem LF   *In a laissez-faire economy, banks maximize the household expected utility*

$$\max_{D_t} \int_0^{\theta_t^*} \left\{\theta_t \ln\left(w_t + L_t\right) + (1 - \theta_t) \ln\left[w_{t+1} + R_t\left(D_t - L_t\right)\right]\right\} f(\theta_t)\, d\theta_t$$
$$+ \int_{\theta_t^*}^1 \left[\theta_t \ln\left(w_t + X\right) + (1 - \theta_t) \ln\left(\underline{w}\right)\right] f(\theta_t)\, d\theta_t,$$

subject to (8), (9) and (12).

The banks choose their leverage levels according to the following first-order condition:

$$\left[\theta_t^* \log\left(\frac{w_t + X}{\theta_t^* m_t^*}\right) + (1 - \theta_t^*) \log\left(\frac{\underline{w}}{R_t^*\left(1 - \theta_t^*\right) m_t^*}\right)\right] f(\theta_t^*) \frac{d\theta_t^*}{dD_t} \tag{14}$$
$$= \int_0^{\theta^*} \left[\frac{1}{m_t}\left(1 - \frac{w_{t+1}}{R_t^2} \frac{dR_t}{dD_t}\bigg|_{LF}\right) + (1 - \theta_t) \frac{1}{R_t} \frac{dR_t}{dD_t}\bigg|_{LF}\right] f(\theta_t)\, d\theta_t.$$

The marginal changes in θ_t^* with respect to D_t can be derived from (12). Likewise, $dR_t/dD_t|_{LF}$ is given by

$$\frac{dR_t}{dD_t}\bigg|_{LF} = \frac{\theta_t}{L'/q_{t+1} + \theta_t w_{t+1}/R_t^2} > 0, \tag{15}$$

10

where L' is the derivative of L_t with respect to R_t/q_{t+1}.

Equation (14) provides an economic interpretation that is in line with broad intuitions. Note that the terms in brackets on the left-hand side of (14) represent the loss of utility in a crisis compared to the threshold. Using (13), we also note that the term outside the brackets, $f(\theta_t^*) \, d\theta_t^*/dD_t$ equals $-d\pi_t/dD_t$, which indicates the marginal changes in a crisis probability with respect to bank leverage. The left-hand side of the equation consists of the expected loss of utility and the marginal change in the crisis probability. Simply put: the left-hand side of (14) is the marginal cost of increasing D_t.

The right-hand side of (14) consists of the effects of increasing leverage on the expected household's lifetime income and the interest rate in the liquidity market. On the one hand, the increase in D_t has an outright positive effect on the household's income: the higher the leverage, the larger the withdrawal, allowing households to enjoy more consumptions. On the other hand, the increase in D_t leads to a higher interest rate via liquidity shortage, discounting the households' labor income in period $t+1$ and reducing returns of forbearing from withdrawal until period $t+1$. Hence, as far as the outright effect on the lifetime income exceeds the effect on the interest rate, the higher leverage is beneficial to households: Simply put, the right-hand side of (14) is the marginal benefit of increasing D_t.

The following proposition establishes the uniqueness of the choice of D_t under our assumptions.

Proposition 1 *The optimal bank leverage is unique.*

Proof. See the Appendix. ∎

While we leave the formal proof to the Appendix, the intuition behind the proposition can be explained as follows. Throughout this paper, we assume that the outright effect of increasing D_t exceeds the effect on the interest rate, since otherwise no deposit offer provides benefits to households and thus there would be no point in discussing banks. On the other extreme, leveraging up to an infinitely large level is clearly not optimal, because such an infinitely large D_t decreases θ_t^* to zero, which means a financial crisis should certainly take place. Thus, once again there would be no benefit in issuing deposits. Then, the strict concavity of the log utility ensures the uniqueness of the optimal leverage within $[X, \infty)$.

Finally, we define the equilibrium in the laissez-faire economy as follows.

Definition *An equilibrium consists of allocations and prices $\{g_t, D_t, L_t, K_t, I_t, H_t, R_t, q_t, w_t\}_{t=0}^{\infty}$ such that (i) withdrawal decisions are given by (5) for $\theta_t \le \theta_t^*$; (ii) the banks' leverage satisfies (14); (iii) the bank's liquidity supply is determined by (6); and (iv) all markets clear.*

3 Systemic Risks and Welfare

3.1 Social Planning Bankers

In the previous section, banks maximize households' expected utility as a result of the perfect competition. In this section, however, we *assume* that banks can choose allocations as the social planner. Under this assumption, we demonstrate that the market equilibrium, defined in the previous section, cannot replicate the social optimum that will be characterized in this section. To lead off the analysis, we clarify the constraint to which the social planning bankers are subject. They must make all their decisions before observing θ_t. After realization of θ_t, they are left with no options. In other words, the planners are subject to the constraint that they cannot control households' behaviors or choose their outright consumption levels because households can react to any realized value of θ_t. This assumption is made for an explicit reason. While we examine the social planner's problem in this section, nonetheless, we aim primarily to see the constrained optimum rather than the unconstrained optimum. The unconstrained, first-best optimum is conceptually easy to understand. By assuming that banks (or anyone else) can issue the Arrow security that pays off contingent on all possible realizations of θ_t, households can enjoy the maximum utility without experiencing any financial crisis. But as we already discussed, banks are, by definition, entities engaged in a maturity mismatch and issue non state-contingent precommitted debt (e.g., demand deposits). Otherwise, banks are no longer banks and should be regarded as other types of financial intermediaries (e.g., private equity funds). Because, in our model, we cannot disregard the special business line of banks, a maturity mismatch, we assume that banks pre-commit to payment on their debt regardless of the states realized following their commitment. The extra capacity given to the social planning bankers compared to the price-taking competitive banks is that the former can internalize all price effects in all markets when they make their decisions regarding their leverage.

The social planning bankers do not take the factor prices as given, but now take into account their changes reflecting the marginal product. Formally, we replace q_t and w_t with $F_{K,t}$ and $F_{H,t}$, respectively. Note that, nonetheless, social planning bankers take the households' behaviors as given, as they cannot make their contract contingent on θ_t. In other words, households always choose their consumption and withdrawal levels given the precommitted D_t by banks.

We summarize the social planning bankers' problem:

Problem SP *Social planning bankers maximize the household expected utility,*

$$\max_{D_t} \int_0^{\theta_t^*} \left\{ \theta_t \ln\left(F_{H,t} + L_t\right) + (1 - \theta_t) \ln\left[F_{H,t+1} + R_t\left(D_t - L_t\right)\right] \right\} f\left(\theta_t\right) d\theta_t$$
$$+ \int_{\theta_t^*}^1 \left[\theta_t \ln\left(F_{H,t} + X\right) + (1 - \theta_t) \ln \underline{F}_H\right] f\left(\theta_t\right) d\theta_t$$

subject to

$$L_t = \theta_t \left(\frac{F_{H,t+1}}{R_t} + D_t \right) - (1 - \theta_t) F_{H,t} \tag{16}$$

$$D_t = A \left(\frac{R_t^*}{F_{K,t+1}^*} \right) \tag{17}$$

$$\theta_t^* = \frac{L \left(R_t^*/F_{K,t+1}^* \right) + F_{H,t}}{F_{H,t} + D_t + F_{H,t+1}^*/R_t^*}. \tag{18}$$

The capital good resource constraint (11) remains the same as in Problem LF except that we replace q_{t+1} with $F_{K,t+1}$. We emphasize that K_{t+1} depends on the market interest rate in period t and define $\Phi(R_t)$ as a function of R_t. Provided that a crisis is not taking place, K_{t+1} evolves according to,

$$K_{t+1} = \underline{I} + I \left(\frac{R_t}{F_{K,t+1}} \right)$$
$$\equiv \Phi(R_t), \tag{19}$$

where $\Phi' < 0$ represents the total derivative of K_{t+1} with respect to R_t.[9] The solution of Problem (SP) can be considered as the constrained optimum as discussed in Allen and Gale (1998). The next proposition establishes our second main result.

Proposition 2 *The laissez-faire banking sector cannot achieve the constrained optimum.*

Proof. Suppose that the allocation under Problem LF is identical to that under Problem SP. To prove the proposition, it suffices to show that $d\theta_t^*/dD_t$ in (14) in Problem LF deviates from that in Problem SP. By differentiating (18) at θ_t^*, we can write out $d\theta_t^*/dD_t$ in Problem SP as

$$\left. \frac{d\theta_t^*}{dD_t} \right|_{SP} = \frac{1}{m_t^*} \left[\frac{\partial}{\partial R_t^*} (L_t^* - g_t^*) - \tau_\varepsilon^* \right] \left. \frac{dR_t^*}{dD_t} \right|_{SP} - \frac{\theta_t^*}{m_t^*}, \tag{20}$$

where $\tau_\varepsilon^* \equiv \left(\Phi^{*\prime}/R_t^* \right) \left[\left(R_t^*/q_{t+1}^* \right)^2 L^{*\prime} F_{KK,t+1}^* + \theta_t^* F_{HK,t+1}^* \right]$. Furthermore, we compare (20) with

$$\left. \frac{d\theta_t^*}{dD_t} \right|_{LF} = \frac{1}{m_t^*} \left[\frac{\partial}{\partial R_t^*} (L_t^* - g_t^*) \right] \left. \frac{dR_t^*}{dD_t} \right|_{LF} - \frac{\theta_t^*}{m_t^*}. \tag{21}$$

Thus, the proposition is proven by either (i) non-zero τ_ε^*, (ii) discrepancies in the derivatives/slopes of the excess liquidity supply functions $L_t^* - g_t^*$, or (iii) discrepancies in dR_t^*/dD_t across the two problems. While in general τ_ε^* is non-zero, it may also be the case that τ_ε^* is zero, depending on the

---

[9]Solving $\Phi' = (1 - \Phi' R_t F_{KK,t+1}/F_{K,t+1}) I'/F_{K,t+1}$ for Φ' ensures that Φ' is negative.

parameter set. For completeness, we highlight (iii), the discrepancy in dR_t^*/dD_t. By differentiating (17),

$$\left.\frac{dR_t^*}{dD_t}\right|_{SP} = \left.\frac{dR_t^*}{dD_t}\right|_{LF} + \frac{R_t^*}{F_{K,t+1}^*}\frac{d}{dD_t}F_{K,t+1}^* \tag{22}$$

confirms that the second term on the right-hand side of (22) is non-zero, contradicting to the supposition. ∎

3.2 Marginal Systemic Risk (MSR)

As a general concept, we introduce a measure to assess the systemic risk of a financial crisis, that is, marginal systemic risk (MSR) as the deviation of the marginal increase in the crisis probability against a target variable in the social optimum from the laissez-faire economy. MSR can be applied in broad models where a financial crisis takes place as a non-zero probability event. Depending on the focus of studies, MSR can be defined vis-à-vis bank leverage, aggregate credit, bank lending, or potentially, asset prices. In our model, MSR is defined with respect to a 1 percent change in bank leverage. Specifically, let D_{LF} be the level of bank leverage chosen in the laissez-faire economy. Then,

$$MSR_t = \left.\frac{d\pi(D_t)}{d\log D_t}\right|_{D_t=D_{LF}} - \left.\frac{d\hat{\pi}(D_t)}{d\log D_t}\right|_{D_t=D_{LF}}, \tag{23}$$

where $\pi(D_t)$ and $\hat{\pi}(D_t)$ are the functions of D_t that indicate the crisis probabilities for the social planning bankers and for the laissez-faire banking sector. Recall that the latter takes other banks' decisions as given, but, in fact, the crisis probability is affected by the synchronized decisions by the banking sector as a whole. In this regard, $\hat{\pi}(D_t)$ can be interpreted as the probability perceived by the individual price-taking banks, which can be contrasted with the true probability, $\pi(D_t)$. With this definition in mind, a positive MSR indicates that the laissez-faire banking sector underestimates the marginal cost of higher leverage by not taking into account the systemic risks. As the result, the banking sector is likely to be overleveraged at (and around) the laissez-faire equilibrium. The MSR can take a negative value if the laissez-faire banking sector is underleveraged. While we do not discuss policy-related issues in this paper, in principle a positive MSR would provide the ground for government intervention to rein in excessive leverage of banks, because such regulatory risk reduction can improve welfare. In our model, and presumably in general, a marginal increase in banks' leverage raises the probability of a financial crisis. If the economy includes, for example, a pecuniary externality, measuring MSR would indicate the extent, to which the laissez-faire economy is exposed to an excessively high probability of a financial crisis. To better grasp what creates the non-zero MSR, or equivalently, the inefficient allocation, we can inspect the efficiency conditions of our model. As a preparatory step, we make one assumption, $\tau_\varepsilon^* = 0$, to specify the most important source of the systemic risk.

The following proposition characterizes our main result in this section.

Proposition 3 *Under the assumption $\tau_\varepsilon^* = 0$, MSR is strictly positive.*

Proof. See the Appendix. ∎

The assumption that τ_ε^* is zero is made solely to crystallize the main channel creating the positive MSR and to underscore the channel analytically.[10] In fact, τ_ε^* tends to be fairly small because of the two offsetting effects.[11] Proposition 3 provides a foundation for understanding why the crisis probability is higher in the laissez-faire economy than in the social optimum. We discuss the interpretation of Proposition 3 based on a sketch of the proof set out in the Appendix.

MSR in our model can be approximately written out as follows. Bearing in mind that $\pi_t = \int_{\theta_t^*}^1 f(\theta_t) \, d\theta_t$,

$$
\begin{aligned}
MSR_t &= f(\theta_t^*) \left(\frac{d\hat{\theta}^*(D_t)}{d \log D_t} - \frac{d\theta^*(D_t)}{d \log D_t} \right) \Bigg|_{D_t = D_{LF}} \\
&\simeq \frac{f(\theta_t^*)}{m_t^*} \frac{\partial}{\partial R_t^*} (L_t^* - g_t^*) \left(\frac{dR^*}{dD_t} \Bigg|_{LF} - \frac{dR^*}{dD_t} \Bigg|_{SP} \right) \times D_{LF}.
\end{aligned}
\tag{24}
$$

Again, $\theta^*(D_t)$ and $\hat{\theta}^*(D_t)$ are the functions of D_t and $\hat{\theta}^*(D_t)$ refers to the tightness of the liquidity market perceived by the laissez-faire banks. Note that all functions are evaluated at the laissez-faire equilibrium. In the above equation, $f(\theta_t^*)/m_t^*$ and the excess liquidity supply function denoted as $\partial(L_t^* - g_t^*)/\partial R_t^*$ are both positive. We are left with the deviation of the changes in R_t^* with respect to bank leverage. Evidently, (24) indicates that MSR is zero if the marginal changes in R_t^*, where banks remain solvent, do not vary across the two economies. However, the discrepancy of the marginal changes in R_t^* can be shown to be strictly positive. To see this, we investigate how banks in each economy recognize their own solvency constraints. In the laissez-faire economy, by differentiating the solvency constraint (8), we obtain

$$
\frac{dR_t^*}{dD_t} \Bigg|_{LF} = \frac{q_{t+1}^*}{A^{*\prime}} < 0,
\tag{25}
$$

which points exactly to the marginal change in R_t^* against bank leverage. Since $A(\cdot)$ is monotonically decreasing in the relative price, the total derivative A' is always negative. Bearing this in mind, we note that a marginal increase in banks' leverage reduces R_t^*. A higher leverage means a higher

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[10] In a later section, we will show that even in the absence of the assumption, the numerical results indicate that the probability of a crisis is higher in the laissez-faire economy than in the social optimum.

[11] The distortion, τ_ε^* includes two offsetting effects on demand and supply for liquidity. In particular, a marginal increase in D_t reduces R_t^* and the reduction in R_t^* affects marginal products of capital and labor via the capital accumulations on the threshold. On the one hand, such an increase in capital on the threshold generates an increase in the marginal product of labor, resulting in higher liquidity demand from households. On the other hand, the increase in the threshold capital gives rise to a decrease in marginal product of capital on the threshold. This effect tightens the solvency condition, resulting in the higher liquidity supply. These two effects raise both demand and supply for liquidity, respectively. Consequently, the net effect on R_t^* tends to be small.

likelihood of insolvency. The issue on which we focus here is how quickly banks fall short of their committed payment. This information is translated into the changes in R_t^* against banks' leverage. If banks can behave as the social planner, taking into account the changes in capital prices, the marginal change in R_t^* is written as

$$\left.\frac{dR_t^*}{dD_t}\right|_{SP} = \left(\frac{1}{1 - R_t^* q_{t+1}^{*\prime} \Phi^{*\prime}/q_{t+1}^*}\right) \frac{q_{t+1}^*}{A^{*\prime}} < \left.\frac{dR_t^*}{dD_t}\right|_{LF} < 0, \tag{26}$$

where q_{t+1}' is the total derivative of q_{t+1} with respect to K_{t+1} (i.e., $dF_{K,t+1}/dK_{t+1}$). Comparing (26) with (25) indicates clearly that an individual price-taking bank underestimates the marginal changes in R_t^* against its own leverage.

Equations (26) and (25), together with (24), confirm that the positive MSR arises from the banks' "rational over-optimism" of the marginal changes in the threshold interest rate R_t^* with respect to their own leverage. In general, an increase in leverage reduces R_t^*, because highly leveraged banks would be more likely to default under a lower interest rate. The positive MSR in our model is closely associated with the size of reduction in R_t^*. The banks in our model make rational decisions with full information (while admittedly, they need to decide D_t before observing the random shock θ_t). Despite their rational decisions, banks always look over-optimistic about the marginal cost of higher leverage and turn out to be more frequently insolvent than expected. To understand the bank's over-optimism, note that $q_{t+1}^{*\prime}{}^1 \Phi^{*\prime}$ is included in (26). This term points to a side effect arising from higher leverage: In general equilibrium, the reduction in R_t^* increases K_{t+1}^*. That is, the lower R_t^* stimulates capital accumulation on the threshold and this increase in K_{t+1}^* triggers the decline in the threshold capital price q_{t+1}^* via the lower marginal product of capital. With this side effect, the lower capital price further undermines the bank's solvency, compared to the case without the side effect of increasing the leverage. Because the price-taking competitive banks, however, have no incentive to take into account this side effect, the lower-than-expected capital price and the undermined banks' solvency raise the probability of a financial crisis.

Looking at the real-world experience of past financial crises, it may be pointed out that, with hindsight, outlooks regarding asset prices frequently tended to be overly optimistic in the run-ups to crises. Some argue that such over-optimism arises from irrationality. While we do not claim that irrational behavior is irrelevant, our model suggests that despite the full rationality, a pecuniary externality can result in seemingly irrational over-optimism. The key to understanding the externality lies in the synchronized decisions by the individual banks in a competitive sector. For each bank, capital prices are given. Recall that the financial market is incomplete in the absence of the Arrow securities. Incompleteness of the financial market means, not surprisingly, that a competitive market equilibrium can result in a distorted allocation rather than a Pareto-optimal one. In the case of our model, the distortion shows up as the overleveraged banking sector with a higher crisis risk because of the side effect as shown in equations (26) and (24), which can

be interpreted broadly in line with the real-world observations.

4   Numerical Results

4.1   Solving the Model

Analytical results in the previous sections can be translated into numerical examples. We provide numerical solutions of the model in this section to address the following quantitative questions: (i) How frequently does a financial crisis arise?; (ii) To what extent does the laissez-faire banking sector deviate from the social optimum, and to what extent does the laissez-faire banking sector act over-optimistically regarding the cost of a crisis?; (iii) What can the calibrated model tell us about crises which should be prevented by government intervention in the banking sector?; and (iv) How can we compare the numerical results with existing empirical studies on the probability of crises?

Our calibration mostly follows Diamond and Rajan (2009). We set the value of prematurely liquidated project X at 0.95. The distribution of entrepreneurs' projects is assumed to be uniform over $[\omega_L, \omega_H] = [0.5, 3.5]$, similar to the original calibration of Diamond and Rajan (2009). The degree of banks' special collection skill γ is set at 0.9. In addition to parameterization of Diamond and Rajan (2009), we need to set some more parameters. We calibrate the capital share in the production function, α, to 1/3, the capital good endowment received by entrepreneurs, $\underline{I}$, to one. More importantly, we assume that the liquidity shock θ_t follows the beta distribution with a mean of 0.50 and a standard deviation of 0.07. This parameterization indicates a symmetric bell-shaped distribution. To simulate the model, we numerically solve the nonlinear system of the equations consisting of the first-order conditions and resource constraints.

Before interpreting the numerical results, we reconfirm the economic interpretations of D_t. In the context of our model, D_t represents the pre-committed gross return on bank deposits. On the other hand, the model does not specify the length of each period of time (e.g., one year or one quarter) and because of the absence of a time unit, D_t cannot be translated into an annual percentage rate (APR) or an interest rate per annum. To better focus on the economic interpretations, D_t needs to be translated into a timeless measure such as the bank leverage. This is the exact reasoning that we have relied on this interpretation of D_t, that is, bank leverage.

4.2   The Probability of Financial Crises

A notable feature of the model is that the probability of a financial crisis varies endogenously. A primary quantitative question to be addressed here is how frequently a financial crisis can take place.

Our simulation results are summarized in Table 2, where the realization of θ_t is assumed to take the mean of 0.50. The upper panel of the table reconfirms that the laissez-faire banking sector takes

on more risks than the social planning banks, indicating a higher crisis probability. Our calibration points to a 6.59 percent crisis probability in the laissez-faire economy compared to 4.50 percent in the social optimum. The results should not be translated into a duration of non-crisis periods, because of the absence of the time unit. Hence, we argue that, in an arbitrary period, out of 1,000 simultaneous attempts, about 66 attempts would trigger crises. The overleverage can be confirmed by the values of D_t in the upper panel of the table. In fact, D_t is 1.2 percentage points higher in the laissez-faire banking sector than in the social planning banks. Figure 1 plots the level of the expected utility against D_t. As Proposition 1 indicates, laissez-faire banks choose their leverage uniquely, shown at point B in the figure. In line with Proposition 2, however, the laissez-faire banking sector cannot achieve the constrained optimum. In fact, our computation results clearly reaffirm that this is the case.

4.3 The Size of Distortions

We next examine to what extent the laissez-faire banking sector deviates from the allocations achieved by the social planning banks. More broadly, we discuss the quantitative implications of the higher leverage in the laissez-faire economy for the welfare.

We have argued that MSR is a useful measure in assessing the inefficiency of the economy and have shown in Proposition 3 that laissez-faire banks in our model generate a strictly positive MSR under the assumption $\tau_\varepsilon^* = 0$. If we remove this assumption, it could be argued that the proposition may not always be the case. Our simulation results confirm that the MSR, in fact, takes a positive value under a plausible parameter set. The lower panel of Table 2 reports the MSR evaluated under the allocations in the laissez-faire economy. The value is 0.48 percentage point, and this extra increase in crisis probability arises solely from pecuniary externalities. To evaluate the size of pecuniary externalities, suppose that banks increase the leverage by 1 percent. The MSR obtained implies that each bank underestimates the crisis probability by 0.48 percentage point. Since the laissez-faire economy generates the semi-elasticity of a crisis probability, $d\pi_t/d\log D_t$, of 1.64 percent, this MSR means that each bank expects the crisis probability to increase by only 1.16 (i.e., 1.64 - 0.48) percentage points. In other words, each bank makes the individually rational assessment that, by increasing 1 percent of leverage, they are exposed to 7.75 percent crisis probability. However, the true probability is 8.23 percent. Figure 2 also confirms the positive MSR. The red and blue lines in the figure plot $\hat{\pi}(D_t)$ and $\pi(D_t)$, respectively, around the value of D_t chosen under laissez-faire banking. As discussed, the former corresponds to the case when banks take factor prices as given and the latter corresponds to the case where pecuniary externalities are internalized. The figure shows that the red line is flatter than the blue line, reflecting the positive MSR.

Figure 3 plots the marginal cost and benefit in the two economies. The marginal cost is represented by solid lines, and the marginal benefit by dashed lines. The intersections A and B represent

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the choices of leverage under the social optimum and the laissez-faire economy, respectively. This numerical computation reconfirms that a bank's underestimation of the marginal cost leads to the overleverage. In this computation, we also observe a relatively small shift in the marginal benefit curve across the two economies, which slightly exacerbates the bank's overleverage.[12] However, the shift in the marginal cost curve dominates the shift in the marginal benefit curve.

The lower panel of Table 2 compares the bank capital ratio defined as $(A_t - D_t)/A_t$ and the output of the consumption goods Y_{t+1} under the laissez-faire economy and the social optimum. The laissez-faire banking sector is undercapitalized by 1.1 percent compared to the social optimum. Nevertheless, it may be surprising that the production does not substantially vary across the two allocations, provided that a financial crisis does not take place. We also compute the levels of consumption for households in a generation. The household's consumption is $(C_{1,t}, C_{2,t+1}) = (2.21, 2.61)$ for the laissez-faire banking sector in comparison to $(C_{1,t}, C_{2,t+1}) = (2.21, 2.59)$ in the social optimum.

The above exercise indicates that the welfare loss primarily arises from the inefficiently elevated crisis probability. Given that crises are considered to be rare events that we cannot observe frequently, the inefficiency or welfare loss may not be easily detected by looking at the volatility of consumption or output in normal times. In this sense, assessing the inefficiency or welfare loss with MSR appears more appropriate than with the volatility of the consumption or output.

4.4 Crisis Prevention

The prediction that the laissez-faire banking sector is overleveraged implies that the laissez-faire economy undergoes crises created by banks' overoptimism, some of which could be avoided under the social optimum. To illustrate this, we run the model over 100 periods by generating liquidity shocks randomly. Figure 4 plots the dynamic paths of output Y_t and θ_t^* under the two allocations. The red line corresponds to the case of laissez-faire economy, while the blue line points to the case of the social optimum.

The upper panel of Figure 4 shows that the dynamic paths of the output are almost identical except that the production under the laissez-faire economy sharply declines more frequently. Crises take place in period 5, 16, and 94 and output sharply falls in each subsequent period. We note that this simulation result indicates that the latter two crises could have been prevented if the banks were taking the risks at the optimal level, implying a need for government interventions to forestall the crises. However, the first crisis takes place even in an economy in which the social planning

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[12]This shift results mainly from the laissez-faire bank's overestimation of wages in period $t + 1$, conditional on a financial crisis not taking place. A marginal increase in leverage leads to greater consumption by households and a higher interest rate. A higher interest rate implies a lower amount of the completed project (i.e., lower capital accumulation) in the laissez-faire economy. The lower capital accumulation also reduces the marginal product of labor, resulting in lower wages in period $t + 1$. Since the laissez-faire banks disregard this income-reducing effect, the banks overestimate the marginal benefit of increasing D_t.

banks strike the right balance between the costs and benefits of increasing the leverage. Therefore, this crisis should not be avoided as discussed in Allen and Gale (1998) in the context of the optimal financial crises.

To better understand how differences arise in the two economies, the lower panel of the figure shows two different dynamic paths of θ_t^*. Recall that θ_t^* is defined as the threshold value of the liquidity shock that satisfies the solvency constraint on the brink of a default. This threshold level of θ_t^* is always lower in the laissez-faire economy, implying that the solvency constraint is tighter and the economy is more exposed to the macroeconomic fragility. The dashed line represents the realized θ_t in the simulations, which is identical across the two economies. The realized θ_t exceeds both high and low θ_t^* in the first crisis and reaches only lower θ_t^* in the last two crises. Although the difference in the realized θ_t appears to be fairly small among the three crises, relative to the volatility of θ_t, the small difference in the liquidity shock affects the economic performance, through the risk-taking of the banking sectors.

4.5 Comparison with Empirical Studies

The numbers included in Table 2 may be compared to some empirical works on (i) catastrophe (CAT) risks and on (ii) the optimal level of bank capitals. Among a number of works on CAT risks, Barro (2009) can provide a comparable benchmark. He sets the "disaster probability" at 2 percent per year, arguing that a "disaster" could reduce GDP by 30 percent on average. By incorporating such disaster risks into the Epstein-Zin type asset pricing model, he concludes that the 2 percent CAT risk can account for an equity premium of around 4-6 percent. He also argues that the size of scarring effects of a disaster can range from 15 to over-60 percent of GDP.

A more straightforward study by Basel Committee on Banking Supervision (2010, BCBS) assesses impacts of changes in bank capital on the probability of systemic banking crises; it aims to explore the optimal level of bank capital in the context of the yet-to-be enacted Basel III, the new global regulation framework for banking systems. BCBS estimates that a marginal increase in 1 percent bank capital from the pre-reform cross-country average level could reduce the crisis probability by 1.0-1.6 percentage points.[13] We can underscore the proximity of the BCBS estimates and the numerical results included in Table 2.[14] Beyond such simple numerical comparisons, we emphasize that MSR can be applied in line with a broad empirical exercise as typically demonstrated by BCBS. Using reduced-form econometric models, a marginal increase in the crisis probability is evaluated at around a certain point in the data. But empirical estimates, in general, cannot

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[13]Its estimates are based on multiple empirical methodologies, but it could be said that, primarily, reduced form econometric models, such as probit/logit models, are used to yield those estimates.

[14]In addition to the evident proximity of the marginal changes in the crisis probabilities, the level of bank capital in BCBS does not substantially differ from that in our model. BCBS argues that the pre-reform cross-country average of the TCE/RWA (tangible common equity divided by Basel II risk-weighted assets) ratio is 7 percent. The TCE is an extremely narrow definition of bank capital which, by and large, could be doubled or even tripled (i.e. 14 - 21 percent) if measured in more conventional measures for bank capital, such as the Tier I ratio.

provide information on how distant the economy is located from the social optimum. Assessing to what extent the observed equilibrium could be improved by policy interventions may be of more interest. To this end, comparison of empirically estimated marginal changes in crisis probabilities with MSRs calculated using structural models would promote discussion on the desirable size and design of regulations in the banking sector down the road.

5 Relationship to the literature

Contrasting our model with the existing literature would crystallize the contribution of this work. The primary aspect of our model is the endogenously varying crisis probability. With this utmost factor in mind, we discuss how our model could be aligned in comparison with the four strands of literature: (i) models of bubbles, (ii) models of banks or theories of banking, (iii) models of pecuniary externalities and (iv) macroeconomic models with financial sectors.

5.1 Bubbles

While, evidently, our model does not discuss bubbles and is not related to the bubble literature on the surface, our model shares a few aims and interests with models on rational bubbles: Both aim to explain medium- and longer-term economic fluctuations, typically acknowledged as boom-bust cycles. In the aftermath of the 2008-09 global financial crisis, studies on bubbles gained renewed momentum. Martin and Ventura (2011a, 2011b) build models where bubbles are held by rational agents and the bubbles promote investment for as long as they persist.[15] They pave the way to model realistic bubbles compared with early works by incorporating financial frictions and investor sentiment shocks.[16] They basically argue that rational investors are willing to hold bubbly assets because they could improve the distribution of wealth as long as the bubble does not burst. Despite their elegant explanations for why bubbles emerge, they remain relatively silent on why and when bubbles burst, by typically assuming a constant probability of a bubble burst. In their model, bubbly episodes end with no specified reason. In reality, the probability of a sudden ending of asset market booms or a financial crisis is affected by the degree of risk-taking by investors and, among others, banks. Our model highlights and addresses this why-and-when question regarding the collapse of bubbles or financial crises. In a related context, we note that, in reality, it is extremely difficult to distinguish whether an ongoing boom is in line with fundamentals or a bubble on a real time basis. Despite the recurrent nature of boom-bust cycles, it has been quite frequently argued that a boom is driven by fundamentals or "this time is different" (Reinhart and Rogoff 2009). With this lesson in mind, it could be noted that our model predicts that a laissez-faire banking

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[15]Samuelson (1958) and Tirole (1985) are early works on deterministic rational bubbles that reduce investment.

[16]Aoki and Nikolov (2011) argue that collapse of bubbles is severer when financial intermediaries hold bubbly assets than when other investors do.

system tends to end up with inefficient crises and thus calls for remedial measures, including macro-prudential regulations, regardless of whether bubbles arise or not.

5.2 Theories of Banking

Our model is a straightforward extension of Allen and Gale (1998) and Diamond and Rajan (2009) in terms of the basic modeling approach of banks or a banking system. The well-thought-out micro-foundations of banks in those models are essential. A notable feature of the banking systems in those models is that the banks can achieve socially optimal allocation in the absence of government interventions. In fact, Diamond and Rajan (2009) argue that expectations of some sort of bailout (e.g., protracted low interest policy at a time of crisis) can ill-incentivize banks to be overleveraged. While we agree with Diamond and Rajan (2009), we additionally explore inner sources of inefficiency within banking systems as we detect possibilities that even if there were no expectations for a bailout, bank over-leverage and the resulting financial crisis could take place. The fragility of banking systems explored in our work differs from sun-spot driven multiple equilibria, originally suggested by Diamond and Dybvig (1983) and later developed by others. As noted earlier, in our model financial crises are precipitated by fundamental shocks (liquidity preferences) rather than self-fulfilling expectations. We emphasize that financial crises in our model are not unpredictable, entirely random events, but the consequence of excess risk-taking by banking systems.

5.3 Macroeconomic Models with Banks

Macroeconomic models with financial intermediaries have primarily focused on how financial frictions amplify business cycles (Bernanke and Gertler 1989, Kiyotaki and Moore 1997) rather than how and why devastating financial crises could take place sporadically beyond the business cycle frequency. Nonetheless, a large number of macroeconomic models with "banks" make remarkable progress by extending those state-of-the-art frameworks in interpreting the real economic fluctuations in multiple dimensions.[17] Recently, Gertler and Karadi (2011) and Gertler and Kiyotaki (2011), motivated by the fact that real economic activities were vastly disrupted by the banking sector oriented crisis, discussed how banks with limited commitment exacerbate economic downturns. On this front, among others, Gertler and Karadi (2011) shed light on the moral hazard problem between financial intermediaries and households. Gertler and Kiyotaki (2011) generalize a similar framework to consider the liquidity management of banks via the interbank market.[18]

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[17]Bernanke, Gertler, and Gilchrist (1999), Carlstrom and Fuerst (1997), and Kato (2006) introduce capital mutual funds (CMF) which merely pool households' funds and lend them to borrowers. As fully discussed in both of those studies, the CMF is not a profit maximizing agent, but an intermediation system that operates fairly mechanically. Using a similar framework, Christiano, Motto and Rostagno (2011) build a model with a broad spectrum of extensions to answer a number of quantitative questions.

[18]Meh and Moran (2010) and Hirakata, Sudo, and Ueda (2009) adopt two-sided agency problems, by extending Bernanke, Gertler, and Gilchrist's (1999) framework. Nishiyama, Iiboshi, Matsumae, and Namba (2011) also consider

They also focus on how unconventional credit policies, along with conventional interest rate policy, affect the amplification mechanism in the model with banks characterized by the agency problem. While these studies successfully incorporated financial intermediaries or "banks" into the canonical dynamic general stochastic equilibrium models, the business of banking translated in those models still appears somehow over-simplified compared to those handled in the theories of banking as mentioned in the previous subsection.

The focus of our model starkly contrasts with those studies in the following aspects: we aim (i) to account for the vast standstill of financial intermediations and subsequent sharp declines in output and investment, both of which the global economy recently experienced, and (ii) to explore a rationale for government intervention based on a solid and explicit welfare assessment. For the first issue, we stress that, while these preceding studies (Gertler and Karadi 2011, Gertler and Kiyotaki 2011) in principle focused on the amplification mechanism within the business cycle frequency, our focus is to model systemic financial crises, which are rare but large events beyond the business cycle frequency, and the subsequent sharp declines in overall economic activity. For the second, it could be said that our motivation is more fundamental. While those early studies analyze the effect of credit policies on business cycle fluctuations, we seek to answer why policy interventions are needed, if at all. To better understand why we need (or do not need) government interventions in banking systems, we explore why a laissez faire banking system can fail to achieve the (constrained) optimal outcome.

Apart from those main differences from the preceding macroeconomic models with banks, we also stress that banks' liquidity shortages in our model play more prominent roles on banks' insolvency if a financial crisis were to unfold. In Gertler and Kiyotaki (2011), for example, the banks are faced with liquidity constraints but liquidity shortages never lead to banks' insolvency or a financial crisis, even in the case that a large adverse shock erodes the quality of banks' real assets. In this context, Angeloni and Faia (2010) take a step forward by incorporating banks' insolvency into the canonical dynamic general equilibrium model. Motivated by the issues on banking sector regulations, they build a model based on Diamond and Rajan (2000) to consider the macroeconomic consequences of the interaction of banking sector regulations and monetary policy, taking into account a bank's optimal choice of capital structure. To make the model tractable, they allow banks to issue state-contingent "deposits." Whereas their model has the endogenous probability of bank insolvency, the probability of the bank insolvency in Angeloni and Faia (2010) can broadly be interpreted as a measure of bank fragility rather than the probability of financial crises where the vast majority of the banking system comes to a standstill. Our model assesses macroeconomic fragility in terms of the probabilities of a sporadic financial crisis rather than perpetually changing fraction of insolvent banks in the system. We also highlight the non state-contingency of banks' liabilities which is essential for the unique business of banking compared to other financial interme-

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a similar two-sided agency problems based on the Gertler and Karadi's (2011) framework.

diaries. The non state-contingency is a key ingredient of our model as it creates the venue where pecuniary externalities arise. We will discuss this issue in the next subsection.

5.4 Pecuniary Externalities

In the wake of the insolvency of the Lehman Brothers in September 2008 and amid the ensuing repercussions, a big question mark was posed over the status quo banking systems. Against this backdrop, we address whether financial crises could be understood as inefficient outcomes arising from over-risk-taking by the banking sector. A number of similar attempts have been made to examine why and how over-credit/over-borrowing can arise under laissez-faire economies with the aim to better focus on tail risks (i.e., financial crises) in a general equilibrium context.

Lorenzoni (2008) set out a formal model where a laissez-faire economy tends to result in over-borrowing and, based on the model, he suggests that inefficient credit booms could be a natural outcome of perfectly competitive financial transactions. Prompted by Lorenzoni's (2008) thought-provoking and generally applicable framework, a number of studies explore the source of over-credit in an attempt to re-affirm the micro-foundation of government interventions, including macro-prudential regulations. In this line of work, Bianchi (2010), Korinek (2010), Jeane and Korinek (2010, 2011), and Stein (2011), among others, have reached a broad consensus that pecuniary externalities, combined with some sort of incomplete market and/or limited commitment, can in general prevent laissez-faire markets from achieving socially optimal outcomes. Many of these models assume that borrowers are constrained by collateral constraints and the constraints depend on some prices (e.g., capital prices) that the agents take as given. Owing to the pecuniary externalities, such price-taking behavior does not achieve the constrained optimum. Our model follows their idea in terms of pecuniary externalities that create inefficiency in financial intermediation. A marked contrast among our model and the other early studies is that, in our model, pecuniary externalities operate on the banks' solvency constraints rather than borrowing constraints of entrepreneurs or other non-bank agents. We agree with Lorenzoni and others on the basic idea that the pecuniary externalities are likely to perform the key roles in creating inefficient boom-bust cycles. With this shared view in place, we suggest two complementary ideas regarding which constraint we need to focus on when we consider the pecuniary externalities. First, the previous work by Bianchi (2010), Jeane and Korinek (2010, 2011) treat the constraint due to the limited commitment as the occasionally binding constraint. By contrast, as long as banks have strictly positive net worth (bank capital), solvency constraints of banks are not binding. They bind at a time of a default or a crisis in the case that the default is systemically taking place. Second, we note that overemphasis on borrowing constraints (or collateral constraint) may sometimes be debatable because evidently some cash-rich investors could be free from such constraints. On the other hand, as long as banks are acting in place, they are all faced with some sort of solvency constraints which naturally depend on prices of some financial assets. In sum, our view is that, while the pecuniary externalities remain

as the key factor, detecting where they perform in the financial system matters and incorporating them into the banking system itself à la Diamond and Rajan (2001a, 2009) appears to be a realistic and straightforward approach to model financial crises.

6 Conclusions

We developed a dynamic general equilibrium model that explicitly includes banks with maturity mismatches. Using the model, we showed that, under the laissez-faire economy, inefficient financial crises are precipitated by a liquidity shortage in the overleveraged banking system. In general, a perfectly competitive banking sector cannot always achieve the first-best allocation because the banking business per se (i.e., maturity transformation via issuance of non state-contingent debt rather than the Arrow security) implies that financial markets are incomplete. Our model demonstrated that a perfectly competitive banking sector cannot achieve the second-best allocations, resulting in inefficient financial crises. In our model, pecuniary externalities arise, distorting the banks' assessment on their own solvency. The banks fail to internalize the side effects from changes in the illiquid asset prices on their own solvency. From the viewpoint of a social planner, because of this failure, the rational banks overestimate their solvency and underestimate the cost of increasing the leverage systemically. This rational over-optimism exposes banks to inefficiently elevated systemic risks. In the light of real world experience, our model could lay a foundation for better understanding of the repeatedly observed financial and economic crises.

We also introduced MSR (marginal systemic risks) as a measure to assess the (in)efficiency of an economy. When MSR takes a positive value, the economy is considered being exposed to systemically excessive risks of financial crises. While our model defined MSR vis-à-vis bank leverage, it can also be defined against some other financial variables, and thus can be applied to a wide class of models explaining financial crises. As discussed by Allen and Gale (1998), financial crises can arise as a socially optimal outcome. In line with their argument, financial crises need to be prevented, only if it is inefficient. To assess the need for government intervention, MSR can be used as an informative measure.

The analysis demonstrated in this paper can be extended in a number of directions. First, we may need to consider policy measures that can eliminates the positive MSR in our model. Second, we may need to examine how changes in a variety of the economic environment (e.g., changes in the stochastic process of the liquidity shock) or newly introduced aggregate shocks (e.g., shocks to the asset side of the banks' balance sheet) affect the economy's exposure to crisis risks.[19] Finally, introducing price stickiness into the model would pave the way for reconstruction of the roles of monetary policy in comparison with similar models which does not include the possibility of a

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[19]Kato and Tsuruga (2011) demonstrate that rational banks can take on more risks, resulting in a higher default probability, in response to changes in the underlying distribution of shocks that reduce exogenous risks of bank defaults.

financial crisis. All of these directions would provide important steps for future research.

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A   Appendix: Proofs

A.1   Proof of Proposition 1

The proof broadly follows Kato and Tsuruga (2011). To prove Proposition 1, we first show that bank leverage $D_t/\left[A\left(R_t/q_{t+1}\right) - D_t\right]$ has a one-to-one relationship with the deposit face value D_t. In Problem LF, banks choose D_t, taking the factor prices q_{t+1} and w_t as given, while they internalize the liquidity market clearing condition (9). This assures that the interest rate R_t increases with D_t. Along with $A'(\cdot) < 0$, the leverage monotonically increases with D_t.

We next show the unique choice of the optimal D_t. In the model, it is natural to assume that the lifetime income m_t is strictly increasing in D_t around the neighborhood of X. Otherwise, the allocation under the financial autarky is at least as good as that under Problem LF. This assumption requires the following condition:

$$\lim_{D \to X} \int_0^1 \frac{w_{t+1}}{R_t} \frac{dR_t}{dD_t}\bigg|_{LF} f\left(\theta_t\right) d\theta_t < 1,$$

where $dR_t/dD_t|_{LF}$ is defined by (15). Then we use m_t to rewrite the household's utility in Problem LF:

$$\max_{D_t} \int_0^{\theta_t^*} \left\{\theta_t \log\left(\theta_t m_t\right) + \left(1 - \theta_t\right) \log\left[\left(1 - \theta_t\right) R_t m_t\right]\right\} f\left(\theta_t\right) d\theta_t$$
$$+ \int_{\theta_t^*}^1 \left[\theta_t \log\left(w_t + X\right) + \left(1 - \theta_t\right) \log\left(\underline{w}\right)\right] f\left(\theta_t\right) d\theta_t.$$

We now consider the two extreme case: (i) $D_t = X$ and (ii) D_t approaching infinity. Suppose that $D_t = X$. In this case, banks always remain solvent. Hence, the household utility can be written as

$$\int_0^1 \left\{\theta_t \log\left(\theta_t m_t\right) + \left(1 - \theta_t\right) \log\left[\left(1 - \theta_t\right) R_t m_t\right]\right\} f\left(\theta_t\right) d\theta_t$$
$$= \int_0^1 \left\{\log\left(m_t\right) + \left(1 - \theta_t\right) \log\left(R_t\right) + \log\left[\theta_t^{\theta_t} \left(1 - \theta_t\right)^{1-\theta_t}\right]\right\} f\left(\theta_t\right) d\theta_t.$$

If $D_t = X$ is optimal, the inequality,

$$\lim_{D \to X} \frac{dU}{dD_t} = \int_0^1 \left(\frac{1}{m_t} \frac{dm_t}{dD_t} + \frac{1 - \theta_t}{R_t} \frac{dR_t}{dD_t}\bigg|_{LF}\right) f\left(\theta_t\right) d\theta_t \leq 0,$$

must hold. However, the first term inside the parentheses is strictly positive and the second term is non-negative for any $\theta_t \in [0, 1]$. This contradicts the above inequality.

Next, suppose that D_t takes an infinitely large value. An infinitely large D_t means that banks

cannot satisfy the solvency constraint $D_t \leq A\left(R_t/q_{t+1}\right)$. Therefore, for any $\theta_t \in [0,1]$, we can write the household's utility as

$$\int_0^1 \left[\theta_t \log\left(w_t + X\right) + \left(1 - \theta_t\right)\log\left(\underline{w}\right)\right] f\left(\theta_t\right) d\theta_t$$

$$= \int_0^1 u_{crisis} f\left(\theta_t\right) d\theta_t < \int_0^1 u_{no-crisis} f\left(\theta_t\right) d\theta_t$$

where $u_{crisis} = \theta_t \log\left(w_t + X\right) + \left(1 - \theta_t\right)\log\left(\underline{w}\right)$ and $u_{no-crisis} = \theta_t \log\left(\theta_t m_t\right) + \left(1 - \theta_t\right)\log\left[\left(1 - \theta_t\right) R_t m_t\right]$, both of which are evaluated at an infinitely large D_t. With any $\theta_t^{**} \in [0,1]$, it is straightforward to show

$$\int_0^1 u_{crisis} f\left(\theta_t\right) d\theta_t < \int_0^{\theta_t^{**}} u_{crisis} f\left(\theta_t\right) d\theta_t + \int_{\theta_t^{**}}^1 u_{no-crisis} f\left(\theta_t\right) d\theta_t$$

Hence, the infinitely large D_t cannot be the optimum.

Since the two extreme cases cannot be the optimum, the strict concavity of log utility assures the uniqueness of the optimal $D_t \in (X, \infty)$. ∎

A.2 Proof of Proposition 3

MSR in our model is defined by (23):

$$MSR_t = \left. \frac{d\pi\left(D_t\right)}{d\log D_t}\right|_{D_t=D_{LF}} - \left. \frac{d\pi\left(D_t\right)}{d\log D_t}\right|_{D_t=LF}.$$

Using the fact that $\pi\left(D_t\right) = \int_{\theta^*\left(D_t\right)}^1 f\left(\theta_t\right) d\theta_t$, we have

$$
\begin{aligned}
MSR_t &= \left[\frac{d\pi\left(D_{LF}\right)}{dD_t} - \frac{d\hat{\pi}\left(D_{LF}\right)}{dD_t}\right] D_{LF} \\
&= f\left(\theta_t^*\right)\left[\frac{d\hat{\theta}^*\left(D_{LF}\right)}{dD_t} - \frac{d\theta^*\left(D_{LF}\right)}{dD_t}\right] D_{LF},
\end{aligned}
$$

where $\hat{\theta}^*\left(\cdot\right)$ and $\theta^*\left(\cdot\right)$ are the tightness of the liquidity market perceived by the laissez-faire banks and the social planning banks, respectively. More specifically, $\hat{\theta}^*\left(D_t\right)$ and $\theta^*\left(D_t\right)$ are given by

$$
\hat{\theta}^*\left(D_t\right) = \frac{L\left(\frac{R^*\left(D_t\right)}{q_{t+1}^*}\right) + w_t}{w_t + D_t + \frac{w_{t+1}^*}{R^*\left(D_t\right)}}
$$

$$
\theta^*\left(D_t\right) = \frac{L\left(\frac{R^*\left(D_t\right)}{F_K\{\Phi[R^*\left(D_t\right)]\}}\right) + w_t}{w_t + D_t + \frac{F_H\{\Phi[R^*\left(D_t\right)]\}}{R^*\left(D_t\right)}},
$$

where $F_K(\cdot)$ and $F_H(\cdot)$ are the marginal product of capital and labor as a function of $K_t = \Phi(R_t)$. Furthermore, $\Phi(\cdot)$ is defined by (19). We take the total derivation of $\hat{\theta}^*(D_t)$ with respect to D_t to get

$$\frac{d\hat{\theta}^*(D_t)}{dD_t} = \frac{1}{m_t^*}\left[\frac{L^{*\prime}}{q_{t+1}^*} + \frac{\theta_t^* w_{t+1}^*}{(R_t^*)^2}\right]\frac{dR_t^*}{dD_t}\bigg|_{LF} - \frac{\theta_t^*}{m_t^*}.$$

Notice that the partial derivative of the excess supply for liquidity can be written as

$$\frac{\partial}{\partial R_t^*}(L_t^* - g_t^*) = \frac{L^{*\prime}}{q_{K,t+1}^*} + \frac{\theta_t^* w_{t+1}^*}{(R_t^*)^2}. \tag{27}$$

which is strictly positive, meaning that the excess supply function of the liquidity is increasing in R_t^*. Using the partial derivative of the excess supply function, we obtain (21):

$$\frac{d\hat{\theta}^*(D_t)}{dD_t} = \frac{1}{m_t^*}\left[\frac{\partial}{\partial R_t^*}(L_t^* - g_t^*)\right]\frac{dR_t^*}{dD_t}\bigg|_{LF} - \frac{\theta_t^*}{m_t^*}.$$

In contrast, the total derivative of $\theta^*(D_t)$ introduces the effect of D_t on the capital good price and the wage rate because we replace q_{t+1} and w_{t+1} with the marginal product. Therefore,

$$\begin{aligned}\frac{d\theta^*(D_t)}{dD_t} &= \frac{1}{m_t^*}\left[\frac{L^{*\prime}}{F_{K,t+1}^*} + \frac{\theta_t^* F_{H,t+1}^*}{(R_t^*)^2}\right]\frac{dR_t^*}{dD_t}\bigg|_{SP}\\ &- \frac{\Phi^{*\prime}}{m_t^*}\left[\frac{R_t^*}{(F_{K,t+1})^2}L^{*\prime}F_{KK,t+1}^* + \frac{\theta_t^* F_{HK,t+1}^*}{R_t^*}\right]\frac{dR_t^*}{dD_t}\bigg|_{SP} - \frac{\theta_t^*}{m_t^*}\end{aligned}$$

Using (27) and the definition of τ_ε^*, we have

$$\frac{d\theta^*(D_t)}{dD_t} = \frac{1}{m_t^*}\left[\frac{\partial}{\partial R_t^*}(L_t^* - g_t^*) - \tau_\varepsilon^*\right]\frac{dR_t^*}{dD_t}\bigg|_{SP} - \frac{\theta_t^*}{m_t^*},$$

Consequently, the MSR under $\tau_\varepsilon^* = 0$ is given by

$$MSR_t = \frac{1}{m_t^*}\left[\frac{\partial}{\partial R_t^*}(L_t^* - g_t^*)\right]\left(\frac{dR_t^*}{dD_t}\bigg|_{LF} - \frac{dR_t^*}{dD_t}\bigg|_{SP}\right)D_{LF}.$$

Therefore, the sign of the MSR depends on that of the discrepancy in the threshold interest rate responds. The sign of the threshold interest rate response to bank leverage can be obtained from (26). Namely,

$$\frac{dR_t^*}{dD_t}\bigg|_{LF} > \frac{dR_t^*}{dD_t}\bigg|_{SP},$$

which proves Proposition 3. ∎

Table 1: Sequence of events for generation t

		Period t
1.		Households receive endowments.
2.		Banks offer deposits to households and loans to entrepreneurs.
3.		Entrepreneurs launch their projects.
4.		Households supply labor and receive wages w_t determined by the labor market condition along with the old generation's labor supply.
5.		Liquidity shock θ_t is realized, and banks receive signals of project outcomes.
6.		Households decide the withdrawal g_t.
7.		Banks decide which projects to continue and supply liquidity L_t.
	(i)	If $g_t > L_t$, a financial crisis is precipitated and households receive repayment of X.
	(ii)	Otherwise, the households can transfer their wealth into the period $t+1$.
8.		All agents consume.
		Period $t+1$
1.		Entrepreneurs receive endowments.
2.		Entrepreneurs' projects are completed, and they sell their capital goods for q_{t+1} and make repayment to banks.
3.		Households fully withdraw deposits, if any.
4.		Households supply labor and receive wages w_{t+1} determined by the labor market condition along with the young generation's labor supply.
5.		All agents consume.

Table 2: Crisis probabilities and allocations
under laissez-faire banking sector and social planning banks

	Social planning banks	Laissez-faire banks
Leverage and crisis probabilities		
D_t	1.05	1.06
π_t (%)	4.50	6.59
MSR_t	0.48	
Bank capital and output		
Bank capital ratio(%)	15.10	13.95
Y_{t+1}	5.46	5.46

Note: Simulation results based on the assumption that the liquidity shock θ_t follows the beta distribution and the mean is materialized as a realized value of θ_t. The level of bank's leverage D_t and the probability of a financial crisis π_t are obtained from Problems LF and SP, respectively. The marginal systemic risk, MSR_t, is given by (23). The bank capital ratio is $(A_t - D_t)/A_t$.

Figure 1: Optimal bank leverage
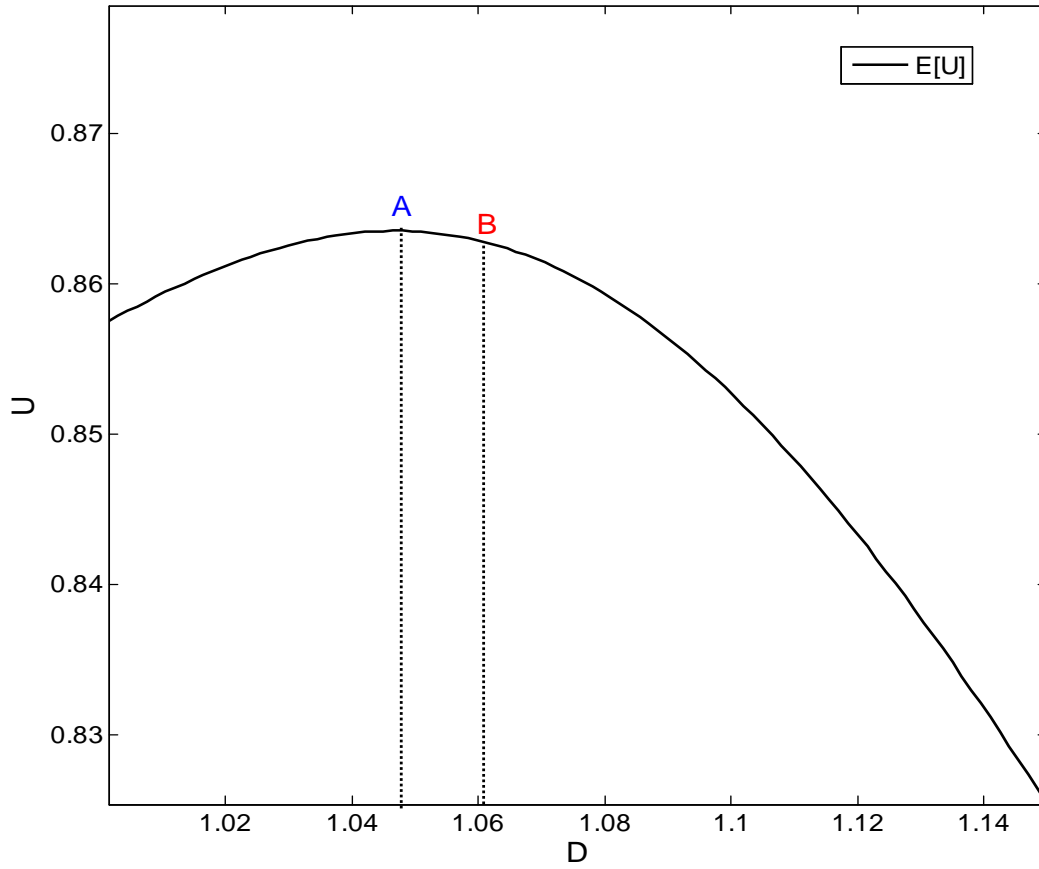
Note: The expected utility against D_t. Point A corresponds to the expected utility level evaluated at D_t chosen by social planning banks, while point B corresponds to the expected utility evaluated at D_t chosen by laissez-faire banking sector.

Figure 2: Marginal systemic risk (MSR)
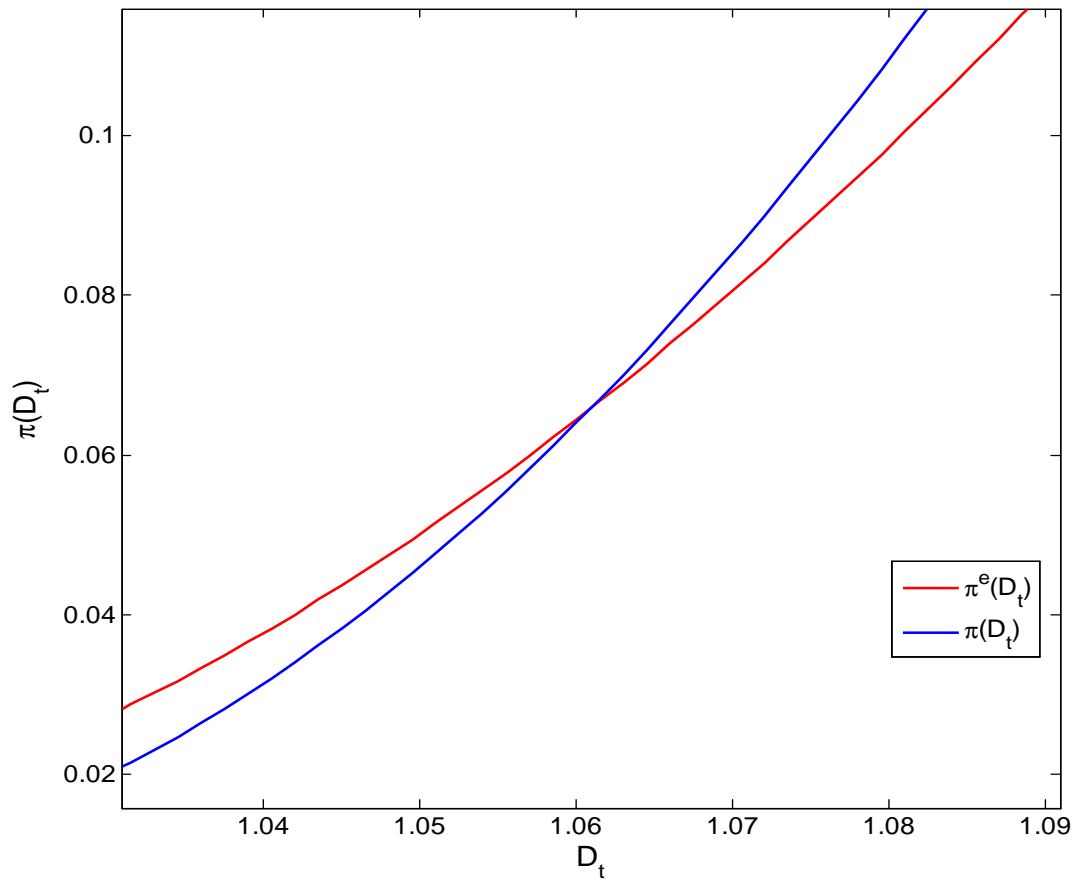
Note: The blue line plots the crisis probability in the general equilibrium allocation. The red line plots the crisis probability that laissez-faire banks recognize.

Figure 3: Marginal cost and benefit of increasing D_t
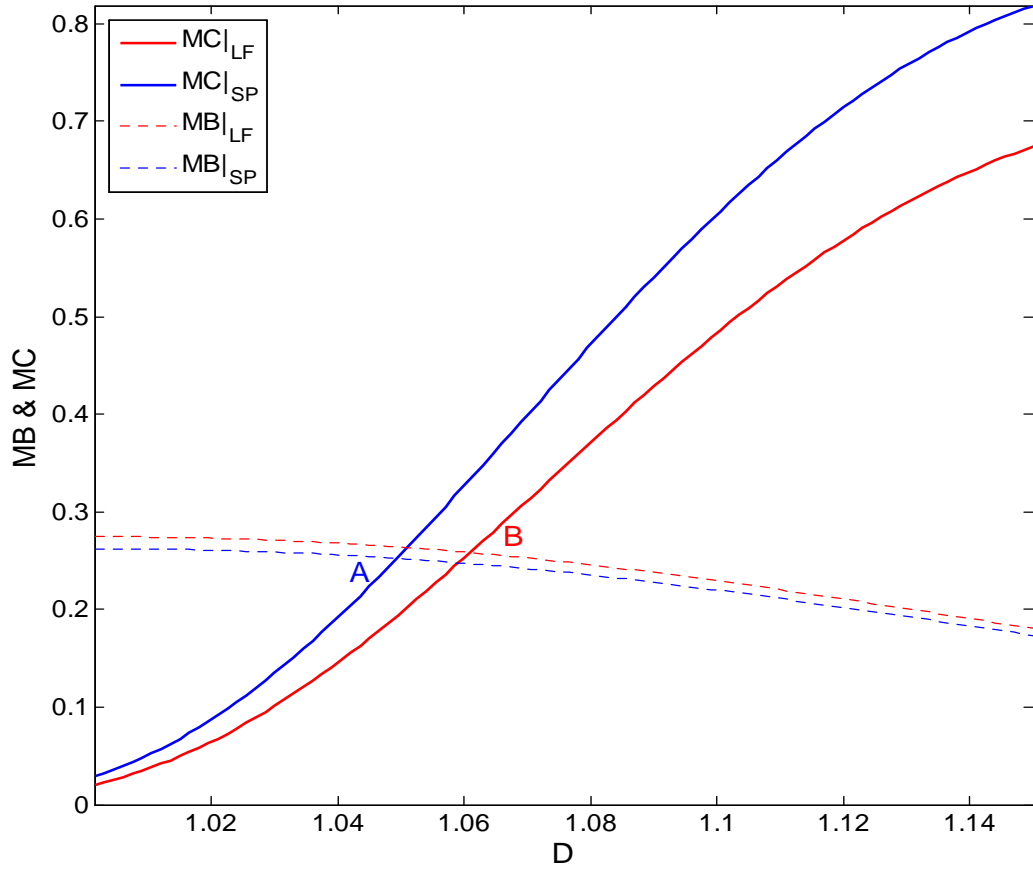
Note: The blue solid line ($MC|_{SP}$) and dashed line ($MB|_{SP}$) represent the marginal cost and benefit under the social optimum, respectively. The red solid line ($MC|_{LF}$) and dashed line ($MB|_{LF}$) are the marginal cost and benefit under the laissez-faire economy, respectively.

Figure 4: Simulated paths of output and the liquidity shock
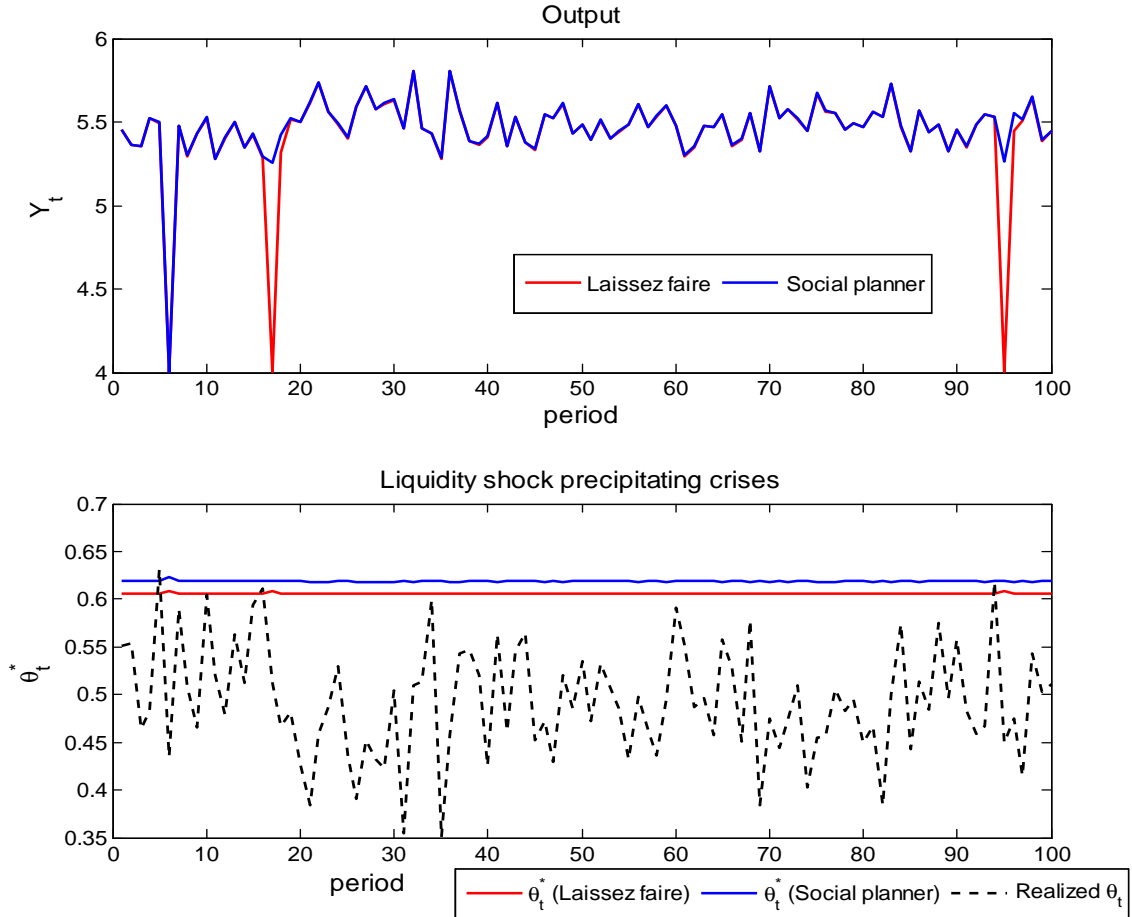
Note: The upper panel shows the simulated dynamic paths of output Y_t under the laissez-faire economy and the social optimum. The liquidity shock plotted as the black dashed line in the lower panel is generated from the beta distribution with a mean of 0.50 and a standard deviation of 0.07. The blue and red solid lines in the lower panel are the threshold level of the liquidity shock that marginally satisfies the solvency condition. The blue line is for the social planning banks and the red line is for the laissez-faire banks.