# IMES DISCUSSION PAPER SERIES

Dynamic Model of Credit Risk in Relationship Lending:
A Game-theoretic Real Options Approach

Takashi Shibata and Tetsuya Yamada

**Discussion Paper No. 2009-E-7**

# IMES

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

# Dynamic Model of Credit Risk in Relationship Lending: A Game-theoretic Real Options Approach

**Takashi Shibata\* and Tetsuya Yamada\*\***

## Abstract

  We develop a dynamic credit risk model for the case that banks compete to collect their loans from a firm falling in danger of bankruptcy. We apply a game-theoretic real options approach to investigate bank's optimal strategies. Our model reveals that the bank with the larger loan amount, namely the main bank, provides an additional loan to support the deteriorating firm when the other bank collects its loan. This suggests that there exists rational forbearance lending by the main bank. Comparative statics show that as the liquidation value is lower, the optimal exit timing for the non-main bank comes at an earlier stage of business downturn and the optimal liquidation timing by the main bank is delayed further. As the interest rate of the loan is lower, the optimal exit timing for the non-main bank comes earlier. These analyses are consistent with the forbearance lending and exposure concentration of main banks observed in Japan.

# 1. Introduction

From the mid 1990s, Japanese banks struggled with nonperforming loan problems. The problems partly emerged from the risk embedded in relationship banking. The disadvantages and advantages of the relationship have been discussed in the banking theory literature (for details, see Boot [2000] and Elyasiani and Goldberg [2004]). As is well known, the first advantage is a reduction of inefficiency stemming from asymmetric information between a bank and a firm. The second is to facilitate an implicit long-term contract through risk sharing such that a bank maintains a stable loan interest rate even if the firm's credit risk fluctuates. These benefits, however, potentially turn into disadvantages. For example, monopolistic lending as a consequence of a long-term relationship between a bank and a firm gives rise to the hold-up problem, that is, the bank's strong bargaining power gives the firm an incentive to borrow from other banks. The firm, therefore, prefers multiple banking relationships despite additional administrative costs. Another disadvantage is the soft-budget problem, which comes from an implicit long-term contract. When Japanese banks struggled with the nonperforming loan problem, it was often cited as a typical example of the problem posed by Japanese banks' forbearance lending to deteriorating firms in order to avoid losses from firms' bankruptcy. Banks' arbitrary policies regarding their support for firms reduced their discipline for credit risk management. [1]

Multiple banking relationships in some countries are combined with a main bank system. In this system, one particular bank that holds the largest share of a firm's debt is defined as a main bank. The main bank faces the responsibility of monitoring the firm's condition, in return for holding the largest share of lending and providing other financial services to the firm. Within the main bank system, a soft-budget problem might be more serious. When the credit condition of the firm worsens, the non-main bank might collect its outstanding loans from the firm. Unless the main bank provides an additional loan to fill the shortage of firm's loan demand, the main bank immediately suffers from the firm's bankruptcy because of a lack of liquidity. We, therefore, name the main bank's additional loan instead of the non-main bank, "debt assumption." Forbearance lending, however, might lead to a large loss for the main bank in the future. An arbitrary lending policy in relationship banking is required for a certain decision rule. As for the non-

---

[1] The soft-budget problem highlights the difference between ex ante efficiency and ex post efficiency. In this paper, ex ante efficiency corresponds to the optimal exit strategy without thought for other banks' lending strategies as discussed later. The equilibrium of the game with consideration of counterparts' strategies represents ex post efficiency. Hence, "arbitrary policy", here, means the main bank's policy without consideration of the other banks' policies.

main bank, it also faces uncertainty regarding the exit timing from lending to the firm. The optimal timing is determined by the trade-off between an increase in credit risk and an opportunity to gain future earnings from their loan to the firm. In this paper, we propose a theoretical model to measure banks' credit risk in a game between a main bank and non-main bank concerning their exit timing and their decisions about the debt assumptions.

First, assuming that a firm borrows from one bank, we examine the optimal exit strategy from the lending to the firm. The strategy can be developed using real options theory using the stochastic process of firm value. This approach is developed by Leland [1994] and Mella-Barral and Perraudin [1997]. Real options theory pays attention to the bank's waiting option to collect its loan from the deteriorating firm considering the firm may avoid bankruptcy. It may be optimal for the bank to exit later. The real options model gives a threshold level of firm value for the bank's decision on exiting under uncertainty of firm value. Baba [2001] developed a theoretical model to investigate optimal timing in bank's writing off its nonperforming loan using real options approach.

Second, we assume that a firm borrows from two banks: one is the main bank and the other is the non-main bank. This setup introduces a game-theoretic view into the real options model. We extend the real options model of Mella-Barral and Perraudin [1997] to a game-theoretic real options model. This approach is developed by Dixit and Pindyck [1994] to explain the optimal entry timing in a market in which another player also waits for his/her optimal entry time. They show that there exists an equilibrium where one of the potential entrants invests earlier than the other. In addition, the investment timing of the first-mover is earlier than the noncompetitive real options case. Grenadier [1996] applied this approach to a real estate market to explain "over-building," a variant of overinvestment as a barrier to new entrants. Weeds [2002] applied this approach to firms' R&D investment and compared the results of a cooperative game with those of a noncooperative game. These studies examined the entry game with real options theory, while our study focuses on the exit game.

Similarly to the entry game, our game-theoretic real options model has a unique equilibrium. The equilibrium analysis shows that a difference in the loan amount between the two banks results in a difference in the optimal timing of exit. The main bank makes debt assumptions in terms of its maximization of the loan value, even if the non-main bank exits earlier.

Our model does not describe rational forbearance lending, but does give both banks' measure of their credit risk based on the outlook of the game. In addition, we examine

through comparative statics the impacts of changes in exogenous variables, such as the liquidation value of the firm, the interest rate of loan and the volatility of firm value, on both banks' exit strategies. Each bank determines its optimal exit strategy by taking into account the other bank's optimal strategy and the equilibrium is given in this game-theoretic situation.

These comparative statics reveal the following.

1) The lower the liquidation value of the firm, the earlier the non-main bank exits. In contrast, the lower the liquidation value, the later the main bank liquidates the firm.
2) The lower the interest rate of the loan, the earlier the non-main bank exits.
3) The higher the volatility of firm value, the later both banks exit and liquidate the firm. However, much higher volatility causes an incentive to exit for the non-main bank.

These results are consistent with the forbearance lending and exposure concentration observed within main banks in Japan.

The remainder of the paper is organized as follows. Section 2 explains the benchmark model of monopoly lending developed by Mella-Barral and Perraudin [1997]. It shows how the real options approach helps us determine the optimal timing of exit. Section 3 extends the benchmark model to a game in which a firm borrows from two banks. Section 4 examines the equilibrium of the model described in Section 3. Section 5 discusses the comparative statics and implications. Section 6 concludes.

## 2. Benchmark model for monopoly lending

First, we examine simple monopoly lending using real options theory, following Mella-Barral and Perraudin [1997]. We explore a model in which a firm finances its business with debt and equity. One bank supplies the loan and a representative equity holder controls the firm.

### (1) Model settings

We denote the sales of the firm as $X_t$ and assume that $X_t$ follows a geometric Brownian motion under a risk-neutral measure:

$$dX_t = \mu\, X_t dt + \sigma\, X_t dz_t\, ,\ X_0 = x\, , \tag{1}$$

where $\mu$ and $\sigma$ are constant and $z_t$ is a standard Brownian motion process.

We assume constant variables for the following parameters:

*w*: the operating costs of the firm,

*M*: the principal of the bank's loan,

*C*: the liquidation value of the firm,

*c*: the ratio of loan value covered by the liquidation (defined by *C/M*), [2]

*r*: the risk-free rate, i.e., the discount rate under the risk-neutral measure, and

*b*: the interest rate of the loan.

We assume that the drift of the firm's sales $\mu$ is less than the risk-free rate $r$. [3] We also assume that the interest rate of the loan $b$ is greater than $r$.

The equity holder and the bank determine their optimal strategies, respectively, based on their common knowledge of the stochastic process of $X_t$ and the current value of $X_t$. The choices of the equity holder are either to run the firm or to go bankrupt at each $t$ under observed $X_t$. In the case of bankruptcy, the firm is owned by the debt holder, i.e., the bank, and after bankruptcy the bank runs the firm. The bank's choice is either to run the firm or to liquidate it at each $t$ after the bankruptcy. We denote the bankruptcy time as $\tau_b$ and the liquidation time as $\tau_c$. We also denote $\mathcal{F}_t$ as the filtration sets of the information on $X_t$, and $\mathcal{T}_t$ as the set of stopping times on the information set $\mathcal{F}_t$. The conditional expectation about $\mathcal{F}_t$ is given by $\mathbb{E}_t(\,\cdot\,) = \mathbb{E}(\,\cdot\,|\mathcal{F}_t)$.

Now, we formulate the optimization problem for the equity holder. The equity holder decides the optimal timing of bankruptcy to maximize the equity value in each $t$, and the maximized value is given by:

$$E(X_t) = \max_{\tau_b \in \mathcal{T}_t} \mathbb{E}_t\left( \int_t^{\tau_b} e^{-r(s-t)}(X_s - w - bM)ds \right) \tag{2}$$

The equity value $E(x)$ equals the maximized present value of the firm's profits before the firm's bankruptcy ($t < \tau_b$). After the bankruptcy ($\tau_b \le t$), the equity holder leaves the firm's ownership to the bank. We denote the optimal $\tau_b$ as $\tau_b^* \in \mathcal{T}_t$.

After bankruptcy occurs, the bank decides the timing of liquidation to maximize the loan value under the given value of $\tau_b^*$. The optimized timing $\tau_c^* \in \mathcal{T}_t$ is determined by the maximization problem as follows:

$$D(X_t) = \max_{\tau_c \in \mathcal{T}_t} \mathbb{E}_t\left( \int_t^{\tau_b^*} e^{-r(s-t)}bMds + \int_{\tau_b^*}^{\tau_c} e^{-r(s-t)}(X_s - w)\,ds + e^{-r(\tau_c - t)}C \right) \tag{3}$$

,

---

[2] We assume *c* is less than 1.

[3] This assumption is a necessary condition for the discounted present value of the firm's profit to converge to a finite value. If $r > \mu$ holds, then the integral $\int_{[0,\infty)} e^{-rt}(X_0 e^{\mu t})\,dt$ converges to $X_0/(r-\mu)$. However, the integral diverges to infinity if $r \le \mu$.

where $D(x)$ represents the maximized loan value at time $t$ before $\tau_b^*$. The first term of the conditional expectation in equation (3) represents the present value of the interest incomes $bM$ before bankruptcy occurs $(t < \tau_b^*)$. The second term represents the present value of the bank's earnings in the $s$ periods from the bankruptcy to the liquidation $(\tau_b^* < t < \tau_c)$. The earnings of the firm's owner are given by $X_s - w$ during the $s$ periods. The third term represents the present value of the liquidation value of the firm.

## (2) The option values of bankruptcy and liquidation

The optimization problem for the equity holder in equation (2) can be solved analytically (see Appendix 1). The maximized equity value $E(x)$ is given by:

$$E(x) = \left( \frac{x}{r-\mu} - \frac{w+bM}{r} \right) + \left( -\frac{x_b}{r-\mu} + \frac{w+bM}{r} \right)\left( \frac{x}{x_b} \right)^{\gamma} \text{, for } x_b < x, \qquad (4)$$

and:

$$x_b = \frac{\gamma}{\gamma-1} \cdot \left( \frac{w+bM}{r} \right) \cdot (r-\mu), \qquad (5)$$

where:

$$\gamma = 1/2 - \mu/\sigma^2 - \sqrt{(\mu/\sigma^2 - 1/2)^2 + 2r/\sigma^2} < 0.$$

The first term of the right-hand side of equation (4) is the present value of the firm's earnings before the bankruptcy. [4] This value is an increasing function of the initial firm's sales $x$ and the drift of the firm's sales $\mu$. It is also a decreasing function of the operating cost $w$, the interest rate $r$, and the loan amount $M$. Note that it does not depend on the volatility $\sigma$ in the stochastic process of $X_t$, because the value is derived as the optimization of the expected value under the risk-neutral measure. The second term represents the option value of the equity holder owning the right to bankrupt the firm. We name it the bankruptcy option. The bankruptcy option depends on the volatility $\sigma$ and increases as $\sigma$ increases. [5] This implies that the bankruptcy option becomes more valuable as the firm's sales become more volatile.

To examine how the option value earnings, we show the interpretation of the threshold value $x_b$ corresponding to $\tau_b^*$. The term $(x/x_b)^{\gamma}$ in equation (4) is the

---

[4] The term $x$ is discounted by $r$–$\mu$ while the term $w + bM$ is discounted by $r$. This is because $X_t$ has the drift $\mu$ as shown in equation (1).

[5] Differentiation of equation (4) with respect to $\sigma$ yields $dE/d\sigma = \{(\partial E/\partial x_b)(\partial x_b/\partial \gamma) + \partial E/\partial \gamma\} \ \partial \gamma/\partial = (\partial E/\partial \gamma)(\partial \gamma/\partial \sigma)$ (Notice that $\partial E/\partial x_b = 0$). Thus, $dE/d\sigma > 0$ because $\partial E/\partial \gamma = \{- x_b /(r-\mu) + (w + bM)/r\} (x/x_b)^{\gamma}\log(x/x_b) > 0$ and $(\partial \gamma/\partial \sigma) > 0$.

probability that $x$ reaches the threshold value $x_b$,[6] that is, the probability of bankruptcy. We can easily check that $(x/x_b)^\gamma \to 0$ as $x \to \infty$ and $(x/x_b)^\gamma = 1$ at $x = x_b$. Thus, the first term in the equity value $E(x)$ dominates as $x \to \infty$ and reaches zero at bankruptcy $x = x_b$. In addition, the marginal value of the equity at bankruptcy $E'(x_b)$ also reaches zero. Furthermore, $E(x)$ smoothly passes zero at $x = x_b$. The equity value function is depicted by Figure 1.

The bankruptcy option increases the equity value, which implies that it, in turn, decreases the value of loan for the bank. Once the optimal timing of the bankruptcy is given by the time when $x$ reaches the threshold value $x_b$ of (5), the maximized loan value $D(x)$ and the threshold value of $x$ for the liquidation are given by:

$$D(x) = \frac{bM}{r} + \left( D\big|_{x \le x_b}(x_b) - \frac{bM}{r} \right) \left( \frac{x}{x_b} \right)^\gamma \quad , \text{ for } x_b < x \tag{6}$$

where:

$$D\big|_{x \le x_b}(x) = \frac{x}{r-\mu} - \frac{w}{r} + \left( cM - \frac{x_c}{r-\mu} + \frac{w}{r} \right) \left( \frac{x}{x_c} \right)^\gamma, \tag{7}$$

and:

$$x_c = \frac{\gamma}{\gamma - 1} \cdot \left( cM + \frac{w}{r} \right) \cdot (r - \mu). \tag{8}$$

The derivations of these are shown in Appendix 2.

Equation (6) is interpreted as follows. The first term represents the present value of the interest incomes and the second term is the negative option value resulting from the equity holder's option on bankruptcy. The latter decreases the loan value $D(x)$ as $x$ declines. $D(x)$ is close to $D\big|_{x \le x_b}(x)$ as $x$ reaches $x_b$, while $D(x)$ is close to the first term in equation (6) as $x \to \infty$. $D\big|_{x \le x_b}(x)$ represents the loan value after bankruptcy, given by (7).

Equation (7) shows the present value of the firm that the bank inherits from the equity holder at the time $\tau_b^*$. The first and second terms represent the value to the bank of running the firm after the bankruptcy. The bank obtains the total profit of the firm, while the equity holder obtains the firm's profit after interest payments, as shown in equation (4). The third term is the option value of the bank having the right to postpone

---

[6] Strictly speaking, $(x/x_b)^\gamma$ is the present value of the probability of bankruptcy, which is given by $\mathbf{E}(\exp(-r\tau_b)) = \int_{[0,\infty)} e^{-rt} \mathbf{E}(t \le \tau_b < t + dt)$. See Dixit and Pindyck [1994].

6

the liquidation of the firm in order to bet on the firm's recovery. The option value is proportional to the probability of liquidation $(x/x_c)^\gamma$ and the loss (or profit) at liquidation, which is given by the difference between the liquidation value $C(=cM)$ and the value of running the firm. $D|_{x \le x_b}(x)$ is an increasing function of $x$. It reaches the liquidation value $C$; as $x$ approaches $x_c$, the threshold value of the liquidation given by equation (8). The smoothness condition $D'(x_c)=0$ is assumed in the maximization of equation (3) to obtain equations (6), (7) and (8), which are required for the optimality of the liquidation threshold $x_c$.

   Figure 1 shows the value functions of the equity and the loan that correspond to equations (2) and (3). The optimal timings of the bankruptcy and the liquidation are given by the points where the value function curves are smoothly pasting to zero and $C$, respectively.[7] The loan value curve has an upper bound given by the present value of interest incomes. The bound corresponds to the value in the case of $x \to \infty$, i.e., bankruptcy probability $\to 0$. As long as $C$ is less than $M$, i.e., $c$ is less than a unit, the equity holder accepts bankruptcy earlier than the bank accepts liquidation.

Figure 1. The value of equity and loan in monopoly lending
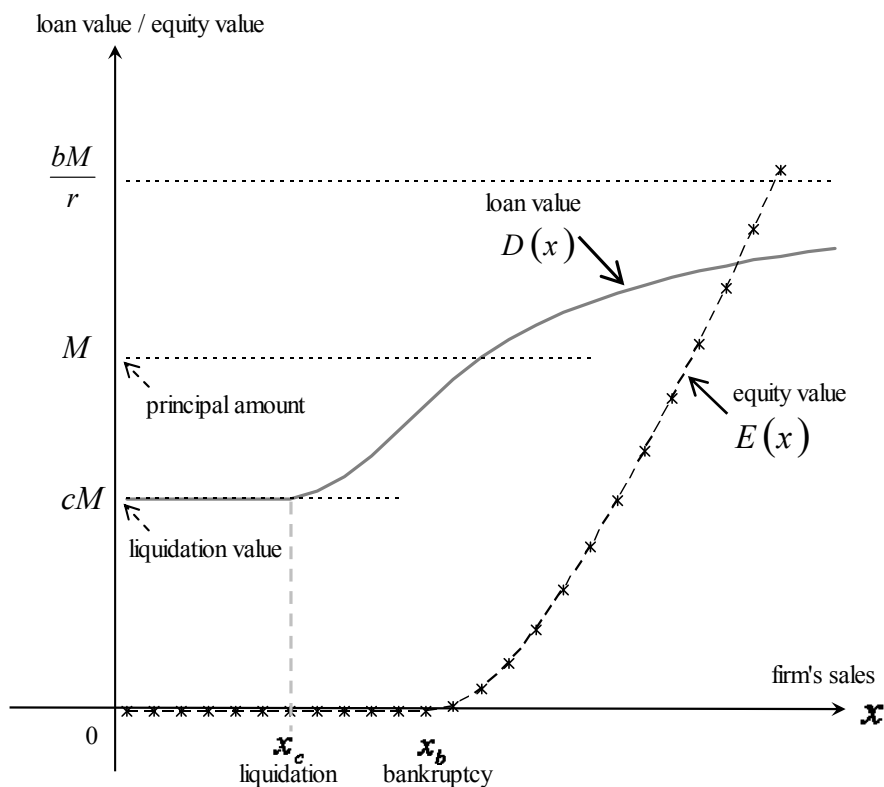
## 3. Extended model for duopoly lending

In this section, we extend the benchmark model to the case where a firm borrows from two banks. In the previous section, one bank provided a loan to the firm and the bank's collecting the loan meant that the firm is liquidated by the bank. In the duopoly lending case, the equity holder runs the firm as long as the total amount of the loan is maintained. In the duopoly case, the optimal bankruptcy timing is earlier than when banks collect their loans in the monopoly case. We, therefore, investigate an exit game between two banks where one of the banks might make a debt assumption when the other bank collects its loan. The decision of the debt assumption depends on the optimization problem for the bank that faces the first action by the other bank. In this section, we examine the optimization problem for the first-mover, and for the other, assigning a bank to the role of either the first-mover or the other.

### (1) Model settings

Neither bank determines its strategy cooperatively. We have two banks: bank A and bank B. Bank $i$ exits earlier than bank $j$ ($i, j \in \{A, B\}$) and thus we refer to $i$ as "the leader" and $j$ as "the follower." The model in this section yields the optimal strategy given the counterpart's strategy, preparing for the noncooperative game in the next section.

All parameters except the loan amount are the same for banks A and B, and denoted as in Section 2. Each loan amount for banks A and B is denoted as $M_A$ and $M_B$, respectively. We also define $m_A$ and $m_B$ as each bank's share of total loan, that is, $m_i = M_i / M$ $i \in \{A, B\}$, where $M_A + M_B = M$. If bank $i$ exits earlier than bank $j$ ($i, j \in \{A, B\}$), then the follower bank $j$ might make a debt assumption of $M_i$, the amount of the loan that the leader bank $i$ collects from the firm. When the follower bank $j$ liquidates the firm after the debt assumption, bank $j$ obtains the liquidation value $C$, where $C < M$. We do not assume whether bank A or B becomes the leader at this stage, but we will show that the difference between $M_A$ and $M_B$ determines the leader, later on.

The optimization problem for the equity holder is the same as in Section 2. Furthermore, the optimal bankruptcy timing $\tau_b^*$ is given as in Section 2.

Given the bankruptcy timing $\tau_b^*$, the optimization problem for the leader bank is given by:

$$D_i^L(X_t) = \max_{\tau_m \in \mathcal{T}_t} \mathbb{E}_t \left( \int_t^{\tau_b^*} e^{-r(s-t)} m_i b M \cdot ds + \int_{\tau_b^*}^{\tau_m} e^{-r(s-t)} m_i (X_s - w) \cdot ds \right.$$
$$\left. + \; e^{-r(\tau_m - t)} m_i M \right),$$

(9)

where $D_i^L(x)$ is the leader's loan value and $\tau_m \in \mathcal{T}_t$ is the exit timing for the leader bank $i$. The first term of the right-hand side in equation (9) represents the present value of the leader's interest incomes before the bankruptcy occurs ($t < \tau_b^*$). The second term represents the present value of the leader's earnings in the periods from the bankruptcy to the exit ($\tau_b^* \leq t < \tau_m$).[8] Because we assume that both banks maintain their exiting loan shares after the bankruptcy, the second term is composed of the firm's profit divided by its lending share after the bankruptcy. The third term represents the present value of the leader's collection of its loan principal $m_i M$ at $t = \tau_m$. $\tau_m^*$ is derived from the optimization in equation (9).

The follower bank determines $\tau_c^*$, the optimal timing of the liquidation after it makes a debt assumption at $\tau_m^*$. $\tau_c^*$ might be close to $\tau_m^*$, but we assume $\tau_c^*$ must be later than $\tau_m^*$. We exclude the possibility of simultaneous exits of both banks, because the continuous $x$ process assures anticoincidence between $x_b$ and $x_m$ for heterogeneous banks with respect to loan amount $M_i$. The optimization problem for the follower bank is given by:

$$D_j^F(X_t) = \max_{\tau_c \in \mathcal{T}} \mathbb{E}_t \left( \int_t^{\tau_b^*} e^{-r(s-t)} m_j b M \cdot ds + \int_{\tau_b^*}^{\tau_m^*} e^{-r(s-t)} m_j (X_s - w) \cdot ds \right.$$
$$\left. - e^{-r(\tau_m^* - t)} m_i M + \int_{\tau_m^*}^{\tau_c} e^{-r(s-t)} (X_s - w) \cdot ds + e^{-r(\tau_c - t)} C \right).$$

(10)

The first term in equation (10) represents the present value of the follower's interest income before the bankruptcy ($t < \tau_b^*$). The second term represents the present value of the follower's earnings after the bankruptcy ($\tau_b^* \leq t < \tau_m$). The third term represents the present value of the follower's new loan instead of the leader's collection at $t = \tau_m^*$. The fourth term represents the present value of the follower's earnings in the period from $\tau_m^*$ to $\tau_c$. The last term represents the present value of liquidation at $t = \tau_c$.

---

[8] If the leader exits before the bankruptcy, the integral interval in the first term has to be changed from $[t, \tau_b^*]$ to $[t, \tau_m]$ and the second term is not needed.

(2) The valuation for each loan value

The maximized leader's loan value $D_i^L(x)$ can be obtained analytically as follows:

$$D_i^L(x) = \frac{m_i bM}{r} + \left( D_i^L \mid_{x \le x_b} (x_b) - \frac{m_i bM}{r} \right) \left( \frac{x}{x_b} \right)^\gamma \text{ , for } x_b < x \text{ ,} \quad (11)$$

where:

$$D_i^L \mid_{x \le x_b} (x) = m_i \left( \frac{x}{r - \mu} - \frac{w}{r} \right) + m_i \left( M - \frac{x_m}{r - \mu} + \frac{w}{r} \right) \left( \frac{x}{x_m} \right)^\gamma \quad (12)$$

and:

$$x_m = \frac{\gamma}{\gamma - 1} \cdot \left( M + \frac{w}{r} \right) \cdot (r - \mu). \quad (13)$$

See Appendix 3 for the derivation.

Equation (11) represents the leader's loan value before the bankruptcy. The first term represents the present value of the interest incomes and the second term corresponds to the negative option value stemming from the equity holder's execution of the bankruptcy option. These formulas are similar to those in the benchmark model (6), while the loan value after the bankruptcy $D_i^L \mid_{x \le x_b} (x)$ includes $x_m$ instead of $x_c$ in the benchmark model (7). The second term represents the positive option value of the leader being able to exit earlier than the follower.

The positive option value is determined by i) the probability that the leader collects its loan at $x$, i.e., $(x/x_m)^\gamma$, and ii) the loan principal minus the loan value measured by the present value of the profits in the case of $x = x_m$. The latter represents the leader's net gain when the leader collects its loan at $x$. The multiple of the gain and the exit probability yields the option value of early exit from lending.

Note that the threshold value of the leader's exit shown in equation (13) is independent of its loan amount. This is because the leader determines the exit timing by the gap between the total loan amount $M$ and the value of total loan at $x_m$. $m_i$ in the second term of the equation (12) operates only as a multiplier.

The option value increases as $x \to x_m$ from above and reaches the maximum at $x = x_m$, where the leader exercises its exit option. At this point, the leader's loan value $D_i^L \mid_{x \le x_b} (x)$ is equal to its loan principal, $m_i M$. We assume that the loan value pastes smoothly to the loan principal by the constraint $D_i^{L'} \mid_{x \le x_b} (x_m) = 0$ at the maximization of equation (9).

The bankruptcy threshold $x_b$ in (5) is larger than the leader's exit threshold $x_m$ in (13),

because we assume the loan interest rate $b$ is greater than the risk-free rate $r$. The leader, therefore, always exits after the bankruptcy.

Figure 2 depicts the leader's loan value function given by equation (11), normalizing the value by its loan principal. The value is always larger than a unit from the assumption that the leader can collect the loan principal at the exit by the follower's debt assumption. The value pastes smoothly to the normalized loan principal value, i.e., a unit. As the firm's sales $x$ increases, the probability of bankruptcy in equation (11) decreases and the lower probability reduces the negative option value in absolute terms, which increases the loan value. The loan value converges to the present value of the interest income, the first term of equation (11) as $x \to \infty$.

The leader's loan value is larger than the loan value of the monopoly lending because the leader holds the option to exit earlier, which the monopoly bank does not. As shown in Figure 2, a decline of $x$ increases the difference between normalized $D_i^L(x)$ and $D(x)$ in the monopoly case.

Figure 2. The value of the leader's loan
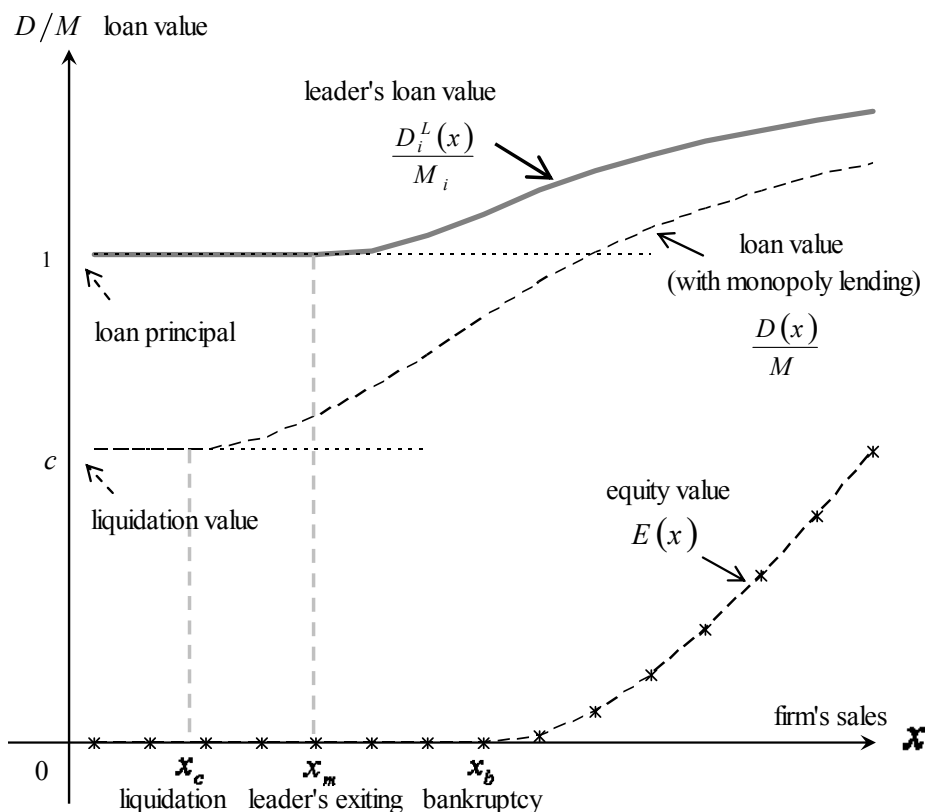
Next, we consider the follower's loan value $D_j^F(x)$. The maximal value is obtained given bankruptcy threshold $x_b$ and exit threshold $x_m$ as follows:

$$D_j^F(x) = \frac{m_j bM}{r} + \left( D_j^F \mid_{x \le x_b} (x_b) - \frac{m_j bM}{r} \right) \left( \frac{x}{x_b} \right)^\gamma \quad , \text{ for } x_b < x, \qquad (14)$$

where $D_j^F \mid_{x \le x_b} = m_j \left( \frac{x}{r-\mu} - \frac{w}{r} \right) + \left( D_j^F \mid_{x \le x_m} (x_m) - m_j \left( \frac{x_m}{r-\mu} - \frac{w}{r} \right) \right) \left( \frac{x}{x_m} \right)^\gamma \qquad (15)$

and:

$$D_j^F \mid_{x \le x_m} = \left( \frac{x}{r-\mu} - \frac{w}{r} - m_i M \right) + \left( cM - \frac{x_c}{r-\mu} + \frac{w}{r} \right) \left( \frac{x}{x_c} \right)^\gamma . \qquad (16)$$

See Appendix 3 for derivation.

The follower's loan value in equation (14) is similar to the leader's value in equation (11), but it is different by the term $D_j^F \mid_{x \le x_b} (x)$ corresponding to the follower's loan value after the bankruptcy. The second term of $D_j^F \mid_{x \le x_b} (x)$ in equation (15) is the "negative" option value for the follower. The leader's option to exit yields a negative option for the follower. [9] $D_j^F \mid_{x \le x_m} (x)$ is defined by equation (16). The first term of equation (16) is the present value of the firm's profits minus the leader's loan principal, which is identical to the amount of the debt assumption for the follower. The second term is the option value for the follower of liquidating the firm. The gap between the liquidation value $cM$ and the firm's discounted profits measured by $x = x_c$ represents the gain from the liquidation. The amount of the gain and the probability of liquidation $(x/x_c)^\gamma$ yield the liquidation option value.

The relation $x_c \le x_m$ holds by the assumption $C < M$. This implies that the optimal liquidation timing is later than the leader's exit. It is rational for the follower to make the debt assumption in this model and to run the firm by itself until $x$ reaches $x_c$. This is because that the follower cannot recover its loan principal immediately by the assumption $C < M$ and it thus has the incentive to make a debt assumption to wait for the firm's recovery.

Figure 3 depicts the follower's loan value function by a solid bold line. It is always less than the monopoly loan value because of the "negative" option value. Naturally, banks A and B both want to avoid being the follower. In the next section, we investigate

---

[9] The negative option is defined by using the gap between the loan value after the leader's exit $D_j^F \mid_{x \le x_m} (x)$, and the loan value measured by $x = x_m$ before the debt assumption. The latter is greater than the former from the former before the debt assumption for $x > x_m$.

the equilibrium in the game between the two banks.

Figure 3. The value of the follower's loan
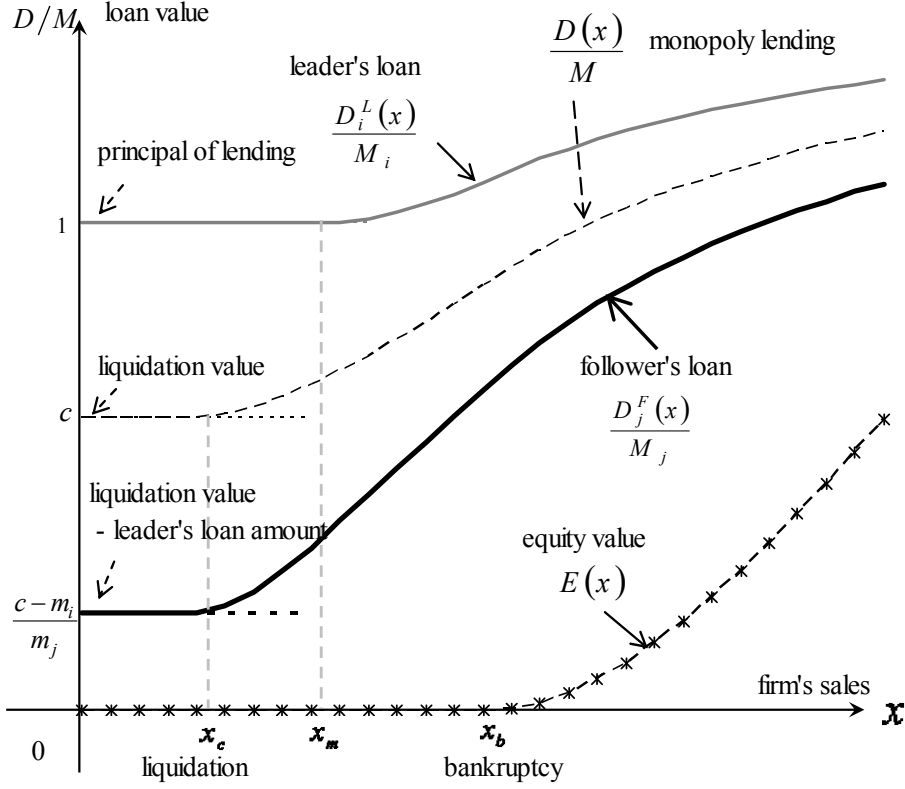


## 4. The equilibrium

In this section, we examine the equilibrium of the model described in Section 3. Without loss of generality, we assume that bank A has a larger loan than bank B. We regard bank A as the "main bank" and bank B as the "non-main bank." The only difference between the main and non-main bank is the loan amount.

First, we examine the follower's loan value assuming each bank is in the position of the follower. Figure 4 depicts the follower's value for the main bank, $D_A^F(x)$, by a solid thin line under the assumption that for the non-main bank, $D_B^F(x)$, by solid bold line under the assumption that $m_A = 80\%$ and $m_B = 20\%$. The follower's value for the non-main bank is less than that for the main bank, because the burden of the debt assumption on the non-main bank is larger than that on the main bank in the case that the firm is finally liquidated. The larger $m_iM$ is in equation (16), the larger the negative option value in equation (15) in absolute value. The larger negative option value for the non-

13

main bank makes the follower's value curve for the bank lower than that for the main bank.

When the non-main bank collects its loan principal, the bank abandons the future profits given by $D_B^F(x)$. $x_B^P$ in Figure 4, the cross point of the $D_B^F(x)$ curve and the principal, both normalized by $M_B$, gives the threshold value of $x$ for the optimal exit timing for the non-main bank. However, $x_B^P$ is not an optimal threshold in the equilibrium of the exit game. If the main bank is a follower, the same discussion holds. $x_A^P$ corresponds to the threshold for the main bank. Note that for $x_A^P < x < x_B^P$, the non-main bank deduces that the main bank does not exit earlier and it is optimal to continue the lending. In short, the non-main bank should exit at $x = x_A^P$.[10] For $x_A^P < x < x_B^P$, both banks have an incentive to maintain their lending. If the non-main bank exits at that point, it is optimal for the main bank to continue lending by making a debt assumption, because the liquidation value corresponding to $x_c$ in Figure 4 is the new comparative value for the main bank to decide its strategy either to run the firm or to liquidate it. The discussion above gives the game equilibrium, 1) for $x > x_A^P$, both banks maintain their lending, 2) at $x = x_A^P$, the non-main bank collects its loan and the main bank makes a debt assumption, 3) for $x_c < x < x_A^P$ the main bank runs the firm, and 4) at $x = x_c$, the main bank liquidates the firm. We denote the equilibrium exit threshold $x_A^P$ as $x^P$.

---

[10] In the strict sense, the exit threshold is slightly larger than $x_A{}^P$. At $x_A{}^P$, the non-main bank is indifferent between exiting and continuing operations.

Figure 4. The equilibrium
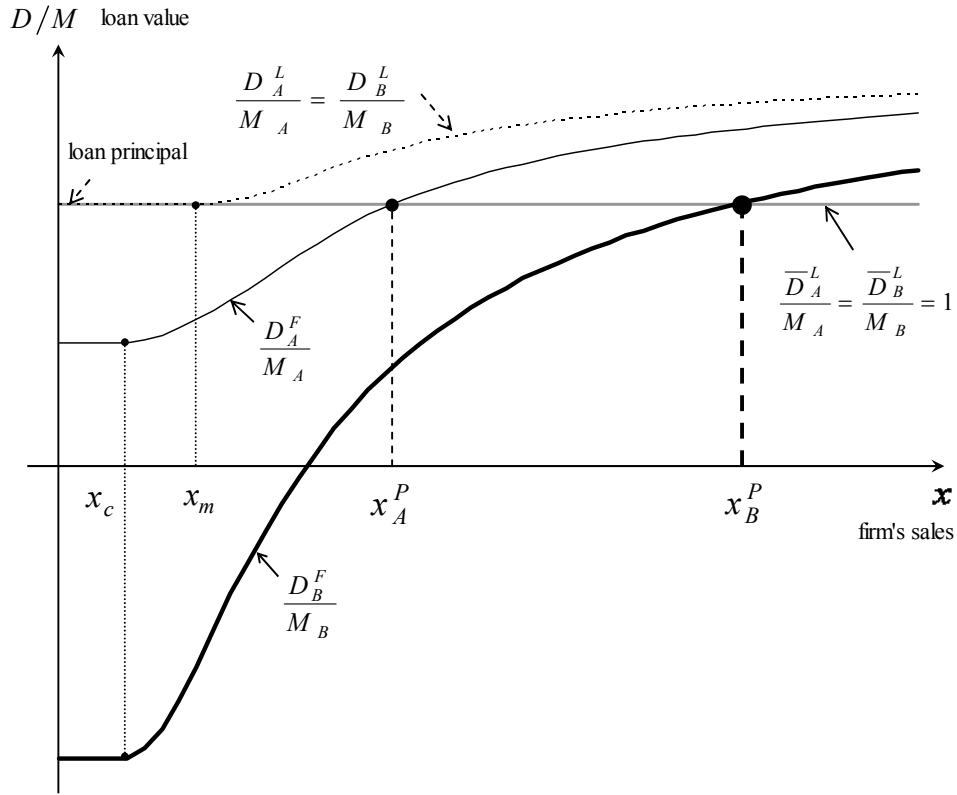


## 5. Comparative statics and implications

In this section, we consider how exogenous conditions affect the (non-) main bank exit strategy. We change the firm's exogenous parameters such as 1) the liquidation value of the firm, 2) the interest rate of the loan, 3) the volatility of the firm's sales.

The benchmark parameters are given as follows.

| | | | |
|---|---|---|---|
| Drift of the process $X_t$ : $\mu$ | 0% | Volatility of $X_t$ : $\sigma$ | 10% |
| Firm's sales: $x$ | 0.5~1.5 | Operating cost: $w$ | 1 |
| Interest rate $b$ | 5% | Risk-free rate $r$ | 2% |
| Principal $M$ | 10 | Principal of main bank $M_A$ | 7 |
| Liquidation value $C$ | 6 | Recovery ratio $c$ (= $C/M$) | 60% |

## (1) Comparative statics on the firm's liquidation value

First, we examine how the banks' exit strategies vary with the liquidation value. In Figure 5, the vertical axis shows the recovery rate of the loan principal at the liquidation instead of the liquidation value, and the horizontal axis represents current firm's sales $x$, i.e., the initial value for future development of the $x$ process. The threshold points, $x^P$, $x_b$ and $x_c$, are shown for the case of the benchmark. The main results are as follows.
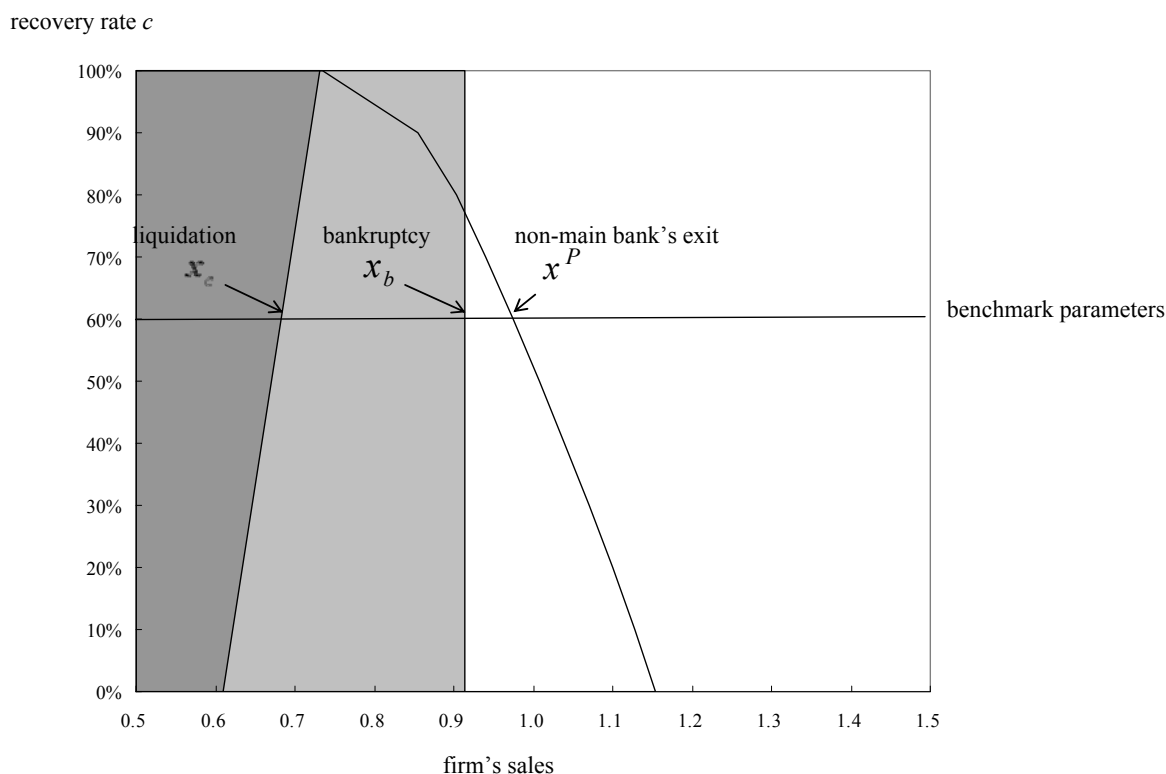
i) The exit threshold for the non-main bank $x^P$ increases as the recovery rate $c$ decreases. This implies that the non-main bank tends to exit earlier when the recovery rate is lower. [11]

ii) The liquidation threshold for the main bank $x_c$ decreases as the recovery rate $c$ decreases. This implies that the main bank tends to hesitate regarding liquidation when the liquidation value decreases. Because the option value of delaying liquidating the firm is higher for the lower recovery rate, it is rational for the main bank to wait for the recovery of the firm's sales $x$.

iii) The bankruptcy threshold $x_b$ for the equity holder is independent of the liquidation value. Because the liquidation value $C$ does not exceed the loan amount $M$, this condition provides the equity holder with less incentive to bet on the sales recovery than the main bank, discussed in Section 2.

In Figure 5, we can also observe that the bankruptcy threshold might be larger than the non-main bank's exit threshold at a high recovery rate, such as more than 80%. In this case, both of the banks continue lending after the bankruptcy.

---

[11] In the game-theoretic situation, the non-main bank determines the exit strategy comparing the follower's value with the loan principal (the value of exiting as a leader) and thus the follower's value plays an important role, whereas the non-main bank always exits as a leader at the equilibrium.

Figure 5. The banks' exit strategies and the firm's liquidation value

recovery rate $c$



(2) Comparative statics on the interest rate

Next, we examine how the banks' exit strategies vary with the interest rate $b$. The main results shown in Figure 6 are as follows.

i) The exit threshold of the non-main bank $x^P$ increases as the loan interest rate decreases. The bank tends to exit earlier when the loan rate is lower. The lower the incomes from loan, the lower the value of both banks' loan. [12] The liquidation threshold $x_c$ is independent of the interest rate $b$ (the $x_c$ curve is a vertical line in Figure 6). This is because the firm's profit belongs to the banks after the bankruptcy and the decision of the liquidation is independent of the interest rate.

ii) The bankruptcy threshold $x_b$ increases as the loan rate increases. The equity holder of the firm tends to bankrupt the firm earlier, because the higher interest payment decreases the equity value. For loan rates of more than 6% in Figure 6, the bankruptcy occurs before the non-main bank's exit. For this reason, the non-main

---

[12] The declines in the loan values hasten the non-main bank to collect its loan, while the timing of liquidation by the main bank does not change. Once the main bank makes its debt assumption, all profits belong to the main bank and therefore the threshold $x_c$ is independent of the lending rate.

17

bank's exit threshold $x^P$ is independent of the loan rate $b$, i.e., the $x^P$ curve is vertical. Because the profits of the firms are shared by both banks in proportion to their loan amount, the loan rate does not matter.

These results suggest that low loan rates tend to increase the loan exposure concentration to the main bank and provide a strong incentive for the equity holder to lower the firm's sales $x$.

Figure 6. The banks' exit strategies and the loan interest rate
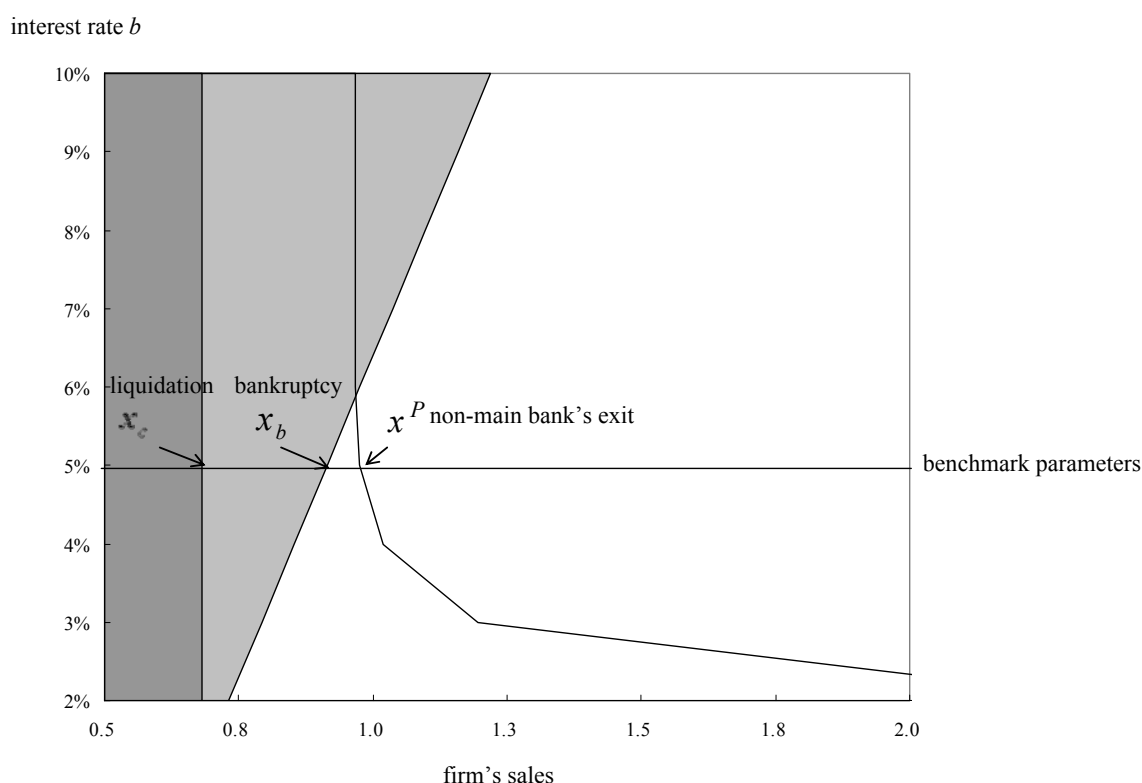


(3) Comparative statics on the volatility of firm sales

Finally, we examine how the banks' exit strategies vary with the volatility of the firm's sales $\sigma$. The main results shown in Figure 7 are as follows.

i) The non-main bank tends to exit later as sales volatility increases, because the option value to bet on the recovery increases. However, in the case of higher volatility such as that greater than 25%, the exit threshold $x^P$ bends backward. We provide reasons for this below.

ii) As the volatility increases, the equity holder tends to reduce firm sales. This is a

similar incentive to the main bank, which has an incentive to delay the liquidation of the firm. This is because the waiting option value to bankrupt and liquidate the firm increases as the volatility increases.

Figure 7. The banks' exit strategies and the volatility of the firm's sales
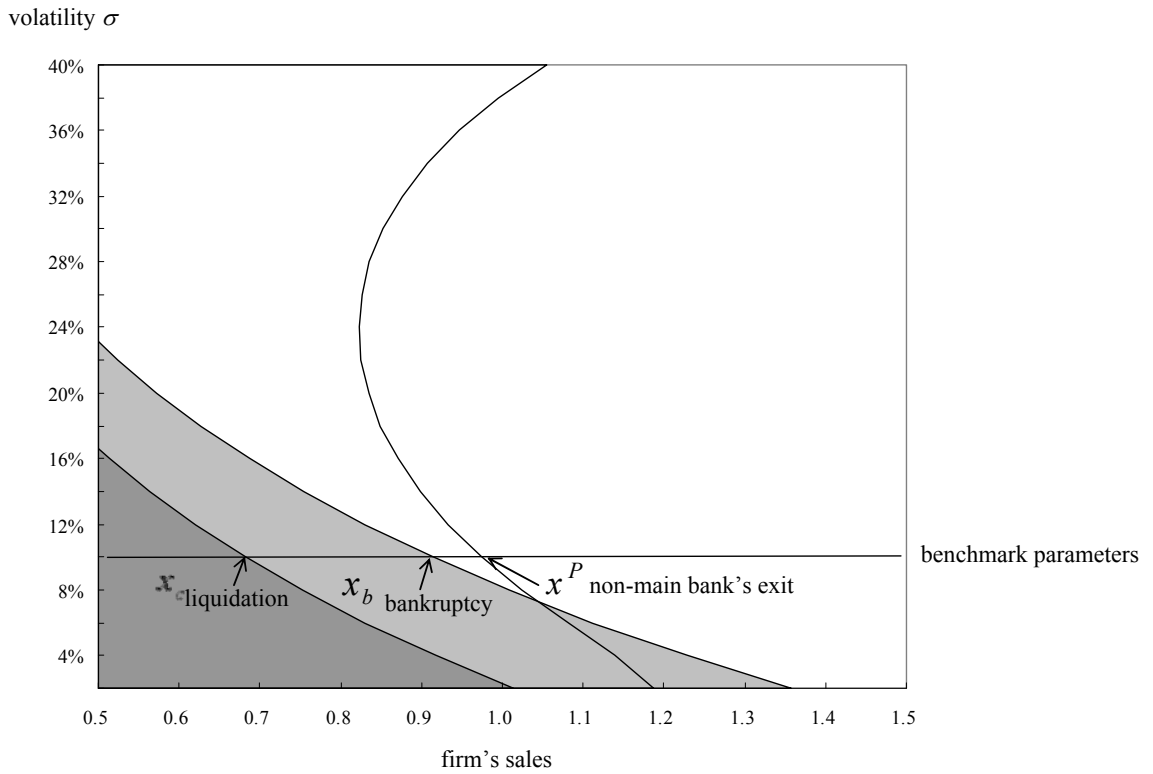


In the standard real options model, the optimal threshold is a monotonically increasing function of sales volatility as shown by the bankruptcy curve in Figure 7. On the other hand, in the game-theoretic real options model, the optimal threshold is not always a monotonically increasing function of sales volatility, as shown in Kijima and Shibata [2005]. The reason is as follows.

Note that the follower's value plays an important role in determining the exit strategy. As explained in (15)–(17), the follower's value is composed of the two negative option values and one positive option value such that:

The follower's option value

= negative option value suffering from the equity's option to bankrupt the firm

+ negative option value suffering from the leader bank's option to exit early

19

+ positive option value generated by the follower bank's option to liquidate the firm.

The relative sizes of the positive and negative option values make the exit threshold curve backward bending. The increase in the volatility heightens all three options values for the main bank, i.e., the follower, on an absolute value basis. For lower volatility, the increase in the positive option value exceeds the increase in the negative one, which makes the exit threshold for the follower lower. This, in turn, lowers the optimal exit threshold for the leader. For higher volatility, in contrast, the increase in the positive option value is less than the increase in the negative one. The exit threshold, therefore, rises as the volatility increases.

## 6. Conclusion

This paper developed a dynamic model for credit risk in relationship lending. It considered the case in which the main bank and the non-main bank play a game of exit from their deteriorating lending, and examined their optimal exit strategies by applying a game-theoretic real options approach.

Our model showed that each bank determines the optimal exit strategy by taking into account the other bank's optimal strategy and that the equilibrium of the game depends on the difference in the loan amount between the two banks. The main bank with a larger loan amount makes a debt assumption rationally in a sense of its maximization of the loan value.

The paper also used comparative statics to examine the effect of exogenous variables, such as the liquidation value of the firm and the loan interest rate, on the banks' strategies. First, a low liquidation value makes the non-main bank exit earlier, whereas it enhances the main bank's incentive to delay the liquidation while waiting for the firm's recovery. Second, a low loan rate leads to the early exit of the non-main bank. These mechanisms accelerate the concentration of the main bank's exposure to the deteriorating firm.

Finally, we illustrated how our model can be further developed. First, it would be interesting to investigate asymmetric information about the firm between the two banks. The main banks may have different information on the stochastic process of initial sales, $x$. Second, it is possible for the main bank to revitalize the firm once the bank owns the firm. The bank may reduce the firm's operating cost and improve the growth rate of the firm's sales. It would be interesting to investigate these extensions to our model.

## Appendix 1. The optimization problem for the equity holder

In this appendix, we show the solution of the optimization problem for the equity holder in equation (2). The equation is equivalent to the following HJB (Hamilton–Jacobi–Bellman) equation:

$$E(X_t) = e^{-rdt} \max \left\{ (X_t - w - bM)dt + \mathbb{E}_t[E(X_{t+dt})], 0 \right\}$$

(A-1)

where $E(X_t)$ represents the value of the firm for the equity holder. Applying Ito's lemma to the HJB equation

(A-1), we obtain the following ordinary differential equations of $E(x)$:

$$rE(x) = x - w - bM + \frac{1}{2}\sigma^2 x^2 E''(x) + \mu x E'(x) ,$$

(A-2)

with boundary conditions:

$$\begin{cases} E(x \to \infty) \to x/(r-\mu) - (w+b)/r \\ E(x_b) = 0 \\ E'(x_b) = 0 . \end{cases}$$

(A-3)

The first condition in (A-3) requires that $E(x)$ converges to the present value of the firm's profits as $x$. The condition excludes "a bubble condition." The second and third conditions require that $E(x)$ pastes smoothly to zero at $x_b$. These conditions are called the "value matching condition" and "smooth pasting condition," respectively.

Equation (A-2) is an Euler differential equation and can be solved analytically with the conditions (A-3), which determine the value of the firm for the equity holder and the bankruptcy threshold $x_b$ by:

$$E(x) = \begin{cases} \dfrac{x}{r-\mu} - \dfrac{w+bM}{r} + \left(-\dfrac{x_b}{r-\mu} + \dfrac{w+bM}{r}\right)\left(\dfrac{x}{x_b}\right)^\gamma & , \text{ for } x_b < x \\ 0 & , \text{ for } x \le x_b , \end{cases}$$

(A-4)

$$x_b = \frac{\gamma}{\gamma-1} \cdot \left(\frac{w+bM}{r}\right) \cdot (r-\mu) ,$$

(A-5)

where $\gamma$ is the negative root of the characteristic equation $\sigma^2/2\,\gamma(\gamma-1) + \mu \cdot \gamma = r$. $\gamma$ is defined by $\gamma = 1/2 - \mu/\sigma^2 - \sqrt{(\mu/\sigma^2 - 1/2)^2 + 2r/\sigma^2}$ .

## Appendix 2. The optimization problem for the monopoly bank

This appendix shows the solution of the optimization problem for the bank in the case of monopoly lending, given the bankruptcy threshold $x_b$ derived in Appendix 1. Equation (3) is equivalent to the HJB equation:

$$D(X_t) = \begin{cases} e^{-rdt}\left( bM \cdot dt + \mathbb{E}_t\left[D(X_{t+dt})\right] \right), & \text{for } t < \tau_b^* \\[2mm] e^{-rdt}\max\left\{ (X_t - w)\cdot dt + \mathbb{E}_t\left[D(X_{t+dt})\right], C\right\}, & \text{for } \tau_b^* \leq t, \end{cases}$$

(A-6)

where $D(X_t)$ represents the value of loan for the monopolist bank. Applying Ito's lemma to (A-6), we obtain the differential equation of $D(x)$:

$$rD(x) = bM + \frac{1}{2}\sigma^2 x^2 D''(x) + \mu x D'(x) \qquad \text{for } x_b < x,$$

(A-7)

with boundary conditions:

$$\begin{cases} D(x \to \infty) \to 0 \\ D(x_b) = D\big|_{x \leq x_b}(x_b), \end{cases}$$

(A-8)

and:

$$rD(x) = (x - w) + \frac{1}{2}\sigma^2 x^2 D''(x) + \mu x D'(x) \quad \text{for } x \leq x_b,$$

(A-9)

with boundary conditions:

$$\begin{cases} D(x \to \infty) \to x/(r - \mu) - w/r \\ D(x_c) = C \\ D'(x_c) = 0. \end{cases}$$

(A-10)

Equations (A-7) and (A-9) are Euler differential equations and can be solved analytically. Equation (A-9) is solved first, and the solution is applied for equation (A-7). The value of the loan and liquidation threshold $x_c$ are derived by the boundary conditions (A-10) as follows:

$$D(x) = \begin{cases} \dfrac{bM}{r} + \left(D\big|_{x \leq x_b}(x_b) - \dfrac{bM}{r}\right)\left(\dfrac{x}{x_b}\right)^\gamma, & \text{for } x_b < x \\[4mm] \dfrac{x}{r - \mu} - \dfrac{w}{r} + \left(C - \dfrac{x_c}{r - \mu} + \dfrac{w}{r}\right)\left(\dfrac{x}{x_c}\right)^\gamma, & \text{for } x_c < x \leq x_b \\[4mm] C, & \text{for } x \leq x_c, \end{cases}$$

(A-11)

$$x_c = \frac{\gamma}{\gamma - 1}\cdot\left(cM + \frac{w}{r}\right)\cdot(r - \mu).$$

(A-12)

## Appendix 3. The optimization problem for the leader bank

This appendix shows the solution of the optimization problem for the leader bank in the case of a duopoly. Equation (9) is equivalent to the HJB equation:

$$D_i^L(X_t) = \begin{cases} e^{-rdt}\left( m_i bM \cdot dt + \mathbb{E}_t\left[D_i^L(X_{t+dt})\right] \right) & , \text{ for } t < \tau_b^* \\ e^{-rdt} \max\left\{ m_i(X_t - w)\cdot dt + \mathbb{E}_t\left[D_i^L(X_{t+dt})\right], m_i M \right\} & , \text{ for } \tau_b^* \le t \end{cases}$$

(A-13)

where $D_i^L(X_t)$ represents the value of loan for the leader bank. Applying Ito's lemma to (A-13), we obtain the differential equation for $D_i^L(X_t)$:

$$rD_i^L(x) = m_i bM + \frac{1}{2}\sigma^2 x^2 D_i^{L''}(x) + \mu x D_i^{L'}(x) \quad , \text{ for } x_b < x$$

(A-14)

with boundary conditions:

$$\begin{cases} D_i^L(x \to \infty) \to 0 \\ D_i^L(x_b) = D_i^L\mid_{x \le x_b}(x_b), \end{cases}$$

(A-15)

and:

$$rD_i^L(x) = m_i(x - w) + \frac{1}{2}\sigma^2 x^2 D_i^{L''}(x) + \mu x D_i^{L'}(x) \quad , \text{ for } x \le x_b$$

(A-16)

with boundary conditions:

$$\begin{cases} D_i^L(x \to \infty) \to m_i(x/(r - \mu) - w/r) \\ D_i^L(x_m) = m_i M \\ D_i^{L'}(x_m) = 0. \end{cases}$$

(A-17)

Equations (A-14) and (A-16) can be solved analytically, and the value of loan and the exit threshold for leader $x_m$ are derived by the boundary conditions (A-17) as:

$$D_i^L(x) = \begin{cases} \dfrac{m_i bM}{r} + \left( D_i^L\mid_{x_m < x \le x_b}(x_b) - \dfrac{m_i bM}{r} \right)\left(\dfrac{x}{x_b}\right)^{\gamma} & , \text{ for } x_b < x \\[2ex] m_i\left(\dfrac{x}{r - \mu} - \dfrac{w}{r}\right) + m_i\left( M - \dfrac{x_m}{r - \mu} + \dfrac{w}{r} \right)\left(\dfrac{x}{x_m}\right)^{\gamma} & , \text{ for } x_m < x \le x_b \\[2ex] m_i M & , \text{ for } x \le x_m, \end{cases}$$

(A-18)

$$x_m = \frac{\gamma}{\gamma - 1}\cdot\left( M + \frac{w}{r} \right)\cdot(r - \mu).$$

(A-19)

## Appendix 4. The optimization problem for the follower bank

This appendix shows the solution of the optimization problem for the follower bank. Equation (10) is equivalent to the HJB equation:

$$D_j^F(X_t) = \begin{cases} e^{-rdt}\left( m_j bM \cdot dt + \mathbb{E}_t\left[D_j^F(X_{t+dt})\right]\right) & , \text{ for } t < \tau_b^* \\[2ex] e^{-rdt}\left( m_j(X_t - w)dt + \mathbb{E}_t\left[D_j^F(X_{t+dt})\right]\right) & , \text{ for } \tau_b^* \leq t < \tau_m^* \\[2ex] e^{-rdt}\max\left\{(X_t - w - rm_iM)dt + \mathbb{E}_t\left[D_j^F(X_{t+dt})\right], \right. \\[2ex] \qquad\qquad\qquad\qquad\qquad\left. C - m_iM\right\} & , \text{ for } \tau_m^* \leq t , \end{cases} \tag{A-20}$$

where $D_j^F(X_t)$ represents the value of loan for the follower bank. Applying Ito's lemma to (A-20), we obtain the differential equation of $D_j^F(X_t)$:

$$rD_j^F(x) = m_j bM + \frac{1}{2}\sigma^2 x^2 D_j^{F''}(x) + \mu x D_j^{F'}(x) \text{ , for } x_b < x \tag{A-21}$$

with boundary conditions:

$$\begin{cases} D_j^F(x \to \infty) \to 0 \\ D_j^F(x_b) = D_j^F \mid_{x \leq x_b}(x_b), \end{cases} \tag{A-22}$$

$$rD_j^F(x) = m_j(x - w) + \frac{1}{2}\sigma^2 x^2 D_j^{F''}(x) + \mu x D_j^{F'}(x), \text{ for } x_m < x \leq x_b \tag{A-23}$$

with boundary conditions:

$$\begin{cases} D_j^F(x \to \infty) \to m_j(x/(r - \mu) - w/r) \\ D_j^F(x_m) = D_j^F \mid_{x \leq x_m}(x_m), \end{cases} \tag{A-24}$$

and:

$$rD_j^F(x) = (x - w - rm_iM) + \frac{1}{2}\sigma^2 x^2 D_j^{F''}(x) + \mu x D_j^{F'}(x) \text{ , for } x \leq x_m \tag{A-25}$$

with boundary conditions:

$$\begin{cases} D_j^F(x \to \infty) \to (x/(r - \mu) - w/r) - m_iM \\ D_j^F(x_c) = cM - m_iM \\ D_j^{F'}(x_c) = 0 . \end{cases} \tag{A-26}$$

The above equations can be solved backward analytically from (A-25) to (A-23) and

then to (A-21). The liquidation threshold for follower $x_c$ is derived by the boundary conditions (A-26) as follows:

$$
D_j^F(x) = \begin{cases}
\dfrac{m_i bM}{r} + \left( D_j^F \big|_{x_m < x \le x_b} (x_b) - \dfrac{m_i bM}{r} \right) \left( \dfrac{x}{x_b} \right)^{\gamma} & , \text{ for } x_b < x \\[2em]
m_j \left( \dfrac{x}{r-\mu} - \dfrac{w}{r} \right) + \\[1em]
\quad \left( D_j^F \big|_{x_c < x \le x_m} (x_m) - m_j \left( \dfrac{x_m}{r-\mu} - \dfrac{w}{r} \right) \right) \left( \dfrac{x}{x_m} \right)^{\gamma} & , \text{ for } x_m < x \le x_b \\[2em]
\dfrac{x}{r-\mu} - \dfrac{w}{r} - m_i M + \left( cM - \dfrac{x_c}{r-\mu} + \dfrac{w}{r} \right) \left( \dfrac{x}{x_c} \right)^{\gamma} & , \text{ for } x_c < x \le x_m \\[2em]
cM - m_i M & , \text{ for } x \le x_c \ ,
\end{cases}
\tag{A-27}
$$

$$
x_c = \frac{\gamma}{\gamma - 1} \cdot \left( cM + \frac{w}{r} \right) \cdot (r - \mu) .
\tag{A-28}
$$

## References

Baba, N., "Optimal timing in banks' write-off decisions under the possible implementation of a subsidy scheme: A real option approach," *Monetary and Economic Studies*, 19 (3) , pp. 113−142, 2001.

Boot, A., "Relationship banking: What Do We Know," *Journal of Financial Intermediation*, 9, pp. 7−25, 2000.

Dixit, A. and R. Pindyck, *Investment under Uncertainty*, Princeton University Press, 1994.

Elyasiani, E. and L. Goldberg, "Relationship lending: a survey of the literature," *Journal of Economic and Business*, 59, pp. 315−330, 2004.

Grenadier, R., "The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets," *Journal of Finance*, 51, pp. 1653−1679, 1996.

Kijima, M. and T. Shibata, "Real Options in An Oligopoly Market," *Kyoto Economic Review*, 74, pp. 47−64, 2005.

Leland, H., "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," *Journal of Finance*, 49, pp. 1213−1252, 1994.

Mella-Barral, P. and W. Perraudin, "Strategic Debt Services," *Journal of Finance*, 52, pp. 531−556, 1997.

Weeds, H., "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies*, 69, pp. 729−747, 2002.