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Jouchi Nakajima*

Abstract

This paper proposes the EGARCH model with jumps and heavy-tailed errors, and studies the empirical performance of different models including the stochastic volatility models with leverage, jumps and heavy-tailed errors for daily stock returns. In the framework of a Bayesian inference, the Markov chain Monte Carlo estimation methods for these models are illustrated with a simulation study. The model comparison based on the marginal likelihood estimation is provided with data on the U.S. stock index.

Keywords: Bayesian analysis; EGARCH; Heavy-tailed error; Jumps; Marginal likelihood; Markov chain Monte Carlo; Stochastic volatility

JEL classification: C11, C15, G12

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1 Introduction

The time-varying volatility models have been widely used in various contexts of a time series analysis. Two main streams of modeling a changing variance, the ARCH (autoregressive conditional heteroskedasticity) model and stochastic volatility (SV) model, are well established in financial econometrics. Numerous papers develop their generalization, and these specifications are more and more often applied for empirical analyses in financial economics and macroeconomics.

Among such generalizations of specification, this paper focuses on the jump components and heavy-tailed errors incorporated in the EGARCH (Exponential Generalized ARCH) model and SV model. In many empirical studies, it has been pointed out that asset return data has heavier tails than those of normal distributions. Recent articles show the jump components work well to describe the excess returns observed rarely in financial market. In addition, many empirical results show the existence of a leverage effect, which refers to the increase in volatility following a previous drop in stock returns (Black (1976)).

The GARCH specification, firstly proposed by Bollerslev (1986), formulates the serial dependence of volatility and incorporates the past observations into the future volatility (Bollerslev et al. (1994)). Nelson (1991) proposes the EGARCH model which formulates the leverage effect. Glosten et al. (1993) extends the GARCH model with leverage effect in another way, called the GJR model. These models came to be standards of the asymmetric GARCH model. As for the jump specification, Jorion (1988) firstly introduces the GARCH model with jumps, and later, more complicated jump dynamics are developed by several papers (e.g., Chan and Maheu (2002), Maheu and McCurdy (2004)). We extend the EGARCH model to the one with jumps and heavy-tailed errors in this paper.

In the other stream, the SV models, based on the continuous-time probability process, have also been well studied in financial econometrics (e.g., Ghysels et al. (2002), Shephard (2005)). Among their generalizations, the leverage effect, jump components and heavy-tailed errors in stock returns are well-known to be important in the recent literature (Chib et al. (2002), Jacquier et al. (2004), Berg et al. (2004), Yu (2005), Omori et al. (2007), Nakajima and Omori (2008)). The SV model with Student-t errors is one of the most popular models to account for heavier tailed returns. However, it has been found insufficient to express the tail fatness of returns, and the jump components have recently been introduced to explain the tail behavior (Eraker et al. (2003), Nakajima and Omori (2008)). Various specifications of the SV-jump models are compared in empirical studies (Chernov et al. (2003), Raggi and Bordignon (2006)).

The model comparisons have been studied for the EGARCH models and the SV models separately in literature (e.g., Lehar et al. (2002)). Several works (Kim et al. (1998), Giot and Laurent (2004)) investigate the model comparison among the models in these two classes. Instead, this paper proposes the model comparison among the models in the class of the

EGARCH models and the SV models with jumps and heavy-tails. The comparison for the performance of these models would be beneficial for modeling jumps and heavy-tails on these volatility models in the study of the behavior of the stock returns and option pricing.

It is well known that we have the difficulty in estimating the discrete-time SV model, whose likelihood function is not easily available. It is possible to compute the likelihood using a simulation-based method for a given set of parameters, but it requires a computational burden since we need to repeat the filtering procedure for many times to evaluate the likelihood function for each set of parameters until it reaches the maximum. To overcome this difficulty, we take Bayesian estimation approach with the MCMC methods (e.g., Chib and Greenberg (1996)) for a precise and efficient estimation of the SV model. Kim et al. (1998) develops a fast and reliable MCMC algorithm, called *mixture sampler*. Using this method, the jumps and heavy-tails (Chib et al. (2002)), the leverage and heavy-tails (Omori et al. (2007)), and the leverage, jumps and heavy-tails (Nakajima and Omori (2008) are incorporated into the SV model. As for the EGARCH and the GARCH models, Bauwens and Lubrano (1998), Vrontos et al. (2000) and Nakatsuma (2000) develop the MCMC estimation method for the simple GARCH and EGARCH model. In this paper we develop the MCMC algorithm for the EGARCH model with jumps and heavy-tails. Based on these method, this paper adopts the Bayesian model comparison for both the EGARCH and the SV models by the marginal likelihood, which can be computed by the technique of Chib (1995), Chib and Jeliazkov (2001, 2005).

The rest of paper is organized as follows. In Section 2 we develop the MCMC estimation method for the EGARCH and the GARCH model with jumps and heavy-tails. In Section 3 we review the MCMC estimation scheme for the SV model with leverage, jumps and heavy-tails. Section 4 illustrates our method on simulated data. In Section 5, we show the estimation result of the Bayesian model comparison among these models using the daily returns of the U.S. stock index. Finally, Section 6 concludes.

2 Bayesian inference for the EGARCHJt model

2.1 The model

We introduce the EGARCH(1, 1) model with jumps, heavy-tails (EGARCHJt model) formulated as

$$y_t = k_t\gamma_t + \sqrt{\lambda_t}\sigma_t\varepsilon_t, \qquad \varepsilon_t \sim N(0,1), \tag{1}$$

$$\log\sigma_t^2 = \omega + \beta\log\sigma_{t-1}^2 + \theta\left|\frac{y_{t-1}}{\sigma_{t-1}}\right| + \alpha\left(\left|\frac{y_{t-1}}{\sigma_{t-1}}\right| - \zeta\right). \tag{2}$$

where $0 < \beta < 1$, $\zeta = \mathrm{E}(|e_t|)$, e_t follows a Student-t distribution with degrees of freedom ν and y_t is a stock return. The $\{\sigma_t^2\}_{t=1}^n$ is the conditional variance of the process. If the coefficient

θ is negative, it measures the leverage effect, which implies the increase in volatility following a previous drop in stock returns. The $k_t\gamma_t$ represents a jump component in the measurement equation (1). Following Jorion (1988), the γ_t is a jump flag defined as a Bernoulli random variable such that

$$\pi(\gamma_t = 1) = \kappa, \quad \pi(\gamma_t = 0) = 1 - \kappa, \quad 0 < \kappa < 1,$$

and the k_t is a jump size specified as $k_t \sim N(0, \delta^2)$, where the jump parameter, κ and δ, are unknown and to be estimated. The measurement error $\sqrt{\lambda_t}\varepsilon_t$ is assumed to follow the heavy-tailed Student-t distribution with unknown degrees of freedom ν by letting

$$\lambda_t^{-1} \sim \text{Gamma}(\nu/2, \nu/2).$$

When $\lambda_t \equiv 1$ for all t, the model reduces to the EGARCH model with normal errors. For the EGARCH model with normal errors, we assume that the e_t follows a standard normal distribution. For simplicity, we assume the unconditional variance for the initial as $\log \sigma_1^2 = \omega/(1-\beta)$.

We also consider the GARCH(1,1) model with jumps, heavy-tails (GARCHJt model) formulated as

$$\begin{aligned} y_t &= k_t\gamma_t + \sqrt{\lambda_t}\varepsilon_t\sigma_t, \qquad \varepsilon_t \sim N(0,1), \\ \sigma_t^2 &= \omega + \beta\sigma_{t-1}^2 + \alpha y_{t-1}^2, \end{aligned}$$

where $\omega > 0$, $\beta, \alpha \geq 0$, $\beta + \alpha < 1$, $\log \sigma_1^2 = (\omega + \alpha\kappa\delta^2)/(1 - \alpha - \beta)$. The model comparison is provided including this GARCHJt models in Section 5.

2.2 MCMC algorithm

As shown in many papers, the parameter estimates of the EGARCH and the GARCH model can be obtained by maximum likelihood estimation. In this paper, however, we estimate them by a Bayesian inference using the MCMC algorithm to provide the model comparison including the SV class whose likelihood is not easily available in the maximum likelihood estimation.

Let $y = \{y_t\}_{t=1}^n$, $\gamma = \{\gamma_t\}_{t=1}^n$, $\lambda = \{\lambda_t\}_{t=1}^n$, $\vartheta = (\omega, \beta, \theta, \alpha)$. We set the prior probability density, $\pi(\vartheta)$, $\pi(\kappa)$, $\pi(\delta)$, $\pi(\nu)$ for ϑ, κ, δ and ν. Deriving the posterior distribution of the EGARCHJt model,

$$\pi(\vartheta, \kappa, \delta, \nu, \gamma, \lambda | y),$$

we develop the procedure to sample from this posterior distribution by the MCMC technique. This procedure is also applied for the GARCHJt models. We propose the following algorithm.

Algorithm 1: MCMC algorithm for the EGARCHJt model.

1. Initialize $\vartheta, \kappa, \delta, \nu, \gamma$ and λ.
2. Sample $\vartheta|\delta, \gamma, \lambda, y$.
3. Sample $(\delta, \gamma)|\vartheta, \kappa, \lambda, y$ by
   (a) Sampling $\delta|\vartheta, \kappa, \lambda, y$,
   (b) Sampling $\gamma|\vartheta, \kappa, \delta, \lambda, y$.
4. Sample $\kappa|\vartheta, \delta, \gamma, \lambda, y$.
5. Sample $(\lambda, \nu)|\vartheta, \delta, \gamma, y$ by
   (a) Sampling $\lambda|\vartheta, \delta, \nu, \gamma, y$,
   (b) Sampling $\nu|\lambda$.
6. Go to 2.

We show the details of the MCMC algorithm in Appendix. We note a marginalization of the conditional posterior density enables us to accelerate the convergence of the MCMC sampling. Overall, the likelihood function of the EGARCHJt and the GARCHJt model can be marginalized on the jump variable k_t. In addition, the conditional posterior density of δ can be marginalized on γ in step 3(a). The performance of our algorithm is examined with simulated data in Section 4.

2.3 Alternative jump specification

As an alternative model to incorporate the jump components into the EGARCH model, it is possible to formulate the model as

$$\begin{aligned} y_t &= k_t\gamma_t + \sigma_t\varepsilon_t, \qquad \varepsilon_t \sim N(0,1), \\ \log\sigma_t^2 &= \omega + \beta\log\sigma_{t-1}^2 + \theta\varepsilon_{t-1} + \alpha(|\varepsilon_{t-1}| - E(|\varepsilon_{t-1}|)), \end{aligned}$$

where the jump component does not affect the volatility process. In this specification, however, the conditional posterior distributions for jump variables are not easily computed because the value of the jump variables, k_t and γ_t, are the state variables to be sampled in the MCMC algorithm, while the k_t and γ_t affects all the volatility from time t to n. The draw of $\{k_t\}_{t=1}^n$ and $\{\gamma_t\}_{t=1}^n$ requires so much time that it is unfeasible to implement the MCMC procedure. From this difficulty, we choose the specification of equation (1) and (2) for the EGARCH model with jumps in this paper.

3 Bayesian inference for the SVLJt model

3.1 The model

We consider the discrete-time SV model formulated as

$$\begin{aligned} y_t &= k_t\gamma_t + \sqrt{\lambda_t}\varepsilon_t \exp(h_t/2), \quad t = 1,\ldots,n, & (3)\\ h_{t+1} &= \mu + \phi(h_t - \mu) + \eta_t, \quad t = 1,\ldots,n-1, & (4)\end{aligned}$$

where h_t is an unobserved log-volatility, $|\phi| < 1$, $h_1 \sim N(0, \sigma^2/(1-\phi^2))$,

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \sim N(0, \Sigma), \quad \text{and} \quad \Sigma = \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}.$$

The correlation coefficient ρ measures the leverage effect and $\rho = 0$ implies the SV model without leverage effect. We introduce a jump component $k_t\gamma_t$ in the measurement equation (3). The γ_t is a jump flag defined as a Bernoulli random variable defined in the previous section, and the k_t is a jump size specified by

$$\psi_t \equiv \log(1 + k_t) \sim N(-0.5\delta^2, \delta^2), \tag{5}$$

following Andersen et al. (2002), Chib et al. (2002). Though this specification of the jump size is different from the one incorporated into the EGARCH model in the previous section, the distribution (5) is derived from a discretization of a Lévy process, which is used in the continuous time modeling of financial asset pricing. We label the model defined by equation (3) and (4) as the SVLJt (SV with leverage, jumps and Student-t errors) model.

3.2 Auxiliary mixture sampler

Following Omori et al. (2007), we define $y_t^* = \log(y_t - k_t\gamma_t)^2 - \log\lambda_t$, $d_t = \mathrm{sign}(y_t - k_t\gamma_t) = I(\varepsilon_t > 0) - I(\varepsilon_t \leq 0)$, which rewrites equation (3) as

$$y_t^* = h_t + \xi_t, \tag{6}$$

where $\xi_t = \log\varepsilon_t^2$. Omori et al. (2007) proposes to approximate the bivariate conditional density of $(\xi_t, \eta_t)|d_t$ by a ten-component mixture of bivariate normal distribution, which is an exhaustive extension of Kim et al. (1998) approach The key essence of their approach is that the model (6) and (4) can be approximated to a linear Gaussian state space model conditioned on the mixture component indicator $s_t \in \{1, 2, \ldots, K\}$ as

$$\begin{pmatrix} y_t^* \\ h_{t+1} \end{pmatrix} = \begin{pmatrix} h_t \\ \mu + \phi(h_t - \mu) \end{pmatrix} + \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix}, \tag{7}$$

i	p_i	m_i	v_i^2	a_i	b_i
1	0.00609	1.92677	0.11265	1.01418	0.50710
2	0.04775	1.34744	0.17788	1.02248	0.51124
3	0.13057	0.73504	0.26768	1.03403	0.51701
4	0.20674	0.02266	0.40611	1.05207	0.52604
5	0.22715	−0.85173	0.62699	1.08153	0.54076
6	0.18842	−1.97278	0.98583	1.13114	0.56557
7	0.12047	−3.46788	1.57469	1.21754	0.60877
8	0.05591	−5.55246	2.54498	1.37454	0.68728
9	0.01575	−8.68384	4.16591	1.68327	0.84163
10	0.00115	−14.65000	7.33342	2.50097	1.25049

Table 1: Selection of $(p_i, m_i, v_i^2, a_i, b_i)$ proposed by Omori et al. (2007).

where

$$\left\{ \begin{pmatrix} \xi_t \\ \eta_t \end{pmatrix} | d_t, (s_t = i) \right\} \stackrel{L}{=} \begin{pmatrix} m_i + v_i z_{1t} \\ d_t \rho \sigma (a_i + b_i v_i z_{1t}) \exp(m_i/2) + \sigma\sqrt{1-\rho^2} z_{2t} \end{pmatrix},$$

for $i = 1, 2, \ldots, K$, and $z_t = (z_{1t}, z_{2t})' \sim N(0, I_2)$. Given $s = \{s_1, \ldots, s_n\}$, we can sample the latent variable $h = \{h_1, \ldots, h_n\}$ in one block from its joint distribution using the simulation smoother for a linear Gaussian state space model (de Jong and Shephard (1995), Durbin and Koopman (2002)). The mixture component parameters are provided by Omori et al. (2007) in the case of $K = 10$ (Table 1). Note that (m_i, v_i, a_i, b_i) do not depend on model parameters $\theta \equiv (\phi, \sigma, \rho)$ and μ.

3.3 MCMC algorithm

Let $y^* = \{y_t^*\}_{t=1}^n$, $d = \{d_t\}_{t=1}^n$, $k = \{k_t\}_{t=1}^n$, and we set the prior probability density $\pi(\theta)$, $\pi(\mu)$, $\pi(\kappa)$, $\pi(\delta)$, $\pi(\nu)$ for θ, μ, κ, δ, ν. Then, we draw sample from the posterior distribution

$$\pi(\theta, \mu, \kappa, \delta, \nu, s, h, k, \gamma, \lambda | y),$$

by the MCMC algorithm. Let us reparameterize k_t by $\psi_t \equiv \log(1 + k_t)$ and denote $\psi = \{\psi_t\}_{t=1}^n$, $\psi^{(0)} = \{\psi_t | t = 1, \ldots, n, \text{ s.t. } \gamma_t = 0\}$, $\psi^{(1)} = \{\psi_t | t = 1, \ldots, n, \text{ s.t. } \gamma_t = 1\}$. Following Omori et al. (2007), we use the following sampling algorithm.

Algorithm 2: MCMC algorithm for the SVLJt model.

1. Initialize $\theta, \mu, \kappa, \delta, \nu, s, h, \psi, \gamma$ and λ.
2. Sample $(\theta, \mu, h) | s, y^*, d$ by
   (a) Sampling $\theta | s, y^*, d$,
   (b) Sampling $(\mu, h) | \theta, s, y^*, d$.

3. Sample $\psi^{(1)}|\theta, \mu, \delta, h, \gamma, \lambda, y$.
4. Sample $(\delta, \psi^{(0)})|\psi^{(1)}, \gamma$ by
   (a) Sampling $\delta|\psi^{(1)}, \gamma$,
   (b) Sampling $\psi^{(0)}|\delta, \gamma$.
5. Sample $(\gamma, s)|\theta, \mu, \kappa, h, \psi, \lambda, y$ by
   (a) Sampling $\gamma|\theta, \mu, \kappa, h, \psi, \lambda, y$,
   (b) Sampling $s|\theta, \mu, h, y^*, d$.
6. Sample $\kappa|\gamma$.
7. Sample $(\lambda, \nu)|\theta, \mu, s, h, \psi, \gamma, y$ by
   (a) Sampling $\lambda|\theta, \mu, \nu, s, h, \psi, \gamma, y$,
   (b) Sampling $\nu|\lambda$.
8. Go to 2.

The details of the algorithm is explained by Nakajima and Omori (2008). They show the efficiency of this sampling scheme by providing simulation study. In this paper, we rely on their results to estimate the SVLJt class.

3.4 Alternative jump specification

The EGARCHJt model defined by equation (1) and (2) has the jumps which affect the volatility process, while the SVLJt model defined by equation (3) and (4) has the jumps which do not affect the volatility. As mentioned in section 2.3, it is unfeasible to estimate the EGARCHJt model with jumps which do not affect the volatility. Alternatively, we consider an additional SV model with jumps which do affect the volatility to compare with the EGARCHJt model.

We consider the SV model with correlated jumps (SVLCJ model) given by

$$y_t = k_t\gamma_t + \varepsilon_t \exp(h_t/2), \quad t = 1, \ldots, n, \tag{8}$$
$$h_{t+1} = \mu + \phi(h_t - \mu) + j_t\gamma_t + \eta_t, \quad t = 0, \ldots, n-1. \tag{9}$$

The measurement equation (8) and the state equation (9) have a common jump indicator variable, γ_t, to model the jumps that occur concurrently both in return and in volatility so that the jumps affect the volatility process. The joint distribution of jump sizes is assumed to be

$$\begin{aligned} j_t &\sim \mathrm{Exp}(\mu_J), \\ k_t|j_t &\sim N(\mu_k + \beta_J j_t,\, \sigma_k^2), \end{aligned}$$

where Exp denotes the exponential distribution. The correlation between jump sizes in return and in volatility is considered by the parameter β_J. This type of jumps in the SV model is studied by the recent literature (e.g., Eraker et al. (2003), Kobayashi (2006)). Nakajima and Omori (2008) compares the SVLCJ model with the models in the SVLJt class. To compare the jump models in the EGARCH and the SV specification, we include the SVLCJ model for the model comparison.

4 Illustrative example

This section illustrates our estimation procedure for the EGARCHJt model using the simulated data.

We generate 3,000 observations from the EGARCHJt model given by equation (1) and (2) with $\omega = -0.2$, $\beta = 0.98$, $\theta = -0.05$, $\alpha = 0.15$, $\kappa = 0.01$, $\delta = 0.03$, $\nu = 10$. The following prior distributions are assumed:

$$\begin{aligned} \omega &\sim N(0,1), \quad \beta \sim \text{Beta}(8,1), \quad \theta \sim N(0,1), \quad \alpha \sim N(0,1), \\ \kappa &\sim \text{Beta}(2,100), \quad \delta \sim N(5,0.05), \quad \nu \sim \text{Gamma}(16,0.8). \end{aligned}$$

These prior distributions and the parameters for simulated data reflect the values obtained in the past literature to a certain extent.

We draw $M = 5,000$ sample after the initial 10,000 samples are discarded. The computational results are generated using Ox version 4.02 (Doornik (2006)). Figure 1 shows the sample autocorrelation function, the sample paths and the posterior densities for each parameters. After discarding samples in burn-in period, the sample paths look stable and the sample autocorrelations drop quickly. This indicates our sampling method produces the uncorrelated samples efficiently

Table 2 gives the estimates for posterior means, standard deviations and the 95% credible intervals. All estimated posterior means are close to the true values and the true values are contained in their corresponding 95% credible intervals. The inefficiency factors are also reported to check the performance of our sampling efficiency. The inefficiency factor is defined as $1 + 2\sum_{s=1}^{\infty} \rho_s$ where ρ_s is the sample autocorrelation function at lag s. It is the ratio of

Parameter	True	Mean	Stdev.	95% interval	Inefficiency
ω	-0.02	-0.2194	0.0524	[-0.3338, -0.1244]	34.17
β	0.98	0.9775	0.0058	[0.9649, 0.9881]	34.72
θ	-0.05	-0.0673	0.0125	[-0.0924, -0.0411]	7.57
α	0.15	0.1801	0.0215	[0.1411, 0.2261]	12.11
κ	0.01	0.0270	0.0134	[0.0077, 0.0595]	64.97
δ	0.03	0.0276	0.0074	[0.0137, 0.0436]	38.63
ν	10.0	12.9531	3.5755	[7.2511, 20.921]	122.87

Table 2: Estimation result of the simulated data for EGARCHJt model.

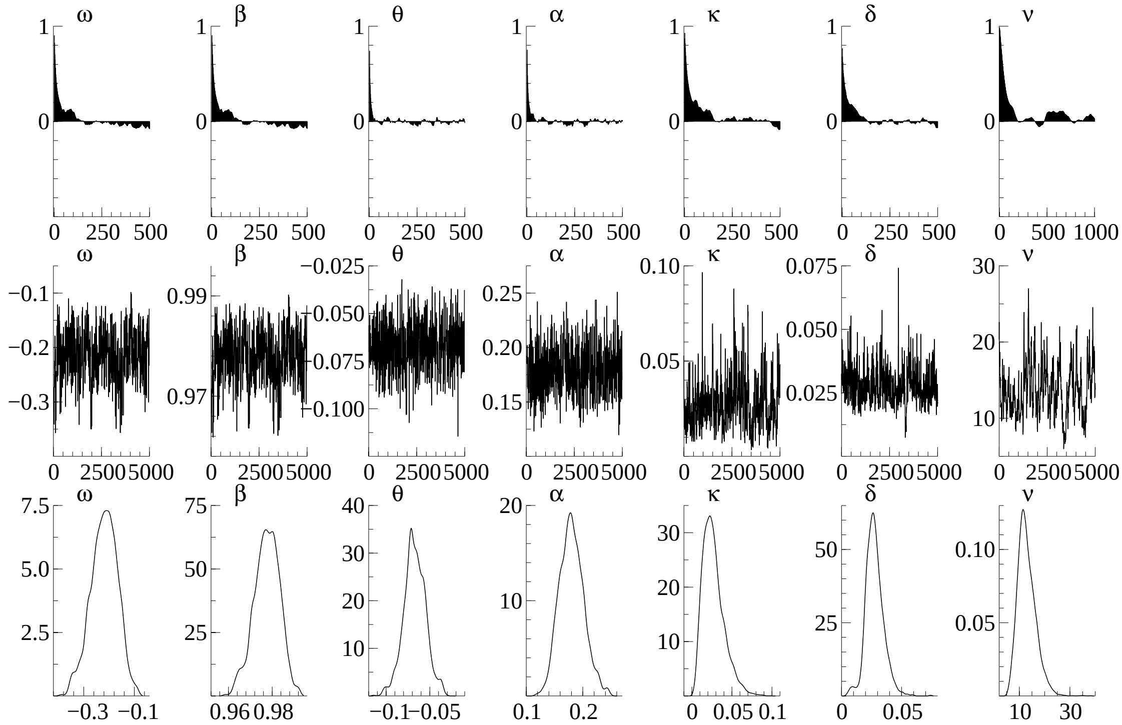

Figure 1: Estimation result of the simulated data (EGARCHJt model). Sample autocorrelations (top), sample paths (middle) and posterior densities (bottom).

variance of the posterior mean from the correlated draws to the one from the hypothetical uncorrelated sample, which measures the loss of sampling efficiency in our correlated MCMC draws (see e.g. Chib (2001)). The inefficiency factors in Table 2 take very low values except ν, which assures the successful sampling without loss of efficiency.

5 Application to stock returns data

5.1 Data

We estimate the EGARCHJt and the SVLJt models for daily stock returns. The series are two U.S. stock price indexes, S&P500 index, and NASDAQ index, from January 2, 1992 to December 29, 2006. The log-difference returns are computed as $y_t = \log P_t - \log P_{t-1}$ where P_t is the closing price on the day t. The sample size is 3,781 for each series. Figure 2 plots the two series and Table 3 summarizes the descriptive statistics of the return data. We find that the skewness of the S&P500 data is slightly negative, while the one of the NASDAQ is almost zero. The kurtosis of each series is around seven or eight, which is larger than a normal distribution.

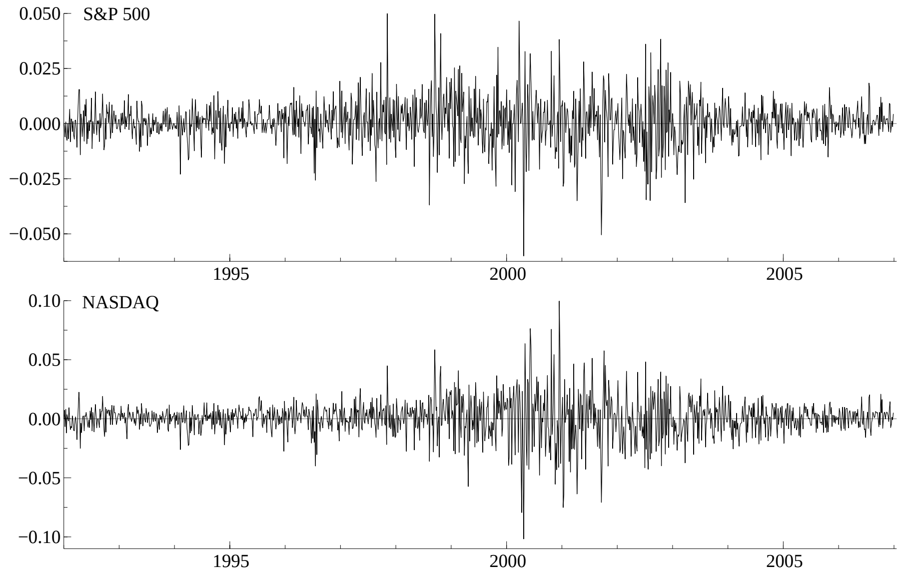

Figure 2: The return data for two stock series (1992/1/2 – 2006/12/29).

	Mean	Stdev.	Skewness	Kurtosis	Max.	Min.
S&P500	0.0003	0.010	-0.109	7.200	0.056	-0.071
NASDAQ	0.0004	0.016	0.002	8.616	0.133	-0.102

Table 3: Summary statistics for the return data (1992/1/2 – 2006/12/29, $n = 3,781$).

5.2 Results for the parameters

We show the results for the parameter estimation of the EGARCHJt model and the SVLJt model using the S&P500 series in this section. For the SVLJt model, we assume the following prior:

$$\begin{aligned}
\frac{\phi+1}{2} &\sim \text{Beta}(20, 1.5), \quad \sigma^{-2} \sim \text{Gamma}(2.5, 0.025), \\
\rho &\sim U(-1, 1), \qquad\qquad \mu \sim N(-10, 1), \\
\kappa &\sim \text{Beta}(2, 100), \quad \log(\delta) \sim N(-2.5, 0.15), \quad \nu \sim \text{Gamma}(16, 0.8).
\end{aligned}$$

As suggested by Kim et al. (1998) for the estimation of the SV model using the mixture sampler, we take $y_t^* = \log((y_t - k_t\gamma_t)^2 + c)$, where c is an offset for the case where $(y_t - k_t\gamma_t)^2$ is too small. We set $c = 10^{-7}$ in this paper. The number of MCMC iterations is same as the simulation study.

Table 4 reports the parameter estimates of the EGARCHJt and the SVLJt model for the S&P500 returns. Figure 3 and 4 plot the sampling results for the EGARCHJt and the SVLJt

Parameter	Mean	Stdev.	95% interval	Inefficiency
EGARCHJt model				
ω	-0.1299	0.0223	[-0.1740, -0.0895]	19.20
β	0.9879	0.0022	[0.9834, 0.9920]	19.14
θ	-0.0880	0.0103	[-0.1097, -0.0693]	10.79
α	0.1051	0.0116	[0.0836, 0.1279]	9.74
κ	0.0307	0.0182	[0.0064, 0.0730]	49.33
δ	0.0098	0.0027	[0.0058, 0.0166]	22.36
ν	13.9744	2.8954	[9.7847, 20.816]	59.79
SVLJt model				
ϕ	0.9864	0.0029	[0.9800, 0.9916]	12.54
σ	0.1373	0.0134	[0.1119, 0.1648]	13.91
ρ	-0.7327	0.0412	[-0.8044, -0.6450]	10.54
$\exp(\mu/2)$	0.0090	0.0006	[0.0079, 0.0103]	1.77
κ	0.0009	0.0007	[0.0001, 0.0026]	3.80
δ	0.0993	0.0525	[0.0334, 0.2192]	62.60
ν	20.8137	4.1579	[14.505, 30.931]	175.81

Table 4: Estimation results for S&P500 returns.

model respectively. The estimates of the volatility parameters $(\omega, \beta, \theta, \alpha)$ and $(\phi, \sigma, \rho, \exp(\mu/2))$ are consistent with the results of the previous literature (e.g., Vrontos et al. (2000), Nakajima and Omori (2008)). For both the two models, the posterior of β or ϕ is close to one, which implies the well-known high persistence of volatility on stock returns. The parameter θ and ρ are estimated negative and the 95% confidence intervals do not contain zero. This indicates that there exists the leverage effect in our stock returns data.

We find that there are specific differences for the estimates of the jump parameters between the EGARCHJt model and the SVLJt model. The posterior mean of κ for the EGARCHLJt model is about 3%, while the one for the SVLJt model is much smaller, 0.9%. This indicates that the EGARCHJt model more often captures the excess returns by the jumps than the SVLJt model. Though we can not compare the jump sizes between the two models directly because the specifications of the jump size are different, if we calculate the standard deviation of the jump size for the SVLJt model using the posterior mean of δ, it is 0.0996 for the SVLJt model and 0.0098 for the EGARCHJt model. The empirical results show the SVLJt model captures larger excess returns by the jumps with the smaller probability than the EGARCHJt model with the larger jump probability. This would be caused from the difference of the specification of the volatility process and the jump component between the two models. The SV models have the disturbance for their volatility process, while the EGARCH models do not have it and the volatility on the next day is determined by the return and the volatility on the current day. In other words, the volatility process of the SV models can move more flexibly than the EGARCH models. In short, when the return marks a certain excess shock, if the effect of the shock is not persistent, the volatility process of the SVLJt model would capture the shock, while the EGARCHJt would capture it by the jump component. The difference of

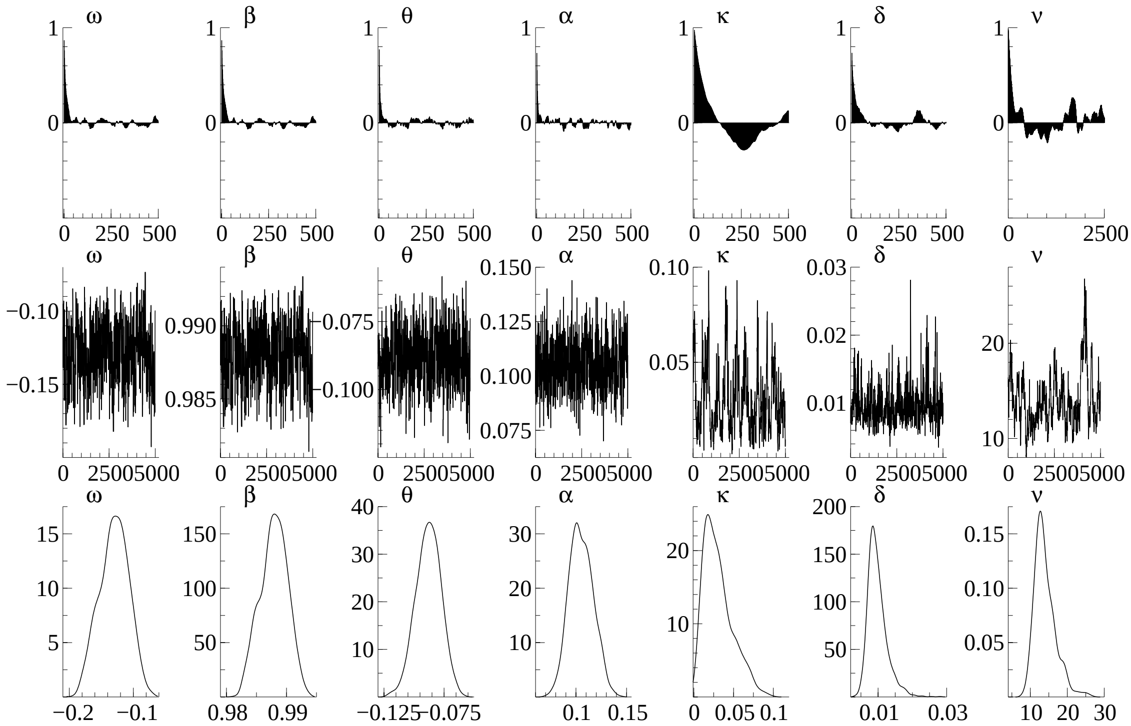

Figure 3: Estimation result of the S&P500 returns (EGARCHJt model). Sample autocorrelations (top), sample paths (middle) and posterior densities (bottom).

the performance to fit the behavior of the stock returns between these two models is examined in the next section.

In addition, the posterior mean of the parameter ν for the EGARCHJt model is smaller than the one for the SVLJt model. As discussed by Nakajima and Omori (2008), the model whose jump probability is estimated to be smaller tends to have a heavier-tailness of the errors. It is considered that the jump component captures less excess returns, when the errors have the heavier-tails. We can find the same results for the relation of the EGARCHJt model and the SVLJt model in this study.

5.3 Results for the model comparisons

In a Bayesian framework, we can compare the model fitting based on the marginal likelihood or Bayes factor, even if the competing models are not nested. When the prior probabilities are assumed to be equal, we choose the model which yields the largest marginal likelihood. In order to compare the competing models in the EGARCH and the SV class with jumps, heavy-tails, we estimate their marginal likelihood for our dataset.

The marginal likelihood is defined as the integral of the likelihood with respect to the prior density of the parameter. Following Chib (1995), we estimate the log of marginal likelihood

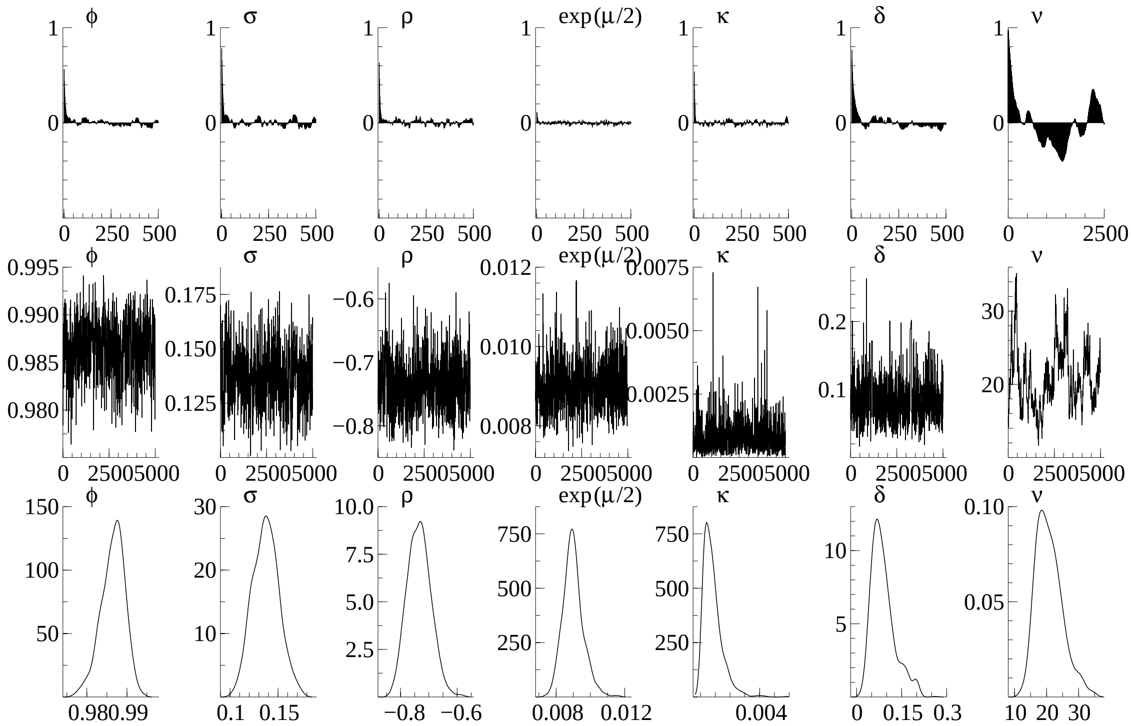

Figure 4: Estimation result of the S&P500 returns (SVLJt model). Sample autocorrelations (top), sample paths (middle) and posterior densities (bottom).

$m(y)$, as

$$\log m(y) = \log f(y|\Theta) + \log \pi(\Theta) - \log \pi(\Theta|y),$$

where Θ is a parameter set in the model, $f(y|\Theta)$ is a likelihood, $\pi(\Theta)$ is a prior probability density and $\pi(\Theta|y)$ is a posterior density. This equality holds for any Θ, but we usually use the posterior mean of Θ to obtain a stable estimate of $m(y)$. The prior probability density is easily calculated, though the likelihood and posterior part requires a simulation evaluation. For the SV class, the likelihood can be estimated by the particle filter (e.g., Pitt and Shephard (1999), Chib et al. (2002), Omori et al. (2007)). For the posterior part, we use the method of Chib (1995), Chib and Jeliazkov (2001, 2005) to compute $\pi(\Theta|y)$ using the sample obtained through the reduced iteration of the MCMC algorithm.

We show the results of the model comparison for the 17 competing models of the EGARCH and the SV class with jumps and heavy-tails, including the SVLCJ model, for two stock returns data. We label the models of the EGARCHJt and the SVLJt class, with 'L' referring to the leverage effect, 'J' referring to jump component, 't' referring to Student-t errors. For the

Series:	**S&P500**		**NASDAQ**	
Model	Log-ML	Ranking	Log-ML	Ranking
GARCH	12578.42 (0.01)	17	11300.17 (0.01)	17
GARCHt	12621.00 (0.26)	14	11321.13 (0.18)	14
GARCHJ	12606.61 (0.02)	16	11317.49 (0.04)	16
GARCHJt	12618.57 (0.22)	15	11319.88 (0.19)	15
EGARCH	12661.70 (0.09)	9	11345.23 (0.06)	11
EGARCHt	12702.52 (0.68)	3	11367.52 (0.93)	5
EGARCHJ	12692.14 (0.12)	7	11369.65 (0.11)	3
EGARCHJt	12703.09 (0.46)	2	11368.57 (0.61)	4
SV	12650.05 (0.43)	12	11349.39 (0.76)	10
SVt	12660.25 (0.80)	10	11350.26 (1.18)	9
SVJ	12644.53 (0.78)	13	11334.23 (2.51)	13
SVJt	12653.52 (0.80)	11	11344.76 (0.94)	12
SVL	12702.21 (0.43)	4	11374.48 (0.83)	2
SVLt	12708.90 (0.66)	1	11375.05 (0.98)	1
SVLJ	12697.26 (1.14)	6	11358.97 (2.42)	7
SVLJt	12702.03 (0.79)	5	11363.84 (0.91)	6
SVLCJ	12691.27 (1.02)	8	11358.49 (0.78)	8

* The values are based on log scale and standard error in parentheses.

Table 5: Marginal likelihood (ML) for stock returns data.

GARCHJt models, we assume the prior as

$$\begin{aligned}
\omega^{-1} &\sim \text{Gamma}(5,\, 5\times 10^{-4}), \quad && \alpha \sim \text{Beta}(3,3), \quad \beta(1-\alpha)|\alpha \sim \text{Beta}(8,1),\\
\theta &\sim U(0, 1-\alpha-\beta), && \kappa \sim \text{Beta}(2,100),\\
\delta &\sim N(5, 0.05), && \nu \sim \text{Gamma}(16, 0.8),
\end{aligned}$$

and for the SVLCJ model, we assume the following prior:

$$\begin{aligned}
&\mu_J \sim \text{Exp}(0.2), \quad \beta_J \sim N(0,1),\\
&\mu_k \sim N(0,1), \quad \sigma_k^{-2} \sim \text{Gamma}(2.5, 0.025).
\end{aligned}$$

We evaluate the posterior density at the point Θ through the reduced MCMC sampling, which is iterated for 5,000 draws. We run 10 replications of the particle filter to estimate the standard error of the likelihood.

Table 5 reports the estimated marginal likelihoods, standard errors and rankings for all the competing models. The estimates shows that the marginal likelihood of the EGARCH models are higher than the GARCH models and the one of the SV models with leverage are higher than the SV models without leverage. This indicates the leverage effect is important to analyze the stock returns as discussed by Omori et al. (2007).

The EGARCHJt models which we propose in this paper perform well and the marginal likelihood is almost equivalent to the SVLJt models. Moreover, the EGARCHt and the

EGARCHJt model are favored over the the SVLJ and SVLJt model. Compared with the SVLCJ model, whose jumps do affect the volatility as the EGARCHJt models, the marginal likelihood of the EGARCHt, the EGARCHJ, and the EGARCHJt model dominate the one of the SVLCJ model.

The marginal likelihood of the SVLt model is the highest among the competing models in the EGARCHJt and SVLJt class. As pointed out by Nakajima and Omori (2008), the SVLt model performs better than the other models in the SVLJt class, and we find that it is also favored over the EGARCHJt models.

The heavy-tails contribute most of the models for their marginal likelihoods. Most of the EGARCH models with Student-t errors are favored over the ones without them. On the other hand, the jumps do not always contribute these models as discussed by Nakajima and Omori (2008). Overall, we find that the jumps and heavy-tails have the large contributions for the EGARCH model based on the marginal likelihood. The ratio of the marginal likelihood of the SVL model to the one of the EGARCH model is quite large, while the one of the SVLJt model to the EGARCHJt model is less than one, which means the EGARCHJt model outperforms the SVLJt model by incorporating the jumps and heavy-tails. These results are robust for the two stock returns series.

6 Conclusion

In this paper, we propose the EGARCH model with jumps and heavy-tailed errors and develop the MCMC estimation algorithm. Our proposed method is illustrated using simulated data. We provided the model comparisons for the EGARCH and the SV models with jumps and heavy-tails using the return data of the U.S. stock index. Based on the estimates of the marginal likelihood, we found the jumps and heavy-tails raises the marginal likelihood of the EGARCH model. The EGARCH model with jumps and heavy-tails and the SV model with heavy-tails and leverage fit to the data better than other competing models for our dataset.

Appendix. MCMC algorithm for EGARCHJt and GARCHJt model

We illustrate the MCMC procedure for the EGARCHJt and the GARCHJt model in this appendix. The proposed algorithm is as follows:

1. Initialize $\vartheta, \kappa, \delta, \nu, \gamma$ and λ.

2. Sample $\vartheta|\delta, \gamma, \lambda, y$.

   To sample ϑ from its conditional posterior distribution $\pi(\vartheta|\delta, \gamma, \lambda, y) \propto \pi(\vartheta) f(y|\vartheta, \delta, \gamma, \lambda)$, we use the Metropolis-Hasting (M-H) algorithm (see e.g., Chib and Greenberg (1995)), because the posterior distribution is not available in the form of an usual distribution such as a normal distribution. We construct the proposal density in the form of a normal distribution for the M-H algorithm by fitting the mean and variance on the target posterior density from the product of second-order Taylor expansion. For the restriction for the parameters in the GARCH models, we consider the transformation of $\vartheta \to \tilde{\vartheta} = (\tilde{\omega}, \tilde{\alpha}, \tilde{\beta})$ such that $\tilde{\omega} = \log \omega$, $\tilde{\alpha} = \log(\alpha/(1-\alpha))$, and $\tilde{\beta} = \log(\beta/(1-\alpha-\beta))$. The posterior density is transformed over the full space of R^3, where we easily implement the proposal density. To draw a candidate of the M-H algorithm, we find $\tilde{\vartheta}^*$ which maximizes (or approximately maximizes) the posterior density, $\pi(\tilde{\vartheta}|\delta, \gamma, \lambda, y)$, and generate the candidate $\tilde{\vartheta}_*$ from the normal distribution $N(\mu_*, \Sigma_*)$, where

   $$\mu_* = \tilde{\vartheta}^* + \Sigma_* \left. \frac{\partial \log \pi(\tilde{\vartheta}|\delta, \gamma, \lambda, y)}{\partial \vartheta} \right|_{\tilde{\vartheta} = \tilde{\vartheta}^*}, \quad \Sigma_*^{-1} = - \left. \frac{\partial \log \pi(\tilde{\vartheta}|\delta, \gamma, \lambda, y)}{\partial \tilde{\vartheta} \partial \tilde{\vartheta}'} \right|_{\tilde{\vartheta} = \tilde{\vartheta}^*}.$$

   which is obtained from the second-order Taylor expansion around $\tilde{\theta}$. Let ϑ_0 denote the current point of ϑ and we reversely transform $\tilde{\vartheta}^*$ to ϑ^*. We accept the candidate ϑ^* with probability

   $$\alpha(\theta_0, \theta^*|\delta, \gamma, \lambda, y) = \min \left\{ \frac{\pi(\theta^*|\delta, \gamma, \lambda, y) q(\theta_0|\delta, \gamma, \lambda, y)}{\pi(\theta_0|\delta, \gamma, \lambda, y) q(\theta^*|\delta, \gamma, \lambda, y)}, 1 \right\},$$

   where q denotes the proposal density. If the candidate θ^* is rejected, we take the current value θ_0 as the next draw.

3. Sample $(\delta, \gamma)|\vartheta, \kappa, \lambda, y$.

   (a) Sample $\delta|\vartheta, \kappa, \lambda, y$.

   The joint posterior distribution of (δ, γ) is given by

   $$\pi(\delta, \gamma|\vartheta, \kappa, \lambda, y) \propto \pi(\delta) \prod_{t=1}^{n} \kappa^{\gamma_t} (1-\kappa)^{1-\gamma_t} \frac{1}{\sqrt{\sigma_t^2 \lambda_t + \gamma_t \delta^2}} \exp \left\{ -\frac{y_t^2}{2(\sigma_t^2 \lambda_t + \gamma_t \delta^2)} \right\},$$

where k_t is marginalized in the likelihood function. To sample δ, we further marginalize this joint posterior distribution over γ. The marginalized conditional posterior distribution is formed as

$$\pi(\delta|\vartheta,\kappa,\lambda,y) \propto \pi(\delta)\prod_{t=1}^{n}\left[\frac{\kappa}{\sqrt{\sigma_t^2\lambda_t+\delta^2}}\exp\left\{-\frac{y_t^2}{2(\sigma_t^2\lambda_t+\delta^2)}\right\}+\frac{1-\kappa}{\sigma_t\sqrt{\lambda_t}}\exp\left(-\frac{y_t^2}{2\sigma_t^2\lambda_t}\right)\right].$$

We can sample δ using the M-H algorithm. We note that this marginalization enables us to accelerate the convergence of the MCMC sampling.

(b) Sample $\gamma|\vartheta,\kappa,\delta,\lambda,y$.

Sampling γ from its posterior distribution requires only to evaluate the Bernoulli distribution $\pi(\gamma_t|\vartheta,\kappa,\delta,\lambda,y)$ where $\gamma_t = 0, 1$. We sample γ_t using the probability mass function of its posterior density as

$$\begin{aligned}
\pi(\gamma_t=1|\vartheta,\kappa,\delta,\lambda,y) &\propto \frac{\kappa}{\sqrt{\sigma_t^2\lambda_t+\delta^2}}\exp\left\{-\frac{y_t^2}{2(\sigma_t^2\lambda_t+\delta^2)}\right\},\\
\pi(\gamma_t=0|\vartheta,\kappa,\delta,\lambda,y) &\propto \frac{1-\kappa}{\sigma\sqrt{\lambda_t}}\exp\left(-\frac{y_t^2}{2\sigma_t^2\lambda_t}\right),
\end{aligned}$$

for $t = 1, \ldots, n$.

4. Sample $\kappa|\vartheta,\delta,\gamma,\lambda,y$.

When we specify the prior as $\kappa \sim \text{Beta}(n_{\alpha 0}, n_{\beta 0})$ for the EGARCHJt model, we sample κ from $\text{Beta}(n_{\alpha 0}+n_1, n_{\beta 0}+n_0)$, where n_1 and n_0 denote the count of $\gamma_t = 1$ and $\gamma_t = 0$ respectively. On the other hand, for the GARCHJt model, we consider the posterior distribution of κ conditional on $(\vartheta,\delta,\gamma,\lambda)$, because we assume $\sigma_1^2 = (\omega+\alpha\kappa\delta^2)/(1-\alpha-\beta-\rho/2)$. The posterior distribution of κ is not written in the form of a beta distribution. We sample κ using the M-H algorithm with the candidate drawn from $\text{Beta}(n_{\alpha 0}+n_1, n_{\beta 0}+n_0)$.

5. Sample $(\lambda,\nu)|\vartheta,\delta,\gamma,y$.

(a) Sample $\lambda|\vartheta,\delta,\nu,\gamma,y$.

The joint posterior distribution of (λ,ν) is given by

$$\begin{aligned}
\pi(\lambda,\nu|\vartheta,\delta,\gamma,y) \propto\ & \pi(\nu)\prod_{t=1}^{n}\frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\lambda_t^{-\left(\frac{\nu}{2}+1\right)}\exp\left(-\frac{\nu}{2\lambda_t}\right)\\
&\times\frac{1}{\sqrt{\sigma_t^2\lambda_t+\gamma_t\delta^2}}\exp\left\{-\frac{y_t^2}{2(\sigma_t^2\lambda_t+\gamma_t\delta^2)}\right\}.
\end{aligned}$$

We sample λ_t from its conditional posterior density,

$$\pi(\lambda_t|\vartheta, \delta, \nu, \gamma, y) \propto \lambda_t^{-(\frac{\nu}{2}+1)} \exp\left(-\frac{\nu}{2\lambda_t}\right) \frac{1}{\sqrt{\sigma_t^2\lambda_t + \gamma_t\delta^2}} \exp\left\{-\frac{y_t^2}{2(\sigma_t^2\lambda_t + \gamma_t\delta^2)}\right\},$$

by the M-H algorithm with the candidate drawn from its prior density as $(\lambda_t^*)^{-1} \sim \text{Gamma}(\nu/2, \nu/2)$, for $t = 1, \ldots, n$.

(b) Sample $\nu|\lambda$.

Finally, the conditional posterior distribution for ν is given by

$$\pi(\nu|\lambda) \propto \pi(\nu) \frac{\left(\frac{\nu}{2}\right)^{\frac{n\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)^n} \prod_{t=1}^{n} \lambda_t^{-\frac{\nu}{2}} \exp\left(-\frac{\nu}{2}\sum_{t=1}^{n}\lambda_t^{-1}\right).$$

We sample ν by the M-H algorithm with the normal proposal density.

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