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Abstract

In this paper, we develop the asymptotic theory of Hwang and Basawa (2005) for explosive random coefficient autoregressive (ERCA) models. Applying the theory, we prove that a locally best invariant (LBI) test in McCabe and Tremayne (1995), which is for the null of a unit root (UR) process against the alternative of a stochastic unit root (STUR) process, is inconsistent against a class of ERCA models. This class includes a class of STUR processes as special cases. We show, however, that the well-known Dickey-Fuller (DF) UR tests and an LBI test of Lee (1998) are consistent against a particular case of this class of ERCA models.

Keywords: Locally Best Invariant Test; Consistency; Dickey-Fuller Test; LBI; RCA; STUR

JEL classification: C12

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1 Introduction

The aim of this article is to show that a locally best invariant (hereafter, LBI) test in McCabe and Tremayne (1995), which is called the McCabe and Tremayne (hereafter, MT) test, is inconsistent against a class of explosive random coefficient autoregressive (hereafter, ERCA) models, developing the asymptotic theory of Hwang and Basawa (2005) for ERCA models.

The MT test is designed for the null of a UR process and against the alternative of a stochastic unit root (hereafter, STUR) process, which has recently gained increasing popularity in empirical financial literature. See for example, Bleaney *et al.* (1999), Bleaney and Leybourne (2003), and Sollis *et al.* (2000), among others. The MT test has been used to distinguish STUR processes from UR processes in the literature cited above. The class of ERCA models includes a class of STUR processes as special cases, and thus our results indicate that the MT test is inconsistent against this class of STUR processes (we give more detailed explanations about this class in Section 2). In contrast to the inconsistency of the MT test, we show that the well-known Dickey-Fuller unit root (DF–UR) tests and the LBI test of Lee (1998)(hereafter, the Lee test), which is for the null of a stationary AR(1) model and against the alternative of a stationary RCA(1) model, are consistent against a particular case of the class of ERCA models. [1]

The rest of this paper is organized as follows: In Section 2, we introduce ERCA and STUR models. In Section 3, we show that the MT test is inconsistent against a class of ERCA models while the DF–UR and Lee tests are consistent against a particular case of this class of ERCA models. Section 4 provides some concluding remarks.

2 The ERCA Model and Related Asymptotic Theory

2.1 ERCA and STUR models

We consider the following RCA(1) model defined on a probability space $(\Omega,\mathcal{F},P)$:

$$X_t = (\phi + \phi_t)X_{t-1} + \varepsilon_t, \quad \text{for} \quad t = 1, 2, ..., \quad X_0 = 0, \tag{1}$$

where $\{\varepsilon_t\}$ and $\{\phi_t\}$ are mutually independent i.i.d.random sequences with $E(\phi_t) = E(\varepsilon_t) = 0$, $E(\phi_t^4) < \infty$, $E(\varepsilon_t^4) < \infty$, $\text{var}(\phi_t) = \sigma_\phi^2$, $\text{var}(\varepsilon_t) = \sigma_\varepsilon^2$ and $\text{var}(\varepsilon_t^2) = \kappa_\varepsilon^2$. When $\phi = 1$, McCabe and Tremayne (1995) call the resulting model a *randomized unit root process*, whereas Granger and Swanson (1997) term the same model a STUR process. Following the latter, we refer to this model as a STUR process.

Define $\tau = \{E[(\phi + \phi_t)^2]\}^{1/2} = (\phi^2 + \sigma_\phi^2)^{1/2}$.[2] After simple calculations, we have, for $n = 1, 2, ...$, $\text{var}(X_n) = (\tau^{2n} - 1)(\tau^2 - 1)^{-1}\sigma_\varepsilon^2$ for $\tau \neq 1$ and $\text{var}(X_n) = n\sigma_\varepsilon^2$ for $\tau = 1$. Note that the variance of X_t increases exponentially when $\tau^2 > 1$, and increases linearly when $\tau^2 = 1$. The RCA(1) model is called the first order ERCA, or ERCA(1) model when $\tau^2 > 1$ (Hwang and Basawa, 2005). STUR processes with $\sigma_\phi^2 > 0$ are special cases of the ERCA(1) model because we have $\tau^2 > 1$ for these STUR processes. Let $\eta \equiv E(\log |\phi + \phi_t|)$. Hwang and Basawa (2005) classify the ERCA(1) model into two subclasses depending on whether the value of η is less than 0. We say that an ERCA(1) model belongs to the class S_1 if it satisfies $\eta < 0$; otherwise it belongs to the class S_2. Of course, any ERCA(1) model belongs to $S = S_1 \cup S_2$. This classification is motivated by properties of the two–sided version of X_t, namely, $X_t = (\phi + \phi_t)X_{t-1} + \varepsilon_t$ for

[1] It should be noted that the results in this paper are largely of theoretical interest and have limited practical relevance to the modeling of actual time series data for the same reason that explosive AR(1) models are rarely applied to actual time series data. The ERCA model typically exhibits explosive behavior similar to that of explosive AR(1) models and this kind of behavior is rarely observed for actual time series data. Recently; however, several studies use explosive AR(1) models for modeling stock bubble behavior.

[2] Our definition for τ is slightly different from that of Hwang and Basawa (2005).

$0, \pm 1, \pm 2, \ldots$. When $\tau^2 < 1$, the two–sided RCA(1) model is strictly stationary and ergodic with finite second moment(Nicholls and Quinn, 1982). The two–sided RCA(1) model is strictly stationary and ergodic if $\eta < 0$ and only if $\eta \leq 0$ (Quinn, 1982). Note that a two–sided STUR process with $\sigma_\phi^2 > 0$ may satisfy that $\eta < 0$, depending on the distribution of ϕ_t.[3] In this case, the two–sided STUR process, which does not satisfy the condition for the existence of a finite second moment, is strictly stationary and ergodic with an infinite second moment. These results for the two–sided process carries over to the one-sided process if X_0 follows the stationary distribution for the two–sided process (and hence the condition $X_0 = 0$ is asymptotically irrelevant). This feature sharpens the contrast between a STUR process and an autoregressive UR process, which is non-stationary. For the estimation and properties of the RCA(1) model with $\eta < 0$ and $\eta \geq 0$, see Aue *et al.* (2006) and Berkes *et al.* (2009), respectively.

2.2 Asymptotic theory for ERCA(1) models

Let $Z_n \equiv \tau^{-n} X_n$, $n = 0, 1, 2, \ldots$. Hwang and Basawa (2005, Theorem 1, p.811, and Lemma 1, p.812) show that, for the ERCA(1) model, as $n \to \infty$:

$$\begin{array}{ll} \text{(i)} & Z_n^2 \xrightarrow{a.s.} Z^2, \text{where } Z^2 \text{ is a random variable such that } E(Z^2) < \infty. \\ \text{(ii)} & \tau^{-2n} \sum_{t=1}^n X_{t-1}^2 \xrightarrow{a.s.} (\tau^2 - 1)^{-1} Z^2. \\ \text{(iii)} & \text{If } \Pr(Z^2 > 0) = 1, \text{ then } \Pr[(\phi + \phi_t)^2 = \tau^2] = 1. \end{array} \tag{2}$$

First, we extend the results in (2) in Proposition 1. Hereafter, the summation is taken from $t = 1$ to n unless otherwise stated.

Proposition 1 *Let $\{\theta_t\}$ be a sequence of i.i.d.real-valued random variables with $E(|\theta_t|) < \infty$. Assume that $\{X_t\}$ follows the RCA(1) model defined as in (1) with $\tau^2 > 1$ and $\sigma_\phi^2 > 0$. Then, for real numbers $r \geq 0$ and $\lambda > 0$, we have, as $n \to \infty$:*

$$\begin{array}{ll} (a) & |Z_n|^r \xrightarrow{a.s.} |Z|^r, \\ (b) & \tau^{-rn} \sum \theta_t |X_{t-1}|^r - |Z|^r \tau^{-rn} \sum \tau^{r(t-1)} \theta_t \xrightarrow{a.s.} 0, \\ (c) & \tau^{-(r+\lambda)n} \sum \theta_t X_{t-1}^r \xrightarrow{a.s.} 0. \end{array} \tag{3}$$

Remark 1 Setting $\theta_t = 1$ in the result (b) in (3), one obtains

$$\tau^{-rn} \sum |X_{t-1}|^r \xrightarrow{a.s.} (\tau^r - 1)^{-1} |Z|^r. \tag{4}$$

The result (ii) in (2) is a special case of (4) with $r = 2$.

We use the following lemma to prove Proposition 1:

Lemma 1 *Let r and τ be real numbers such that $r > 0$ and $\tau > 1$. Let $\{\theta_t\}$ be a sequence of i.i.d.real-valued random variables with $E(|\theta_t|) < \infty$. Define a random variable S_n so that $S_n \equiv \tau^{-rn} \sum \tau^{r(t-1)} \theta_t$. Then, $S_n \xrightarrow{a.s.} S$ as $n \to \infty$, where S is a real-valued random variable such that $S = O_p(1)$, i.e., S is bounded in probability. If* $\Pr(\theta_t \geq 0) = 1$ *and* $\Pr(\theta_t = 0) < 1$, *then* $\Pr(S > 0) = 1$.

Proof It is easy to check that, $S_2, \ldots, S_n$ satisfy that $S_n = \tau^{-r} S_{n-1} + \tau^{-r} \theta_n$ with $S_1 = \tau^{-r} \theta_1$, i.e., S_n follows an asymptotically stationary AR(1) process. Hence, as $n \to \infty$, S_n converges almost surely to its stationary solution S with mean $E(\theta_t)/(\tau^r - 1) < \infty$ (see, for example,

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[3]For example, one can show that if $\phi_t \sim N(0, \sigma_\phi^2)$ with $\sigma_\phi^2 < 2.4212$, then $\eta < 0$. The proof is available from the author upon request.

Brandt, 1986). Because the mean is finite, S is $O_p(1)$. When $\Pr(\theta_t \geq 0) = 1$, we obviously have $\Pr(S \geq 0) = 1$. Moreover, we have

$$\begin{aligned} \Pr(S = 0) = & \ \Pr\left(\lim_{n\to\infty} S_n = 0\right) \\ = & \ \Pr\left(\theta_t = 0 \text{ for all } t \geq s\right) \text{ for some } 1 \leq s \\ = & \ \lim_{n\to\infty} \textstyle\prod_{t=s}^{n} \Pr(\theta_t = 0) \\ = & \ 0. \end{aligned} \tag{5}$$

Thus, we have $\Pr(S > 0) = 1$. □

Here, we prove Proposition 1 based on the result (i) in (2) and Lemma 1.

Proof of Proposition 1 Part (a) follows from the result (i) and the continuous mapping theorem (see van der Vaart, 1998, Theorem 2.3, p.7). The proof of part (b) is similar to that of Theorem 1(ii) in Hwang and Basawa (2005). Following Hwang and Basawa (2005), we repeatedly use the Toeplitz lemma.[4] Let $\{a_t\}$ be a sequence of nonnegative numbers and x_n be an arbitrary converging sequence such that $x_n \to x$ as $n \to \infty$. The Toeplitz lemma states that if $\sum a_t \to \infty$ as $n \to \infty$, then $(\sum a_t)^{-1} \sum a_t x_t \to x$ as $n \to \infty$.

Because the case $r = 0$ is trivial, we consider only the case $r > 0$. The case $\Pr(\theta_t = 0) = 1$ is also trivial, and hence, hereafter, we assume that $\Pr(\theta_t = 0) < 1$. In what follows, we first prove part (b), then prove part (c). We have

$$\left|\tau^{-rn} \sum \theta_t |X_{t-1}|^r - \tau^{-rn} \sum \tau^{r(t-1)} \theta_t |Z|^r\right| \ \leq \tau^{-rn} \sum \tau^{r(t-1)} |\theta_t| \ |u_t|,$$

where $u_t \equiv \tau^{-r(t-1)}|X_{t-1}|^r - |Z|^r$. From part (a), $u_t \xrightarrow{a.s.} 0$ as $t \to \infty$. In other words, $u_t(w) \to 0$ as $t \to \infty$ for almost every $\omega \in \Omega$. From Lemma 1, we have that $\sum \tau^{r(t-1)}|\theta_t(w)| \to \infty$ for almost every $\omega \in \Omega$. Thus, it follows from the Toeplitz lemma that

$$\left(\sum \tau^{r(t-1)} |\theta_t(w)|\right)^{-1} \sum \tau^{r(t-1)} |\theta_t(w)| \ |u_t(w)| \longrightarrow 0, \text{ as } n \to \infty, \tag{6}$$

for almost every $\omega \in \Omega$, or $(\sum \tau^{r(t-1)}|\theta_t|)^{-1} \sum \tau^{r(t-1)}|\theta_t||u_t| \xrightarrow{a.s.} 0$. From (6) and Lemma 1, we have, as $n \to \infty$, that

$$\begin{aligned} \tau^{-rn} \sum \tau^{r(t-1)} |\theta_t||u_t| & = \left(\tau^{-rn} \sum \tau^{r(t-1)} |\theta_t|\right) \left(\sum \tau^{r(t-1)} |\theta_t|\right)^{-1} \sum \tau^{r(t-1)} |\theta_t||u_t| \\ & \xrightarrow{a.s.} 0, \end{aligned}$$

which completes the proof of part (b).

Next, we prove part (c). From Lemma 1, we have, as $n \to \infty$,

$$\tau^{-\lambda n} \tau^{-rn} \sum \tau^{r(t-1)} |\theta_t| \xrightarrow{a.s.} 0. \tag{7}$$

From (7), parts (a) and (b), we have

$$\begin{aligned} \left|\tau^{-\lambda n} \tau^{-rn} \sum \theta_t X_{t-1}^r\right| \leq & \ \tau^{-\lambda n} \tau^{-rn} \sum |\theta_t||X_{t-1}|^r \\ = & \ \tau^{-\lambda n} \left(\tau^{-rn} \sum |\theta_t||X_{t-1}|^r - |Z|^r \tau^{-rn} \sum |\theta_t| \tau^{r(t-1)}\right) \\ & + |Z|^r \tau^{-\lambda n} \tau^{-rn} \sum \tau^{r(t-1)} |\theta_t| \\ \xrightarrow{a.s.} & \ 0, \end{aligned} \tag{8}$$

which completes the proof of part (c). □

[4] See, for example, van der Vaart (1998, Problem 6, p.137) and Shorack (2000, p.205).

3 MT, DF–UR, and Lee Tests

3.1 Definition of the MT test

McCabe and Tremayne (1995) derive a LBI test for the null of $\sigma_\phi^2 = 0$ against the alternative of $\sigma_\phi^2 > 0$ assuming that $\phi = 1$ in (1), i.e., the null hypothesis is a UR process and the alternative hypothesis is a STUR process. The test has been called the MT test. Specifically, we consider the MT test in Corollary 3 of McCabe and Tremayne (1995, p.1021), in which the nuisance parameters σ_ε^2 and κ_ε^2 are replaced by their natural estimators under the null hypothesis. Given n observations, $X_1, ..., X_n$, the MT test statistic MT_n is defined as

$$MT_n \equiv \widetilde{\kappa}_{\varepsilon,n}^{-1}\widetilde{\sigma}_{\varepsilon,n}^{-2}n^{-3/2}\sum X_{t-1}^2\left[(\Delta X_t)^2 - \widetilde{\sigma}_{\varepsilon,n}^2\right], \tag{9}$$

where $\Delta X_t = X_t - X_{t-1}$, $\widetilde{\sigma}_{\varepsilon,n}^2 = n^{-1}\sum(\Delta X_t)^2$ and $\widetilde{\kappa}_{\varepsilon,n} = [n^{-1}\sum(\Delta X_t)^4 - \widetilde{\sigma}_{\varepsilon,n}^4]^{1/2}$. Because $\Delta X_t = \varepsilon_t$ under the null hypothesis, one can interpret the MT test as examining whether or not the covariance between X_{t-1}^2 and ε_t^2 is zero. The asymptotic null distribution of the MT test is non-standard and its critical values are tabulated in Table 1 in McCabe and Tremayne (1995).

3.2 Inconsistency of the MT test against a class of ERCA(1) models

We assume the next condition, which is the same as Condition (C2) in Hwang and Basawa (2005), namely,

(C1) The limiting random variable Z^2 satisfies that $\Pr(Z^2 > 0) > 0$.

Condition (C1) is motivated by the result (iii) in (2), which implies that if $\Pr(Z^2 > 0) = 1$, then a random variable $\phi + \phi_t$ must be a binary random variable that takes τ or $-\tau$. Furthermore, Hwang and Basawa (2005) show that if $\eta < 0$, then $\Pr(Z^2 > 0) = 0$. Hence, Condition (C1) automatically excludes the case of $\eta < 0$. Because of these reasons, Hwang and Basawa (2005) argue that it is reasonable to conduct an analysis under (C1) for ERCA(1) models belonging to S_2, i.e., ERCA(1) models with $\eta \geq 0$.

Let $\Pr^*$ denote the conditional probability measure conditioned on $\{Z^2 > 0\}$, i.e., $\Pr^*(\cdot) = \Pr(\cdot|Z^2 > 0)$. We use notations " $\overset{a.s.(*)}{\longrightarrow}$ " and "$O_p^*(1)$" to emphasize that those statements are made under $\Pr^*$. Whenever we use these notations, it is implicit that the statements are conditioned on $Z^2 > 0$.

To prove the inconsistency of the MT test we use the following lemma, which shows that the results in Proposition 1 and Lemma 1 hold under the conditional probability measure.

Lemma 2 *Assume the same conditions as in Lemma 1 and Proposition 1. Additionally, assume that Condition (C1) is satisfied. Then, all results in Lemma 1 and Proposition 1 hold under* $\Pr^*$*, that is, as* $n \to \infty$*,*

for $r \geq 0$ *and* $\lambda > 0$,
(a) $|Z_n|^r \overset{a.s.(*)}{\longrightarrow} |Z|^r$, (b) $\tau^{-rn}\sum\theta_t|X_{t-1}|^r - |Z|^r\tau^{-rn}\sum\theta_t\tau^{r(t-1)} \overset{a.s.(*)}{\longrightarrow} 0$,
(c) $\tau^{-(r+\lambda)n}\sum\theta_t X_{t-1}^r \overset{a.s.(*)}{\longrightarrow} 0$,
for $r > 0$,
(d) $\tau^{-rn}\sum\theta_t\tau^{r(t-1)} = S_n \overset{a.s.(*)}{\longrightarrow} S$ *such that* $S = O_p^*(1)$,
(e) *If* $\Pr(\theta_t \geq 0) = 1$ *and* $\Pr(\theta_t = 0) < 1$, *then* $\Pr^*(S > 0) = 1$.

Proof For an event E, if $\Pr(E) = 1$ and $\Pr(Z^2 > 0) > 0$, we must have $\Pr(E|Z^2 > 0) = 1$ because $\Pr(E) = \Pr(Z^2 > 0)\Pr(E|Z^2 > 0) + \Pr(Z^2 = 0)\Pr(E|Z^2 = 0)$ and $\Pr(Z^2 > 0) +$

$\Pr(Z^2 = 0) = 1$. Thus, parts (a) $\sim$ (c) follow from the definition of almost sure convergence and Lemma 1 and Proposition 1. It is easy to prove parts (d) and (e) by applying similar arguments. $\square$

From the argument in the above proof, it is clear that $\xrightarrow{a.s.}$ implies $\xrightarrow{a.s.(*)}$. However, the converse is in general not true.

Now, we are ready to show the inconsistency of the MT test against ERCA(1) models that satisfy Condition (C1).

Proposition 2 *Assume that $\{X_t\}$ follows the RCA(1) model defined as in (1) with $\tau^2 > 1$ and $\sigma_\phi^2 > 0$. Assume that Condition (C1) holds for the limiting random variable Z^2. Then, $\lim_{n\to\infty} \Pr(|MT_n| > M) \neq 1$ for any real number M.*

Remark 2 Proposition 2 implies that the MT test defined in (9) is not consistent, at any nominal level, against ERCA(1) models that satisfy Condition (C1). It, in turn, implies that the MT test is inconsistent against STUR processes that belong to this class of ERCA(1) models.

Proof Consistency of the MT test against an alternative implies that

$$\lim_{n\to\infty} \Pr(|MT_n| > M) = 1. \tag{10}$$

Note that

$$\begin{aligned}\lim_{n\to\infty} \Pr(|MT_n| > M) = \ & \Pr(Z^2 > 0) \lim_{n\to\infty} \Pr(|MT_n| > M|Z^2 > 0) \\ & + \Pr(Z^2 = 0) \lim_{n\to\infty} \Pr(|MT_n| > M|Z^2 = 0).\end{aligned} \tag{11}$$

Thus, if $\lim_{n\to\infty} \Pr(|MT_n| > M) = 1$, then we must have $\lim_{n\to\infty} \Pr(|MT_n| > M|Z^2 > 0) = 1$ because $\Pr(Z^2 > 0) + \Pr(Z^2 = 0) = 1$. However, we will show that $\lim_{n\to\infty} \Pr(|MT_n| > M|Z^2 > 0) \neq 1$ when the underlying data generating process is an ERCA(1) model that satisfies Condition (C1).

Write the test statistic MT_n defined in (9) as

$$MT_n = A_n B_n^{-1} C_n^{-1/2}, \tag{12}$$

where $A_n \equiv \tau^{-4n} \sum X_{t-1}^2 \left[(\Delta X_t)^2 - \tilde{\sigma}_{\varepsilon,n}^2\right]$, $B_n \equiv \tau^{-2n} \sum (\Delta X_t)^2$ and $C_n \equiv \tau^{-4n} \sum (\Delta X_t)^4 - n^{-1} B_n^2$. Let $\psi_t \equiv \phi + \phi_t - 1$. Then B_n is rewritten as

$$B_n = \tau^{-2n} \sum (\psi_t X_{t-1} + \varepsilon_t)^2 = \tau^{-2n} \sum \psi_t^2 X_{t-1}^2 + 2\tau^{-2n} \sum X_{t-1}\varepsilon_t\psi_t + \tau^{-2n} \sum \varepsilon_t^2. \tag{13}$$

The assumptions in (1) imply that $E(\psi_t^k \varepsilon_t^j) = E(\psi_t^k) E(\varepsilon_t^j) < \infty$ for $0 \leq k \leq 4$ and $0 \leq j \leq 4$. Hence, Lemma 2(d) implies that there exists a limiting random variable, $S_{r,k}$, such that, as $n \to \infty$,

$$\tau^{-rn} \sum \psi_t^k \tau^{r(t-1)} \xrightarrow{a.s.(*)} S_{r,k} \quad \text{for} \quad r > 0 \quad \text{and} \quad 0 \leq k \leq 4. \tag{14}$$

(Note that when $k = 0$, $S_{r,k}$ is degenerated at $(\tau^r - 1)^{-1}$). From (14) and Lemma 2(b), we have, as $n \to \infty$,

$$\tau^{-rn} \sum \psi_t^k |X_{t-1}|^r \xrightarrow{a.s.(*)} |Z|^r S_{r,k}. \tag{15}$$

From (15), the first term in (13) converges so that

$$\tau^{-2n} \sum \psi_t^2 X_{t-1}^2 \xrightarrow{a.s.(*)} Z^2 S_{2,2}. \tag{16}$$

From Lemma 2(c), the second term in (13) converges almost surely $\left(\xrightarrow{a.s.(*)}\right)$ to zero. Because $\tau > 1$ and ε_t^2 is i.i.d.with $E(\varepsilon_t^2) = \sigma_\varepsilon^2 < \infty$, we have $\tau^{-2n}\sum \varepsilon_t^2 = (\tau^{-2n}n)(n^{-1}\sum \varepsilon_t^2) \xrightarrow{a.s.}$ $\left(\text{and hence } \xrightarrow{a.s.(*)}\right)$ 0. Thus, we have

$$B_n \xrightarrow{a.s.(*)} Z^2 S_{2,2}. \tag{17}$$

From (17), we immediately see that the second term of C_n, $n^{-1}B_n^2$, converges almost surely $\left(\xrightarrow{a.s.(*)}\right)$ to zero.

At this point, it is convenient to consider the convergence of $\tau^{-(j+k)n}\sum X_{t-1}^j(\Delta X_t)^k$, where j and k are integers such that $0 \le j \le 4$, $0 \le k \le 4$, $j+k$ are even and $2 \le j+k$. By the binomial theorem, we have

$$\begin{aligned}\tau^{-(j+k)n}\sum X_{t-1}^j(\Delta X_t)^k &= \tau^{-(j+k)n}\sum X_{t-1}^j(\psi_t X_{t-1}+\varepsilon_t)^k \\ &= \tau^{-(j+k)n}\sum \psi_t^k X_{t-1}^{j+k} + k\tau^{-(j+k)n}\sum X_{t-1}^{j+k-1}\psi_t^{k-1}\varepsilon_t \\ &\quad + \cdots + \tau^{-(j+k)n}\sum X_{t-1}^j \varepsilon_t^k.\end{aligned} \tag{18}$$

From Lemma 2(c), all terms, except for the first term, on the right-hand side in (18) converge almost surely to zero. Hence, from (15), we have

$$\tau^{-(j+k)n}\sum X_{t-1}^j(\Delta X_t)^k \xrightarrow{a.s.(*)} Z^{j+k}S_{j+k,k}. \tag{19}$$

The result in (17) is a special case of (19) with $j=0$ and $k=2$.

From (17) and (19), it readily follows that

$$C_n \xrightarrow{a.s.(*)} Z^4 S_{4,4}. \tag{20}$$

and

$$\begin{aligned} A_n &= \tau^{-4n}\sum X_{t-1}^2(\Delta X_t)^2 - \tau^{-2n}\widetilde{\sigma}_{\varepsilon,n}^2\tau^{-2n}\sum X_{t-1}^2 \\ &\xrightarrow{a.s.(*)} Z^4 S_{4,2}. \end{aligned} \tag{21}$$

From (12), (17), (20) and (21), we have

$$MT_n \xrightarrow{a.s.(*)} S_{4,2}S_{2,2}^{-1}S_{4,4}^{-1/2}, \tag{22}$$

which is $O_p^*(1)$. $\square$

3.3 Consistency of the DF–UR and Lee Tests

Let $\widehat{\phi}_n \equiv \left(\sum X_{t-1}^2\right)^{-1}\sum X_t X_{t-1}$, namely, $\widehat{\phi}_n$ is the OLS estimator for ϕ under the null hypothesis of $\sigma_\phi^2 = 0$. We consider the two well–known DF–UR tests, which are proposed by Dickey and Fuller (1979). The first one is defined as $DF_{\phi,n} \equiv n(\widehat{\phi}_n - 1)$. We call this test the "DF ϕ test." The second one is defined as $DF_{t,n} \equiv \widehat{\sigma}_{\varepsilon,n}^{-1}\left[\left(\widehat{\phi}_n - 1\right)\left(\sum X_{t-1}^2\right)^{1/2}\right]$, which is the t test of the OLS estimator, where $\widehat{\sigma}_{\varepsilon,n}$ is the OLS estimate of σ_ε, defined as $\widehat{\sigma}_{\varepsilon,n} \equiv [n^{-1}\sum \widehat{\varepsilon}_{t,n}^2]^{1/2}$ and $\widehat{\varepsilon}_{t,n} \equiv X_t - \widehat{\phi}_n X_{t-1}$.[5] We call this test the "DF t test." Note that here we use the OLS estimate of ϕ unlike the MT test.

[5] Alternatively, $\widehat{\sigma}_{\varepsilon,n}$ is often defined as $[(n-1)^{-1}\sum \widehat{\varepsilon}_{t,n}^2]^{1/2}$. These two definitions are asymptotically equivalent.

We also consider the Lee test, which is proposed by Lee (1998). The Lee test statistic L_n is defined as[6]

$$L_n \equiv \widehat{\kappa}_{\varepsilon,n}^{-1}\widetilde{\nu}_n^{-1}n^{-1/2}\sum X_{t-1}^2(\widehat{\varepsilon}_{t,n}^2 - \widehat{\sigma}_{\varepsilon,n}^2),$$

where $\widetilde{\nu}_n \equiv [n^{-1}\sum X_{t-1}^4 - (n^{-1}\sum X_{t-1}^2)^2]^{1/2}$ and $\widehat{\kappa}_{\varepsilon,n} \equiv [n^{-1}\sum \widehat{\varepsilon}_{t,n}^4 - \widehat{\sigma}_{\varepsilon,n}^4]^{1/2}$. The Lee test is not a test for a UR; however, it is closely related to the MT test. The Lee test is derived as a LBI test for the null of a stationary AR(1) model against the alternative of a stationary RCA(1) model and can be used as a complement of the MT test (see Nagakura (2009) for more details). Thus, it is of interest to investigate the behavior of the Lee test when the true data generating process follows the ERCA(1) model.

Throughout this subsection, we assume the following condition:

(C2) The limiting random variable Z^2 satisfies that $\Pr(Z^2 > 0) = 1$.

The result (iii) in (2) implies that, under this condition, a random variable $\phi + \phi_t$ is a binary random variable such that $\phi+\phi_t = \tau$ with probability α and $\phi+\phi_t = -\tau$ with probability $1-\alpha$, where $0 < \alpha < 1$. In this case, the conditions $\tau < 1$, $\tau = 1$ and $\tau > 1$ are equivalent to $\eta < 0$, $\eta = 0$ and $\eta > 0$, respectively. Note that, under Condition (C2), there is no difference between $\xrightarrow{a.s.(*)}$ and $\xrightarrow{a.s.}$.

Hwang *et al.* (2006) consider the above binary RCA(1) model with $\tau = 1$. With the additional assumption that ε_t is symmetrically distributed, Hwang *et al.* (2006) show that $\widehat{\phi}_n$ is a consistent estimator for $E(\phi + \phi_t) = 2\alpha - 1$. They also show the asymptotic normality of a weighted least square estimator and propose a test regarding the criticality parameter τ. We consider the case of $\tau > 1$. For the usual RCA(1) model in (1) with $\tau > 1$, Hwang and Basawa (2005) has shown that $\widehat{\phi}_n$ is an inconsistent estimator for ϕ. Our result in (23) complements the results of Hwang and Basawa (2005) and Hwang *et al.* (2006) by showing the asymptotic distribution of $\widehat{\phi}_n$ for the binary RCA(1) model with $\tau > 1$.

Proposition 2 implies that the MT test is inconsistent even against the ERCA(1) models that satisfy Condition (C2). By contrast, Proposition 3 below shows that the two DF–UR and Lee tests are, at any nominal level, consistent against the ERCA(1) models.

Proposition 3 *Assume that $\{X_t\}$ follows the RCA(1) model defined in (1) with $\tau^2 > 1$ and $\sigma_\phi > 0$. Assume that Condition (C2) holds for the limiting random variable Z^2. Then, $\lim_{n\to\infty} \Pr(|DF_{\phi,n}| > M) = 1$, $\lim_{n\to\infty} \Pr(|DF_{t,n}| > M) = 1$ and $\lim_{n\to\infty} \Pr(|L_n| > M) = 1$ for any real number M.*

Proof
(DF ϕ test) From (19), it follows that

$$n^{-1}DF_{\phi,n} = \widehat{\phi}_n - 1 = \left(\tau^{-2n}\sum X_{t-1}^2\right)^{-1}\left(\tau^{-2n}\sum X_{t-1}\Delta X_t\right) \xrightarrow{a.s.} (\tau^2-1)S_{2,1}, \tag{23}$$

which completes the proof of the first part of Proposition 3.
(DF t test) From (19) and (23), we have

$$\begin{aligned}\tau^{-2n}\sum\widehat{\varepsilon}_{t,n}^2 &= \tau^{-2n}\sum[\Delta X_t - (\widehat{\phi}_n - 1)X_{t-1}]^2\\ &= \tau^{-2n}\sum(\Delta X_t)^2 + (\widehat{\phi}_n-1)^2\tau^{-2n}\sum X_{t-1}^2 - 2(\widehat{\phi}_n-1)\tau^{-2n}\sum X_{t-1}\Delta X_{t-1}\\ &\xrightarrow{a.s.} Z^2S_{2,2} + (\tau^2-1)^2S_{2,1}^2(\tau^2-1)^{-1}Z^2 - 2(\tau^2-1)S_{2,1}Z^2S_{2,1}\\ &= Z^2S_{2,2} - (\tau^2-1)Z^2S_{2,1}^2.\end{aligned} \tag{24}$$

[6]We consider a simplified form of the Lee test, considered in Nagakura (2009), that ignores asymptotically negligible terms.

Thus, we have

$$\begin{aligned} n^{-1/2}DF_{t,n} &= (\widehat{\phi}_n - 1)\left(\tau^{-2n}\sum X_{t-1}^2\right)^{1/2}\left[\tau^{-2n}\sum \widehat{\varepsilon}_t^2\right]^{-1/2} \\ &\xrightarrow{a.s.} (\tau^2-1)^{1/2}S_{2,1}\left[S_{2,2} - (\tau^2-1)S_{2,1}^2\right]^{-1/2} \end{aligned} \tag{25}$$

which completes the proof of the second part of Proposition 3.
(Lee test) L_n is rewritten as

$$L_n = n^{1/2}\widehat{A}_n\widehat{C}_n^{-1/2}\widehat{D}_n^{-1/2}, \tag{26}$$

where $\widehat{A}_n \equiv \tau^{-4n}\sum X_{t-1}^2[\widehat{\varepsilon}_{t,n}^2 - \widehat{\sigma}_{\varepsilon,n}^2]$, $\widehat{C}_n \equiv \tau^{-4n}\sum \widehat{\varepsilon}_{t,n}^4 - n^{-1}\widehat{B}_n^2$, $\widehat{D}_n \equiv \tau^{-4n}\sum X_{t-1}^4 - n^{-1}(\tau^{-2n}\sum X_{t-1}^2)^2$, and $\widehat{B}_n \equiv \tau^{-2n}\sum\widehat{\varepsilon}_{t,n}^2$. From (19) and (24), we have

$$\widehat{D}_n \xrightarrow{a.s.} (\tau^4-1)^{-1}Z^4. \tag{27}$$

From (19) and (23), we have

$$\begin{aligned} \tau^{-4n}\sum\widehat{\varepsilon}_{t,n}^4 &= \tau^{-4n}\sum[\Delta X_t - (\widehat{\phi}_n-1)X_{t-1}]^4 \\ &= \tau^{-4n}\sum(\Delta X_t)^4 - 4(\widehat{\phi}_n-1)\tau^{-4n}\sum X_{t-1}(\Delta X_t)^3 + 6(\widehat{\phi}_n-1)^2\tau^{-4n}\sum X_{t-1}^2(\Delta X_t)^2 \\ &\quad -4(\widehat{\phi}_n-1)^3\tau^{-4n}\sum X_{t-1}^3(\Delta X_t) + (\widehat{\phi}_n-1)^4\tau^{-4n}\sum X_{t-1}^4, \\ &\xrightarrow{a.s.} Z^4S_{4,4} - 4(\tau^2-1)S_{2,1}Z^4S_{4,3} + 6(\tau^2-1)^2S_{2,1}^2Z^4S_{4,2} \\ &\quad -4(\tau^2-1)^3S_{2,1}^3Z^4S_{4,1} + (\tau^2-1)^4S_{2,1}^4(\tau^4-1)^{-1}Z^4. \end{aligned}$$

Thus, because $n^{-1}\widehat{B}_n \xrightarrow{a.s.} 0$ by (24), we have

$$\begin{aligned} \widehat{C}_n \xrightarrow{a.s.}\ & Z^4[S_{4,4} - 4(\tau^2-1)S_{2,1}S_{4,3} + 6(\tau^2-1)^2S_{2,1}^2S_{4,2} \\ & -4(\tau^2-1)^3S_{2,1}^3S_{4,1} + (\tau^2-1)^4(\tau^4-1)^{-1}S_{2,1}^4]. \end{aligned} \tag{28}$$

From (19), we have

$$\begin{aligned} \widehat{A}_n &= \tau^{-4n}\sum X_{t-1}^2\{[\Delta X_t - (\widehat{\phi}_n-1)X_{t-1}]^2 - \widehat{\sigma}_{\varepsilon,n}^2\} \\ &= \tau^{-4n}\sum X_{t-1}^2(\Delta X_t)^2 - 2(\widehat{\phi}_n-1)\tau^{-4n}\sum X_{t-1}^3\Delta X_t \\ &\quad +(\widehat{\phi}_n-1)^2\tau^{-4n}\sum X_{t-1}^4 - \tau^{-2n}\widehat{\sigma}_{\varepsilon,n}^2\tau^{-2n}\sum X_{t-1}^2 \\ &\xrightarrow{a.s.} Z^4S_{4,2} - 2(\tau^2-1)S_{2,1}Z^4S_{4,1} + (\tau^2-1)^2S_{2,1}^2(\tau^4-1)^{-1}Z^4. \end{aligned} \tag{29}$$

From (26), (27), (28), and (29), we eventually have

$$n^{-1/2}L_n \xrightarrow{a.s.} \frac{(\tau^4-1)^{1/2}\{S_{4,2} - 2(\tau^2-1)S_{2,1}S_{4,1} + (\tau^2-1)^2(\tau^4-1)^{-1}S_{2,1}^2\}}{[S_{4,4} - 4(\tau^2-1)S_{2,1}S_{4,3} + 6(\tau^2-1)^2S_{2,1}^2S_{4,2} - 4(\tau^2-1)^3S_{2,1}^3S_{4,1} + (\tau^2-1)^4(\tau^4-1)^{-1}S_{2,1}^4]^{1/2}},$$

which completes the proof of the last part of Proposition 3. □

4 Simulation

In this section, we conduct a simulation experiment to confirm Propositions 2 and 3. Let $\rho_t = \phi + \phi_t$. We generate samples from a binary RCA model, namely, $\rho_t = \tau$ with probability α and $\rho_t = -\tau$ with probability $1-\alpha$. Here, we set $\tau = 1.1$. We examine three cases of $\alpha = 0.2$, 0.5, and 0.7. The number of samples is set at $n = 200$, 500, or 1000. The number of replications is $10,000$. We perform all tests at the nominal level of 5 %.[7]

Table 1 reports the results. One can see that in fact, MT test does not have power even when $n = 1000$, wheras DF-ϕ, DF-t, and Lee tests increases their powers as n increases, which implies that they are consistent tests. When we normalize the statistics as the proof of Proposition 3 suggests, then the powers of DF-ϕ, DF-t, and Lee tests do not increases, which is consistent with Proposition 3.

[7]For the nominal level of 5%, the critical values of the MT, DF ϕ, DF t, and Lee tests are 0.81, −8.1, −1.95, and 1.64, respectively, which are taken from Table 1 in McCabe and Tremayne (1995), Tables B.5 and B.6 in Hamilton (1994), and the standard normal distribution table, respectively. The rejection regions of the MT and Lee tests are in the upper tail areas, while the rejection regions of the both DF tests are lower tail areas.

Table 1: Rejection Percentages (Power) at 5 % Nominal Level

	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.7$
$n = 200$			
MT_n	0.000	0.000	0.001
$DF_{\phi,n}$	1.000	1.000	0.970
$DF_{t,n}$	1.000	1.000	1.000
L_n	0.892	0.999	0.974
$n^{-1}DF_{\phi,n}$	0.000	0.000	0.000
$n^{-1/2}DF_{t,n}$	0.000	0.000	0.000
$n^{-1}L_n$	0.000	0.000	0.000
$n = 500$			
MT_n	0.000	0.000	0.000
$DF_{\phi,n}$	1.000	1.000	0.981
$DF_{t,n}$	1.000	1.000	1.000
L_n	0.939	1.000	0.992
$n^{-1}DF_{\phi,n}$	0.000	0.000	0.000
$n^{-1/2}DF_{t,n}$	0.001	0.000	0.000
$n^{-1/2}L_n$	0.000	0.000	0.000
$n = 1000$			
MT_n	0.000	0.000	0.001
$DF_{\phi,n}$	1.000	1.000	0.982
$DF_{t,n}$	1.000	1.000	1.000
L_n	0.960	1.000	0.995
$n^{-1}DF_{\phi,n}$	0.000	0.000	0.000
$n^{-1/2}DF_{t,n}$	0.001	0.000	0.000
$n^{-1/2}L_n$	0.000	0.000	0.000

5 Concluding Remarks

In this note, we developed the asymptotic theory for ERCA(1) models considered in Hwang and Basawa (2005). Applying the results, we showed that a LBI test proposed by McCabe and Tremayne (1995) for the null of a UR process against the alternative of a STUR process is inconsistent against a class of ERCA(1) models to which a class of STUR processes belongs. We also showed that the DF–UR and Lee tests are consistent against a particular case of this class of ERCA(1) models. Lastly, it is worth remarking that the simulation results in McCabe and Tremayne (1995) and Nagakura (2009) demonstrate that the power of the MT test is very low and does not go to 1 as T increases against STUR processes with moderately large σ_ϕ, even when those STUR processes satisfy the condition $\eta < 0$. This implies that the condition $\eta \geq 0$ is not a necessary condition for the inconsistency of the MT test.

References

Aue, A., Horvath, L., and Steinebach, J. (2006), Estimation in Random Coefficient Autoregressive Models, *Journal of Time Series Analysis*, 27, 61-76.

Berkes, I., Horvath, L., and Shiqing, L. (2009), Estimation in Nonstationary Random Coefficient Autoregressive Models, *Journal of Time Series Analysis*, 30, 395-416.

Bleaney, M. F. and Leybourne, S. J. (2003), Real Exchange Rate Dynamics Under the Current Float: A Re-examination, *Manchester School*, 71, 156-171.

Bleaney, M. F. and Leybourne, S. J., and Mizen, P. (1999), Mean Reversion of Real Exchange Rates in High-inflation Countries, *Southern Economic Journal*, 65, 839-854.

Brandt, A. (1986), The Stochastic Equation $Y_{n+1} = A_n Y_n + B_n$ with Stationary Coefficients. *Advances in Applied Probability*, 18, 211-220.

Dickey, D. A. and Fuller, W. A. (1979), Distribution of the Estimators for Auto-regressive Time Series with a Unit Root, *Journal of the American Statistical Association*, 72, 427-31.

Granger, C. W. J. and Swanson, N. R. (1997), An Introduction to Stochastic Unit-root Processes, *Journal of Econometrics*, 80, 35-62.

Hamilton, J. D. (1994), *Time Series Analysis*, Princeton University Press, New Jersey.

Hwang, S. Y. and Basawa, I. V. (2005), Explosive Random-coefficient AR(1) Processes and Related Asymptotics for Least-squares Estimation, *Journal of Time Series Analysis*, 26, 807-824.

Hwang, S. Y., Basawa, I. V., and Kim, T. Y. (2006), Least Squares Estimation for Critical Random Coefficient First-order Autoregressive Processes, *Statistics and Probability Letters*, 76, 310-317.

Lee, S. (1998), Coefficient Constancy Test in a Random Coefficient Autoregressive Model, *Journal of Statistical Planning and Inference*, 74, 93-101.

McCabe, B. P. and Tremayne, A. R. (1995), Testing a Time Series for Difference Stationary, *Annals of Statistics, 23*, 1015-1028.

Nagakura, D. (2009), Testing for Coefficient Stability of AR(1) Model When the Null Is an Integrated or a Stationary Process, *Journal of Statistical Planning and Inference*, 139, 2731-2745.

Nicholls, D. F. and Quinn, B. G. (1982), *Random Coefficient Autoregressive Models: An Introduction*, Springer, New York.

Quinn, B. G. (1982), A Note on the Existence of Strictly Stationary Solutions to Bilinear Equations, *Journal of Time Series Analysis*, 3, 249-252.

Shorack, G. R. (2000), *Probability for Statisticians*, Springer, New York.

Sollis, R., Leybourne, S. J., and Newbold, P. (2000), Stochastic Unit Roots Modelling of Stock Price Indices. *Applied Financial Economics*, 10, 311-315.

van der Vaart, A. W. (1998), *Asymptotic Statistics*, Cambridge University Press, Cambridge.