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Abstract

This paper derives a generalized optimal interest rate rule that is optimal even under a zero lower bound on nominal interest rates in an otherwise basic New Keynesian model with inflation inertia. Using this optimal rule, we investigate optimal entrance and exit strategies of the zero interest rate policy (ZIP) under the realistic model with inflation inertia and a variety of shocks. The simulation results reveal that the timings of the entrance and exit strategies in a ZIP change considerably according to the forward- or backward-lookingness of the economy and the size of the shocks. In particular, for large shocks that result in long ZIP periods, the time to the start (end) of the ZIP period is earlier (later) in an economy with inflation inertia than in a purely forward-looking economy. However, these outcomes are surprisingly converse to small shocks that result in short ZIP periods.

Keywords: Zero Interest Rate Policy; Optimal Interest Rate Rule
JEL classification: E52, E58

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1 Introduction

Central banks implement a low interest rate where the scope for cutting the policy rate is very limited. For example, the Japanese economy has faced a deflationary environment for a prolonged period. The Bank of Japan (BOJ) set their operational short-term interest rate -the uncollateralized overnight call rate- virtually equal to zero for almost seven years from February 1999 to June 2006. Moreover, a low interest rate environment, where the policy interest rate equals 0.5 percent, has continued up to now (July 2008), as shown in Figure 1. In the United States, the Federal Reserve Board (FRB) temporarily set the federal funds rate as low as one percent in 2003 and 2004, which was a historical low. In Switzerland, the Swiss National Bank reduced its policy rate to almost zero percent from 2003 to 2005.[1] Central banks can no longer ignore the possibility of hitting the zero (percent) lower bound on nominal interest rates.

In a situation in which the zero lower bound on nominal interest rates binds, many studies, such as Reifschneider and Williams (2000), Eggertsson and Woodford (2003a, b), and Jung, Teranishi and Watanabe (2005), outline the characteristics of desirable monetary policies.[2] Reifschneider and Williams (2000) investigate a desirable monetary policy of the US in a low interest rate environment. Their conclusion is that a central bank must preemptively start a ZIP and enough prolong a ZIP with history dependence in a situation where the policy interest rates hit zeros. Their analysis is very powerful and reasonable; however, they do not address the issue of optimal monetary policy. Eggertsson and Woodford (2003a, b) and Jung et al. (2005) assume a standard New Keynesian model consisting

[1] Furthermore, the European Central Bank set overnight rates at two percent, from 2003 to 2005.

[2] Adam and Billi (2006, 2007) and Nakov (2008) assume shocks follow a stochastic process and numerically reveal the properties of optimal monetary policies under a situation in which a zero lower bound on nominal interest rate binds in a standard New Keynesian model consisting of a forward-looking IS curve and forward-looking Phillips curve. Their conclusions are qualitatively the same as in the former studies mentioned above.

of a forward-looking IS curve and forward-looking Phillips curve and derive optimal targeting rules in a purely forward-looking economy. They imply that an important feature of optimal monetary policy in a low interest rate environment is that the ZIP should be continued after the improvement in the economic situation. Because of this commitment to the policy, central banks are able to stimulate the economy by inducing high expected inflation, and therefore, low real interest rates even in a situation where the nominal interest rate is at the zero lower bound. Their analyses, however, are extreme cases using purely forward-looking models and focus on the roles of expectations of agents. Thus, we have to assume a more realistic model with inflation inertia to obtain implications from theory for the implementation of monetary policy. Moreover, their suggestions that the central bank should continue a ZIP even after the inflation rate becomes a positive value or shocks disappear, mainly depend on the effects of large negative shocks in the natural rate of interest that induce a long enough ZIP period. They ignore the roles of price shocks and the effects of the size of the shocks on the nature of the ZIP, and so these papers mainly focus on one of four situations: the case of the Forward-looking Economy, Large Shock, in a ZIP environment, as shown in Table 1.

The first contribution of the paper is to provide an optimal interest rate rule in a low interest rate environment by extending the discussion in Giannoni and Woodford (2002). In other words, we propose a generalized optimal interest rate rule that is valid regardless of whether or not the zero lower bound on nominal interest rates binds. In contrast with Eggertsson and Woodford (2003a, b) and Jung et al. (2005), which show the optimal targeting rule in a low interest rate environment, we propose an optimal interest rate rule that is intuitively comprehensible.[3] Unlike Reifschneider and Williams (2000), we theoretically derive an optimal interest rate rule. We reveal that the optimal interest rate rule should

[3]Sugo and Teranishi (2005) derive other forms of optimal interest rate rules under a zero lower bound on the nominal interest rate in a purely forward-looking economy.

keep proper information on forward- and backward-looking properties using indicator variables regarding the zero lower bound on the nominal interest rate instead of the nominal interest rate itself in a low interest rate environment.

The second contribution is to consider an optimal monetary policy under a more realistic Phillips curve with inflation inertia (hybrid Phillips curve) and a variety of shocks, including price shocks and natural rate of interest shocks, of various sizes, than the former studies do, which assume a forward-looking Phillips curve and large natural rate of interest shocks. Many studies that develop realistic models, such as Smets and Wouters (2003) and Christiano, Eichenbaum and Evans (2005), support the hybrid Phillips curve and the importance of price shocks in explaining the economic dynamics.[4] This realistic setting provides many implications for the conduct of monetary policy, especially for entrance and exit strategies in a ZIP environment. Moreover, both the nature and size of the shocks change the timing of the ZIPs. To summarize, the implications for monetary policy are as follows. For the case of a large-scale shock that induces a long ZIP period, the central bank should continue the ZIP even after the end of the economic contraction in a purely forward-looking economy. We, however, need to carefully consider this result, because the ZIP period is shorter with inflation inertia than without it. In particular, the time to the start (end) of the ZIP period is earlier (later) in an economy with inflation inertia. These properties exist because the central bank has to commit to a long enough ZIP period in response to large shocks to stimulate the economy through the expected inflation channel, which is eventually more likely to induce stronger economic fluctuations after the ZIP period in a hybrid economy than in a forward-looking economy. But, these results are converse for the case of small-scale shocks that induce a ZIP for a few periods. For small-scale shocks, the ZIP is

[4]For example, Amato and Laubach (2003a) and Steinsson (2003) consider optimal monetary policies in an economy with inflation inertia but without a zero lower bound on the nominal interest rate. Our analysis extends their studies in the sense that we explicitly introduce a nonnegativity constraint on the nominal interest rate.

ended well before the economic contractions end. Moreover, the time to the start (end) of the ZIP period is earlier (later) in an economy with inflation inertia. These properties exist because the central bank does not need to care about a large economic boom after ending the ZIP because the central bank does not rely on the expected inflation channel as much.

The rest of the paper is organized as follows. The following section describes the model. In Section 3, we propose a generalized optimal interest rate rule under the zero lower bound on the nominal interest rate. Section 4 investigates the properties of the optimal monetary policy rule relating to the start and end of the policy following large-scale shocks. Section 5 investigates the properties of the optimal monetary policy rule relating to the start and end of policy following small-scale shocks. Section 6 provides the robustness analysis. Finally, in Section 7, we summarize our findings in this paper.

2 The Model

We use the model developed by Clarida, Gali and Gertler (1999) and Woodford (2003). The economy other than the central bank is represented by four equations: an "IS curve", a "Phillips curve", a shock to the natural interest rate, and a cost-push shock.

$$x_t = \mathrm{E}_t x_{t+1} - \sigma \left[(i_t - \mathrm{E}_t \pi_{t+1}) - r_t^n \right], \tag{2.1}$$

$$\pi_t - \gamma \pi_{t-1} = \kappa x_t + \beta \mathrm{E}_t(\pi_{t+1} - \gamma \pi_t) + \varepsilon_t, \tag{2.2}$$

$$r_t^n = \rho^r r_{t-1}^n + \epsilon_t^r, \tag{2.3}$$

$$\varepsilon_t = \rho^\varepsilon \varepsilon_{t-1} + \epsilon_t^\varepsilon. \tag{2.4}$$

Eq. (2.1) represents the forward-looking IS curve. This IS curve states that the output gap in period t, denoted by x_t, is determined by the expected value of the output gap in period t+1 and the deviation of the short-term real interest rate, the nominal interest rate

i_t minus the expected rate of inflation $E_t\pi_{t+1}$, from the natural rate of interest in period t, denoted by r_t^n, which can be interpreted as a shock and follows a first order autoregressive process. Eq. (2.2) is a hybrid Phillips curve. This Phillips curve states that inflation in period t depends on an expected rate of future inflation in period t+1, a lag of inflation in period t-1, and the output gap in period t, and includes price shock given by ε_t that follow a first autoregressive process. Gali and Gertler (1999) and Woodford (2003) show the microfoundations of the Phillips curve that includes inflation inertia. The hybrid Phillips curve is empirically more realistic than the forward-looking Phillips curve, as suggested by Smets and Wouters (2003) and Christiano et al. (2005), and induces important policy implications as shown in the later sections. Here ϵ_t^r and ϵ_t^ε are i.i.d. disturbances and σ, κ, β, γ, ρ^r, and ρ^ε are parameters, satisfying $\sigma > 0$, $\kappa > 0$, $0 < \beta < 1$, $0 \leq \gamma \leq 1$, $0 \leq \rho^r < 1$, and $0 \leq \rho^\varepsilon < 1$. Eq. (2.3) and Eq. (2.4) describe shocks to the economy. It should be noted that the Phillips curve becomes purely forward-looking when $\gamma = 0$. Furthermore, we put a nonnegativity constraint on nominal interest rates.

$$i_t \geq 0. \tag{2.5}$$

We assume that the entire shock process is known with certainty in period 1; namely, a deterministic shock.[5] We know that this assumption is not trivial. However our assumptions about the shock process enable us to analytically investigate the properties of the optimal interest rate rule in the face of a zero lower bound on the nominal interest rate in a simple way. We also assume that, prior to the shock, the model economy is in a steady state where x_t and π_t are zeros and i_t is i^*.

[5] We note that certainty equivalence does not hold in our optimization problem because of the nonlinearity caused by the zero lower bound on the nominal interest rate. Thus, it is impossible to obtain an analytical solution under stochastic shocks. Eggertsson and Woodford (2003a, b) extend the analysis under the special case of stochastic disturbances. Surely, we can extend our analysis by making use of the method suggested by Eggertsson and Woodford (2003a, b); however, the qualitative outcomes do not change.

Next, we present the central bank's intertemporal optimization problem. In the case of the hybrid Phillips curve, Woodford (2003) shows that the period loss function is given by:

$$L_t = (\pi_t - \gamma\pi_{t-1})^2 + \lambda_x x_t^2 + \lambda_i (i_t - i^*)^2, \tag{2.6}$$

where λ_x and λ_i are positive parameters. The central bank chooses the path of the short-term nominal interest rate, starting from period 1, to minimize welfare loss U_1:

$$U_1 = \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} L_t. \tag{2.7}$$

3 The Optimal Monetary Policy Rule in a Low Interest Rate Environment

In this section, we set up the optimization problem to obtain the optimal monetary policy conditions in the low interest rate environment, namely under the zero lower bound on nominal interest rates. In this process, we make use of the Kuhn–Tucker solution. We then propose a generalized optimal interest rate rule in a low interest rate environment.

3.1 Optimization

We assume that the central bank solves an intertemporal optimization problem in period 1, considering the expectation channel of monetary policy, and commits itself to the computed optimal path. This is the optimal solution from a timeless perspective defined by Woodford (2003).

The optimal monetary policy under the zero lower bound on the nominal interest rate in a timeless perspective[6] is expressed by the solution of the optimization problem, which

[6] A detailed explanation of the timeless perspective is provided in Woodford (2003).

is represented by the following Lagrangian form:

$$\mathcal{L} = \mathrm{E}_1 \left\{ \sum_{t=1}^{\infty} \beta^{t-1} \left\{ \begin{array}{c} L_t \\ -2\phi_{1t} \left[x_{t+1} - \sigma(i_t - \pi_{t+1} - r_t^n) - x_t \right] \\ -2\phi_{2t} \left[\kappa x_t + \beta(\pi_{t+1} - \gamma \pi_t) - \pi_t + \gamma \pi_{t-1} \right] \\ -2\phi_{3t} i_t \end{array} \right\} \right\},$$

where ϕ_1, ϕ_2, and ϕ_3 represent the Lagrange multipliers associated with the IS constraint, the Phillips curve constraint, and the nominal interest rate constraint, respectively. We differentiate the Lagrangian with respect to π_t, x_t, and i_t under the nonnegativity constraint on nominal interest rates to obtain the first-order conditions:

$$-\beta\gamma\pi_{t+1} + (\beta\gamma^2 + 1)\pi_t - \gamma\pi_{t-1} - \beta^{-1}\sigma\phi_{1t-1} - \beta\gamma\phi_{2t+1} + (\beta\gamma + 1)\phi_{2t} - \phi_{2t-1} = 0, \quad (3.1)$$

$$\lambda_x x_t + \phi_{1t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t} = 0, \quad (3.2)$$

$$\lambda_i(i_t - i^*) + \sigma\phi_{1t} - \phi_{3t} = 0, \quad (3.3)$$

$$i_t\phi_{3t} = 0, \quad (3.4)$$

$$\phi_{3t} \geq 0, \quad (3.5)$$

$$i_t \geq 0. \quad (3.6)$$

Eqs (3.4), (3.5), and (3.6) are conditions for the nonnegativity constraint on nominal interest rates. The above six conditions, together with the IS (Eq. (2.1)) and hybrid Phillips (Eq. (2.2)) equations, are the conditions governing the loss minimization. In other words, the sequence of the interest rates determined by these conditions is the optimal interest rate setting at each time under the zero lower bound on nominal interest rates. When the nonnegativity constraint is not binding (i.e., $i_t > 0$), the Lagrange multiplier, ϕ_{3t}, becomes zero by the Kuhn–Tucker condition in Eq. (3.3), and then the interest rate is determined by the conditions given by Eqs (2.1), (2.2), (3.1), (3.2) and (3.3) with $\phi_{3t} = 0$. When the nonnegativity constraint is binding (i.e., $i_t = 0$), the interest rate is simply set to zero.

In this case, the interest rate remains zero until the Lagrange multiplier, ϕ_{3t}, becomes zero.[7] It should be noted that the expectation operator, E_t, does not appear in these equations because the future path of shocks is perfectly foreseen, thanks to the assumption of deterministic shocks.

3.2 The Generalized Optimal Interest Rate Rule

In this subsection, we propose the generalized optimal interest rate rule that is valid with any deterministic shock process under the zero lower bound on the nominal interest rate.

The generalized optimal interest rate rule in the face of a zero lower bound on the nominal interest rate can be derived from the optimality conditions in the last subsection, as follows:

$$i_t = Max(0,\ \hat{\imath}_t),$$

$$\begin{aligned} &\psi_1(1-\psi_2 L)(1-\psi_3 L)(1-\psi_4 F)(\hat{\imath}_t - i^*) = \\ &\qquad \phi_\pi^*(-\beta\gamma\pi_{t+1} + (\beta\gamma^2+1)\pi_t - \gamma\pi_{t-1}) + \phi_x^*(-\beta\gamma x_{t+1} + (\beta\gamma+1)x_t - x_{t-1}), \end{aligned} \tag{3.7}$$

where i_t cannot take a negative value, while $\hat{\imath}_t$ can. $\hat{\imath}_t$ is interpreted as an indicator variable that provides the information necessary to implement the optimal monetary policy with the possibility of a ZIP. We can then show the following proposition:

Proposition 1: In a timeless perspective, the interest rate rule given by Eq. (3.7) is the one that remains optimal with any deterministic shock process, regardless of whether the nonnegativity constraint on nominal interest rates binds.

Proof. See Appendix 1. ■

[7] From the Kuhn–Tucker conditions, especially from Eq. (3.4), when ϕ_3 is positive, the nominal interest rate is always zero on the one hand, and when ϕ_3 becomes nonpositive, the nominal interest rate always becomes nonnegative, on the other hand.

Eq. (3.7) is a generalization of the optimal interest rate rules in Giannoni and Woodford (Eq. (2.14), 2002) that does not consider a nonnegativity constraint of the nominal interest rate. Giannoni and Woodford (Eq. (2.14), 2002) shows the optimal monetary policy rule that is valid only under no nonnegativity constraint of the nominal interest rate as:

$$i_t = \rho_1 i_{t-1} + \rho_2 \triangle i_{t-1} + \phi_\pi F_t(\pi) + \frac{\phi_x}{4} F_t(x) - \theta_\pi \pi_{t-1} - \frac{\theta_x}{4} x_{t-1} + (1-\rho_1) i^*, \tag{3.8}$$

where $F_t(\pi)$ and $F_t(x)$ are inflation rates and output gaps from the current period to the infinite future. Our rule given by Eq. (3.7) achieves the same equilibrium as the Giannoni–Woodford rule given by Eq. (3.8) when the zero lower bound does not bind. Therefore, our rule is optimal both with and without the zero lower bound on the nominal interest rate. In this sense, Eq. (3.7) is the generalized optimal interest rate rule under the zero lower bound on nominal interest rates.

Eq. (3.7) can be interpreted as both a precautional (forward-looking) and history-dependent (backward-looking) rule for determining the current value of $\hat{\imath}_t - i^*$ in period t. It is important to note that it depends on forward- and backward-looking values of $\hat{\imath}_{t-j}$ for $j = -1, 1, 2$, and not on values of nominal interest rates themselves when the zero lower bound on nominal interest rates binds. Thus, this optimal interest rate rule can keep proper information on the forward- and backward-looking properties depending on the indicator variables $\hat{\imath}_t$ and endogenous variables such as π_t and x_t, which are free from the nonnegativity constraint of the nominal interest rate, but not on the nominal interest rates that suffer from the constraint.[8]

[8] The optimal rule given by Eq. (3.7) becomes purely backward-looking in the purely forward-looking economy, i.e., $\gamma = 0$.

4 Entrance and Exit Strategies in Large Shocks

In this section, we assume large shocks that induce long ZIP periods.[9] We use the quarterly parameters of Woodford (2003) in Table 2 in all simulations[10] and assume two cases: a purely forward-looking economy ($\gamma = 0$) and a hybrid economy ($\gamma = 0.5$).

4.1 Large Unanticipated Shock

We assume an unanticipated shock in the initial period, i.e., t=1. In particular, we assume -5 percent cost-push and natural interest rate shocks with persistence $\rho^r = \rho^\varepsilon = 0.9$, which induce a long ZIP period in the base case. In this case, the concern for the central bank is how to end the ZIP after the unexpected introduction of the ZIP.

Eggertsson and Woodford (2003a, b) and Jung et al. (2005) consider the relation between the length of the ZIP and the inflation dynamics to highlight the properties of the ZIP. Thus, inflation dynamics is one factor that determines the nature of the ZIP. We follow this view. Figure 2 shows the simulation results. The upper panel shows the case of a purely forward-looking economy and the lower panel shows the case of a hybrid economy. The results show that the central bank continues to set the policy rates at zero percent even after inflation rates become positive in the two cases. This result is consistent with the conclusions of Eggertsson and Woodford (2003a, b) and Jung et al. (2005), which insist that the optimal path of the short-term nominal interest rate is characterized by monetary policy inertia, in the sense that ZIP is continued for a while even after inflation becomes positive.

The time to the end of the ZIP, however, is very different according to the degree of inertia in the economy. The ZIP period is shorter in the hybrid economy than in the purely

[9] For example, the BOJ has continued a ZIP for a long period in Japan, which can be interpreted as the case of a large shock.

[10] We set $i^* = 1$ percentage, which does not follow Woodford (2003).

forward-looking economy. In particular, in the case of the hybrid economy, the ZIP ends immediately after the inflation rate takes a positive value. The reason for the shorter ZIP is that too much monetary stimulation is likely to amplify economic fluctuations, as shown by the larger fluctuations of the inflation rate after the ZIP in the economy with inflation inertia. We can confirm this point from the speed of the policy interest rate change. In the hybrid economy, the policy rate change after the ZIP is faster than that in the purely forward-looking economy. The speed of policy interest rate change is 5.2 in the hybrid economy, but it is only 2.8 in the forward-looking economy in the base case.[11] This result has crucial implications for monetary policy with respect to the timing of the end of the ZIP. The timing of the end of the ZIP depends on the economic structure of each country. Therefore a ZIP that lasts for too long against economic inertia can harm social welfare.

Figure 3 provides a robustness check. We impose both price and natural rate of interest shocks, and change the size of the shocks from -0.1 to -5 percent by 0.1 percent. The figure reports the duration of the ZIP period (denoted by PZIP) in the upper panel, and the difference in the length of time taken for inflation to become positive and for the policy interest rate to become positive (denoted by DIF) in the lower panel.[12] From the upper panel, we can confirm that the ZIP should be longer (more history-dependent) in the purely forward-looking economy than in the hybrid economy for the large-scale shocks from -5 to -1 percent. In the lower panel, we see that the central bank should continue ZIPs even after the inflation rate becomes positive following the large-scale shocks of smaller than -1.4 in the purely forward-looking economy and of smaller than -1.5 in the hybrid economy. There, it should be noted that the natural rate of interest shocks rather than the price shocks are likely to induce a ZIP under positive inflation rates and output gaps. In response to only

[11] We report the speeds of the policy rate changes over six periods (one and half years) after the ZIP (unit is per year) to unanticipated shocks. Thus, the unit is percentage change per period.

[12] DIF is calculated by the time taken for the policy interest rate to become positive minus for inflation to become positive.

price shocks, the ZIP ends in the same time period that the inflation rate becomes positive for all the shocks in our experiments.[13]

4.2 Large Anticipated Shock

We assume an anticipated shock occurred in t = 20. In particular, we assume -5 percent cost-push and natural interest rate shocks with persistence $\rho^r = \rho^\varepsilon = 0.9$ for the base case.

Figure 4 shows the simulation results. The upper panel shows the case of a purely forward-looking economy and the lower panel shows the case of a hybrid economy. The results show that the central bank sets the policy rate equal to zero percent long enough before the economic contraction becomes serious which occurs around t = 20 and keeps the ZIP even after the inflation rate becomes positive, as in the previous subsection. The periods in which the ZIP should be implemented, however, are very different according to the degree of inertia in the economy. Basically, the duration of the ZIP is shorter in the hybrid economy than in the purely forward-looking economy. Because of the economic inertia, the start time of the ZIP is later and the end time is earlier in the economy with inflation inertia than in the economy without inflation inertia. The central bank has to avoid too much monetary easing in the economy with inflation inertia.

Figure 5 provides a robustness check. We impose both price and natural rate of interest shocks and change the size of the shocks from -5 to -0.1 percent by 0.1 percent. The figure plots the time to the start of the ZIP period, denoted as SZIP, and PZIP in the upper panel and DIF in the lower panel. From the upper panel, we can see that PZIP is longer and SZIP is earlier in the purely forward-looking economy than in the hybrid economy in response to the large-scale shocks. This implies that the ZIP in the purely forward-looking economy should start earlier (more precautional) and continue longer. This property holds

[13] Surely, by assuming a larger price shock, the ZIP ends with some lag after the inflation rate becomes positive, although such a large shock may be somewhat unrealistic.

to shocks of smaller than -0.9. From the lower panel, we again see that the central bank should continue ZIPs even after the inflation rates take positive values in response to the large-scale shock. This property holds to shocks of smaller than -1.1 in the purely forward-looking economy and of smaller than -2 in the hybrid economy. It should be noted that the natural rate of interest shocks rather than the price shocks are likely to induce a ZIP under positive inflation rates and output gaps. In response only to the price shocks, the ZIP is ended with only a one-period lag to when the inflation rate turns positive following the anticipated shock; however, the ZIP is implemented for many periods under positive inflation rates and output gaps in response to the natural rate of interest shocks for all the shocks in our experiments.

5 Entrance and Exit Strategies in Small Shocks

In this section, we assume small shocks that induce short ZIPs.

5.1 Small Unanticipated Shock

We assume unanticipated -0.3 percent cost-push and natural interest rate shocks, occurring at t = 1, with persistence $\rho^r = \rho^\varepsilon = 0.9$ in the base case.

Figure 6 shows the simulation results. The upper panel shows the case of a purely forward-looking economy and the lower panel shows the case of a hybrid economy. In contrast with the cases of large shocks, the ZIP period is longer in the hybrid economy than in the purely forward-looking economy. Moreover, the central bank does not continue to set the policy rates at zero percent even after inflation becomes positive or the shock disappears, unlike Eggertsson and Woodford (2003a, b) and Jung et al. (2005), in both economies. This is not a surprising result because the central bank does not need to stimulate inflation expectations to stimulate the economy by committing to a long ZIP period, which ultimately creates large economic booms in the future, in response to small shocks. In this case, the central bank carries out a ZIP only during the periods of serious economic contraction. Thus, there are no history-dependent properties in the policy. The reason for the longer ZIP in the hybrid economy is that a short ZIP does not create a large economic boom in the future and so the central bank can use somewhat longer ZIP periods in response to small shocks. Another reason for longer ZIP periods in the hybrid economy is the fact that the same size shocks create larger economic fluctuations because of economic inertia in the hybrid economy than in the purely forward-looking economy. Thus, the central bank basically reacts more strongly to the shocks in the hybrid economy than in the purely forward-looking economy. This property is hidden in the case of large shocks because of the required strong effect of the inflation expectation channel.

Figure 3 provides a robustness check. From the upper panel, we can confirm that the ZIP period should be shorter in the purely forward-looking economy than in the hybrid economy. The upper panel also shows the transition in the length of the ZIP period from large to small shocks. We see from the threshold of -0.4 percent that the hybrid economy demands a longer ZIP period than does the purely forward-looking economy. The lower panel shows that the central bank should end ZIPs before the inflation rate becomes positive in response to shocks of larger than -1 percent in the purely forward-looking economy and larger than -0.9 in the hybrid economy. It should be noted that the price shocks, rather than the natural rate of interest shocks, are likely to end the ZIP before inflation rates and output gaps become positive. Only in response to the natural rate of interest shocks is the ZIP ended, at least in the same period that the inflation rate turns positive, even in response to the small shocks in our experiments. This result is consistent with findings in Eggertsson and Woodford (2003a, b) and Jung et al. (2005). They, however, do not consider the case of a small price shock.

5.2 Small Anticipated Shock

We assume anticipated -0.5 percent cost-push and natural interest rate shocks, occurring at t = 1, with persistence $\rho^r = \rho^\varepsilon = 0.9$ to produce short ZIP periods in the base case.

Figure 7 shows the simulation results. The upper panel shows the case of a purely forward-looking economy and the lower panel shows the case of a hybrid economy. The outcomes show results in contrast with ones for the case of large shocks. The ZIP period is longer in the hybrid economy than in the purely forward-looking economy. Moreover, we cannot find history dependency and can see only small precautionality in monetary policy through the simulations. The central bank must conduct a ZIP only during periods of serious economic contraction.

Figure 5 provides a robustness check. The upper panel shows that the central bank implements earlier and longer ZIPs in the hybrid economy than in the forward-looking economy. We see from the threshold of -0.6 percent that the hybrid economy demands a longer ZIP period than does the purely forward-looking economy. The lower panel shows that the central bank conducts ZIPs only during the periods when serious economic contractions are occurring in both economies in response to the small shocks. The central bank should end ZIPs before the inflation rate becomes positive in response to shocks of larger than -1.1 percent in the purely forward-looking economy and larger than -1 in the hybrid economy. It again should be noted that the price shocks, rather than the natural rate of interest shocks, are likely to end the ZIP before the inflation rate and output gap become positive.

6 Robustness

We change the elasticity of the real interest rate to the output gap σ since many papers suggest the lower values. We assume two alternative parameters, $\sigma = 3.85$ from Amato and

Laubach (2003b) and $\sigma = 1$ from the conventional value in RBC model, i.e. from the log utility.

Figure 8 reports the PZIP and DIF in the case of $\sigma = 3.85$ to the unanticipated shocks on both the price and natural rate of interest shocks as in Figure 3. The qualitative results shown in the last section do not change even in this case. It, however, is important to note that the threshold that the hybrid economy demands longer ZIP than does the purely forward-looking economy becomes -0.6 in Figure 8 from -0.4 in Figure 3. Figure 9 reports the PZIP, SZIP, and DIF in the case of $\sigma = 3.85$ to the anticipated shocks on both the price and natural rate of interest as in Figure 5. We see that the threshold that the hybrid economy demands longer ZIP than does the purely forward-looking economy becomes -0.7 in Figure 9 from -0.6 in Figure 5.

Figure 10 reports the PZIP and DIF in the case of $\sigma = 1$ to the unanticipated shocks on both the price and natural rate of interest as before. Figure 11 reports the PZIP, SZIP, and DIF in the case of $\sigma = 1$ to the anticipated shocks on both the price and natural rate of interest as before. We can confirm the same qualitative results in Figure 10 and Figure 11 as in the last section. One quantitative difference is that the threshold that the hybrid economy demands longer ZIP than does the purely forward-looking economy becomes smaller as -1.6 in Figure 10 to unanticipated shocks and as -2.2 in Figure 11 to the anticipated shocks in this parameter than in other parameters. As the elasticity of the real interest rate to the output gap becomes smaller, the threshold that the hybrid economy demands longer ZIP than does the purely forward-looking economy becomes smaller.

7 Concluding Remarks

In this paper, we proposed a generalized optimal interest rate rule that is optimal regardless of whether or not the zero lower bound on nominal interest rates binds. The proposed rule is designed to retain information about the ZIP by using variables that are not affected by the nonnegativity constraint, such as the indicator variable that provides the information necessary to implement the optimal monetary policy, inflation rate, and the output gap, so that it is optimal even under the zero lower bound on nominal interest rates. Then, we show the optimal start and end strategies of the ZIP using a realistic model with inflation inertia and a variety of shocks. The nature of the ZIP changes significantly according to the degree of economic inertia and the size of the shocks.

The theoretical suggestion in this paper provides a good guideline for when to start and end the ZIP. Practically, it is difficult for central banks to commit to these particular rules. However, the simulations above provide many implications for designing monetary policy in a low interest rate environment.

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A Proof of Proposition 1

To prove Proposition 1, we make use of the Kuhn–Tucker conditions. When the zero lower bound may be binding, we have the following equation from Eq. (3.2):

$$\phi_{2t} = \kappa^{-1}(\lambda_x x_t + \phi_{1t} - \beta^{-1}\phi_{1t-1}). \tag{A.1}$$

By substituting Eq. (A.1) into Eq. (3.1), we obtain:

$$\begin{aligned} -\beta\gamma\phi_{1t+1} + (\beta\gamma+\gamma+1)\phi_{1t} - (1+\gamma+\beta^{-1}(1+\kappa\sigma))\phi_{1t-1} + \beta^{-1}\phi_{1t-2} \\ = -\kappa(-\beta\gamma\pi_{t+1} + (\beta\gamma^2+1)\pi_t - \gamma\pi_{t-1}) - \lambda_x(-\beta\gamma x_{t+1} + (\beta\gamma+1)x_t - x_{t-1}), \end{aligned}$$

$$\begin{aligned} \Rightarrow \psi_1(1-\psi_2 L)(1-\psi_3 L)(1-\psi_4 F)\phi^*_{1t} = \\ = \phi^*_\pi(-\beta\gamma\pi_{t+1} + (\beta\gamma^2+1)\pi_t - \gamma\pi_{t-1}) + \phi^*_x(-\beta\gamma x_{t+1} + (\beta\gamma+1)x_t - x_{t-1}), \end{aligned} \tag{A.2}$$

where $\phi^*_{1t} = -\sigma\lambda_i^{-1}\phi_{1t}$, $\phi^*_\pi \equiv \kappa\sigma(\beta\gamma\lambda_i)^{-1}$, $\phi^*_x \equiv \sigma\lambda_x(\beta\gamma\lambda_i)^{-1}$, $\psi_1\psi_2\psi_3 = \beta^{-2}\gamma^{-1}$, $\psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3 = (\beta\gamma)^{-1}(1+\beta^{-1}(1+\beta\gamma+\kappa\sigma))$, $\psi_1 + \psi_2 + \psi_3 = (\beta\gamma)^{-1}(1+\beta\gamma+\gamma)$ and $\psi_4 = \psi_1^{-1}$ $(\psi_2 > \psi_3)$. We note that Eq. (A.1) is valid with and without the zero lower bound in the system of equations given by Eq. (3.1) through Eq. (3.6).[14] It should be noted that the expectation operator, E_t, does not appear in these equations because the future paths of shocks are perfectly foreseen thanks to the assumption of deterministic shocks. On the other hand, we have the interest rate rule given by Eq. (3.8), which is optimal without the zero lower bound, as shown in Giannoni and Woodford (2002):

[14] If the zero lower bound is not binding, from the Kuhn–Tucker conditions Eq. (3.3), we can substitute $i_t - i^* = \phi^*_{1t}$ into Eq. (A.1), and we obtain the optimal interest rate rule given by Eq. (A.3). If the zero lower bound is binding, we can set the optimal interest rates by Eq. (A.1) and Eq. (3.3). Therefore, Eq. (A.1) is valid with and without the zero lower bound.

$$\psi_1(1-\psi_2 L)(1-\psi_3 L)(1-\psi_4 F)(i_t - i^*) =$$

$$\phi_\pi^*(-\beta\gamma\pi_{t+1} + (\beta\gamma^2+1)\pi_t - \gamma\pi_{t-1}) + \phi_x^*(-\beta\gamma x_{t+1} + (\beta\gamma+1)x_t - x_{t-1}). \quad \text{(A.3)}$$

From Eq. (A.1) and Eq. (A.3), we obtain:

$$i_t = \phi_{1t}^* + i^*. \quad \text{(A.4)}$$

This relation is true only when the zero lower bound does not bind if i_t cannot take a negative value.[15]

Here, in the case where the zero lower bound binds, the Kuhn–Tucker condition Eq. (3.3) also holds:

$$\lambda_i(i_t - i^*) + \sigma\phi_{1t} - \phi_{3t} = 0,$$

with $i_t = 0$. Then it must be the case that[16]

$$\phi_{3t} = -\lambda_i(\phi_{1t}^* + i^*).$$

This equation implies that the ZIP will be terminated when $\phi_{1t}^* + i^*$ becomes positive in Eq. (A.2) (or equivalently, the ZIP will be implemented while $\phi_{1t}^* + i^*$ takes a negative value). Therefore, from Eq. (A.4), we can confirm that if i_t could take a negative value in Eq. (A.3), then Eq. (A.4) always holds with and without the zero lower bound and i_t becomes positive in Eq. (A.3) at the exact same time as the end of the ZIP, which is indicated by

[15] This is because ϕ_{1t} takes a negative value, but i_t cannot.

[16] If we substitute $\phi_{3t} = 0$ into Eq. (3.3), then we have Eq. (A.4) because

$$\lambda_i(i_t - i^*) + \sigma\phi_{1t} = 0 \Leftrightarrow \; i_t - i^* = -\lambda_i^{-1}\sigma\phi_{1t} = \phi_{1t}^*.$$

:

ϕ^*_{1t} in Eq. (A.2). The above argument can be summarized in the following two equations by redefining $\hat{\imath}_t - i^* = \phi^*_{1t}$, where $\hat{\imath}_t$ can take negative values:

$$i_t = Max(0,\ \hat{\imath}_t),$$

$$\begin{aligned}
&\psi_1(1-\psi_2 L)(1-\psi_3 L)(1-\psi_4 F)(\hat{\imath}_t - i^*) = \\
&\qquad \phi^*_\pi(-\beta\gamma\pi_{t+1} + (\beta\gamma^2+1)\pi_t - \gamma\pi_{t-1}) + \phi^*_x(-\beta\gamma x_{t+1} + (\beta\gamma+1)x_t - x_{t-1}).
\end{aligned}$$

We again emphasize that $\hat{\imath}_t$ can even take a negative value, while i_t cannot under the zero lower bound on nominal interest rates. The above argument completes the proof of *Proposition 1*.

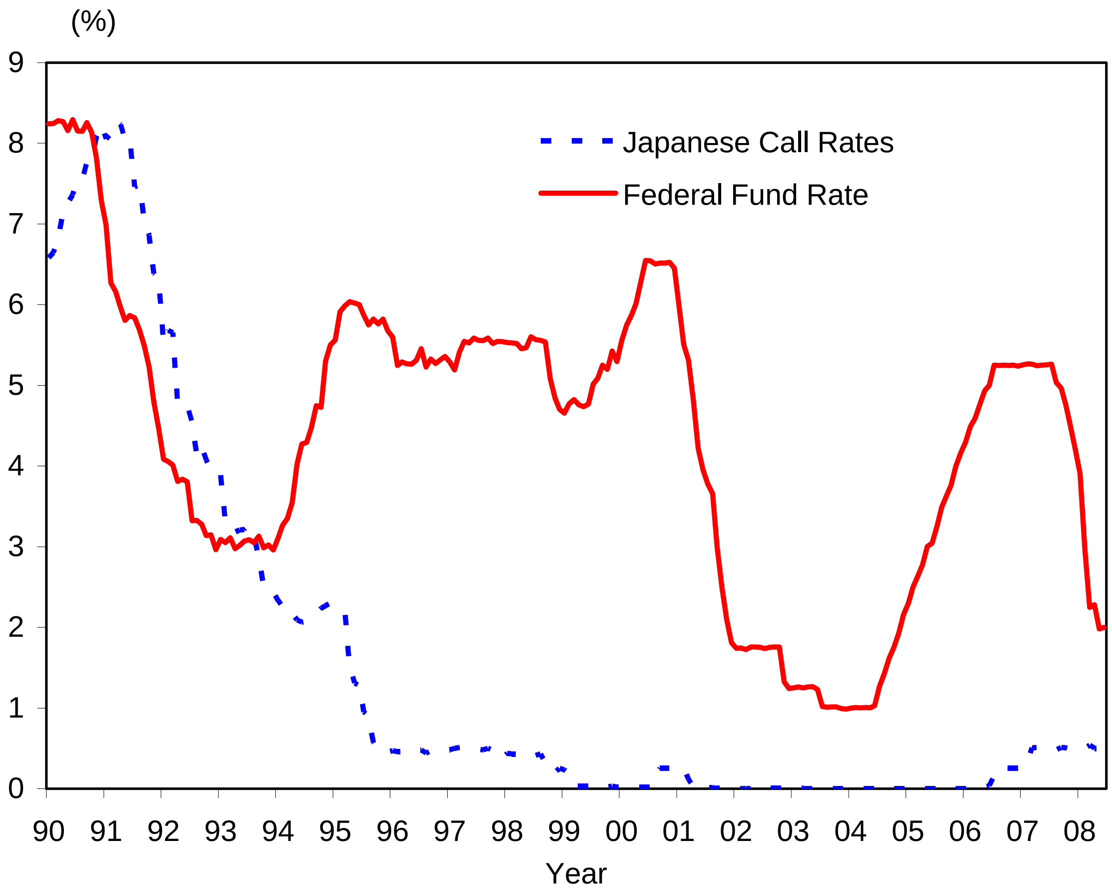
Table 1: Four Situations

Economic Structures\Scale of Shocks	Large Shock (LS)	Small Shock (SS)
Forward-looking Economy (FE)	(FE, LS)	(FE, SS)
Hybrid Economy (HE)	(HE, LS)	(HE, SS)

Table 2: Parameter Values

Parameters	Values	Explanation
β	0.99	Discount Factor
σ	6.25	Elasticity of Output Gap to Real Interest Rate
κ	0.024	Elasticity of Inflation to Output Gap
λ_i	0.077	Weight for Interest Rate
λ_x	0.048	Weight for Output Gap
i^*	1	Steady State Interest Rate

Figure 1



Japanese call rate is given by the Bank of Japan, Financial and Economic Statistics, Financial Markets, Short-term Money Market Rates, Call Rates (Year, Monthly Average). Federal fund rate is given by Federal Reserved Board, Statistics Releases and Historical Data, Selected Interest Rates, Federal Fund Rate (Year, Monthly Average).

Figure 2

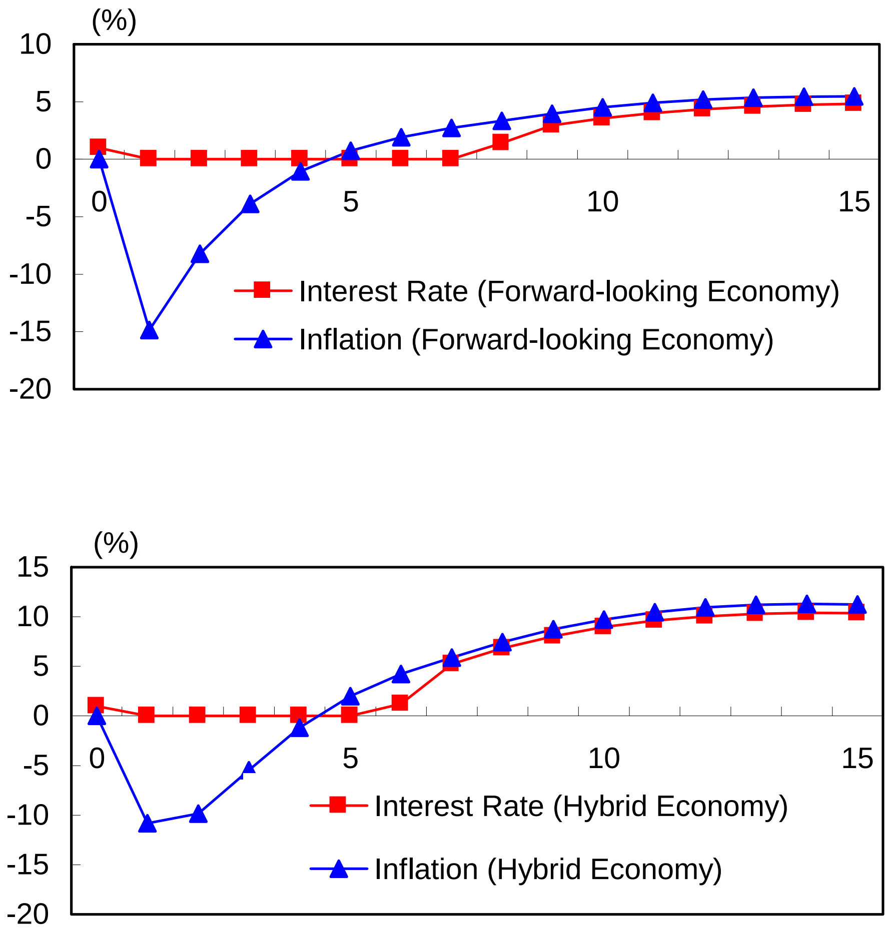
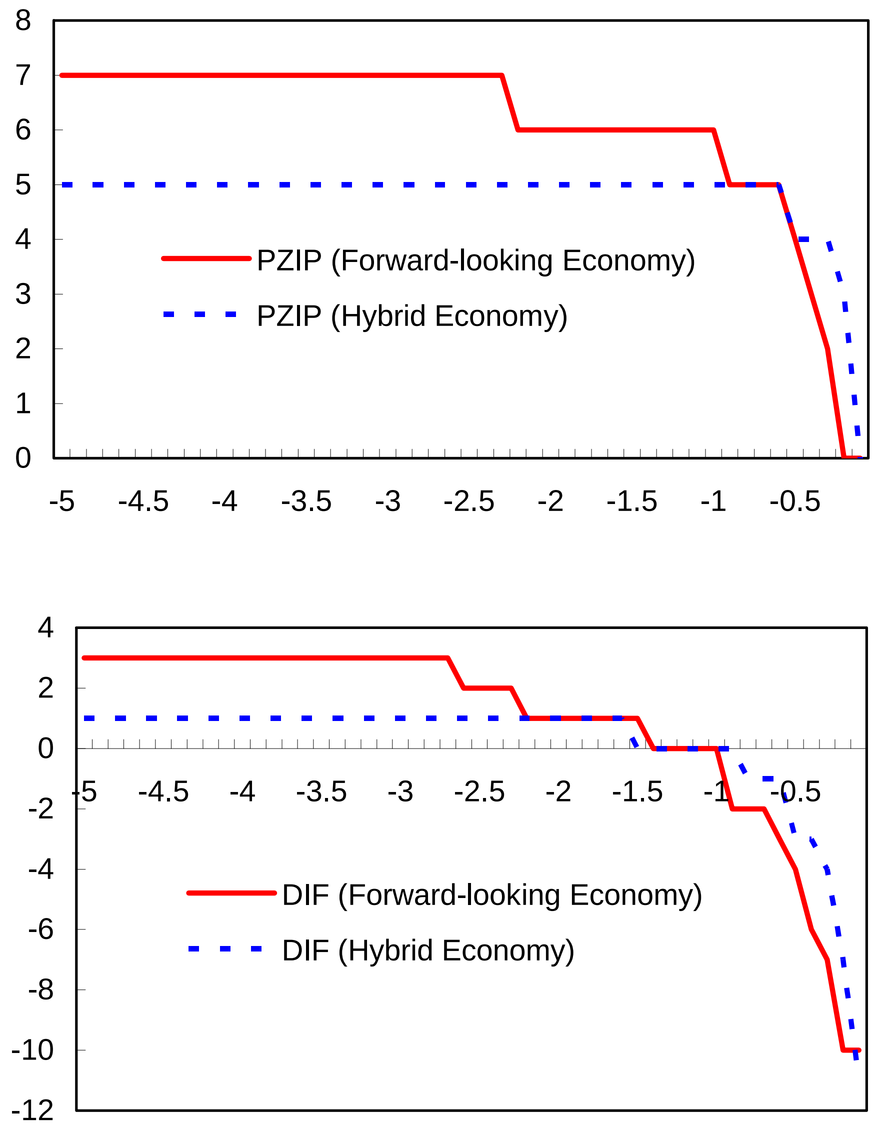
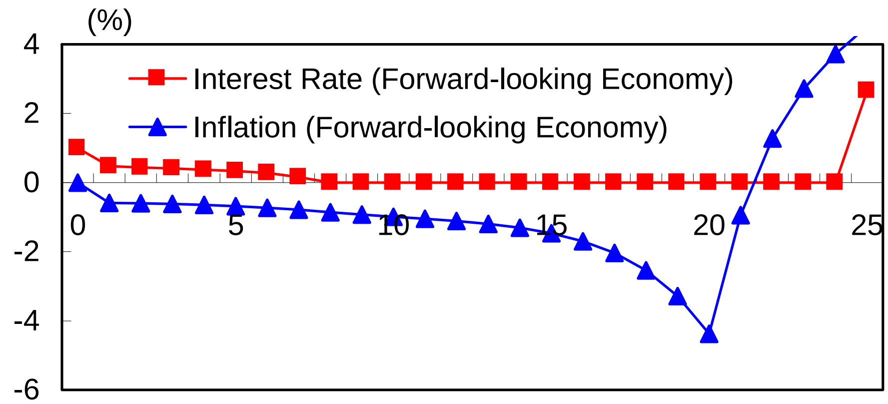

(%)
10
5
0
-5
-10
-15
-20
0
5
10
15
Interest Rate (Forward-looking Economy)
Inflation (Forward-looking Economy)
(%)
15
10
5
0
-5
-10
-15
-20
0
5
10
15
Interest Rate (Hybrid Economy)
Inflation (Hybrid Economy)

Figure 3

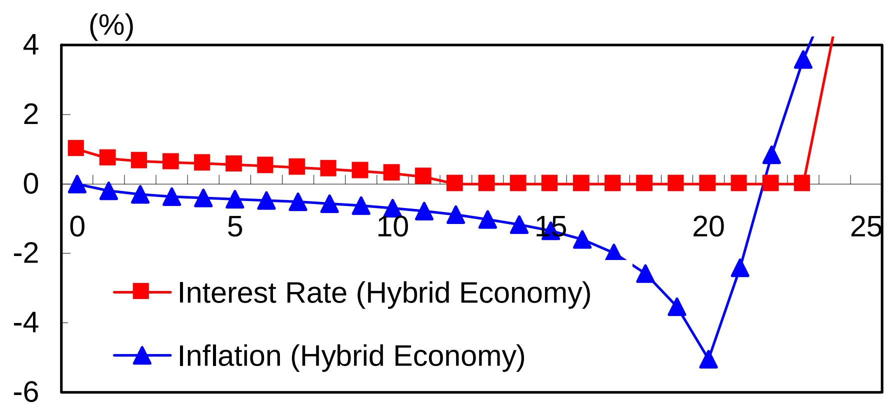


PZIP denotes the duration of the ZIP period. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.

Figure 4

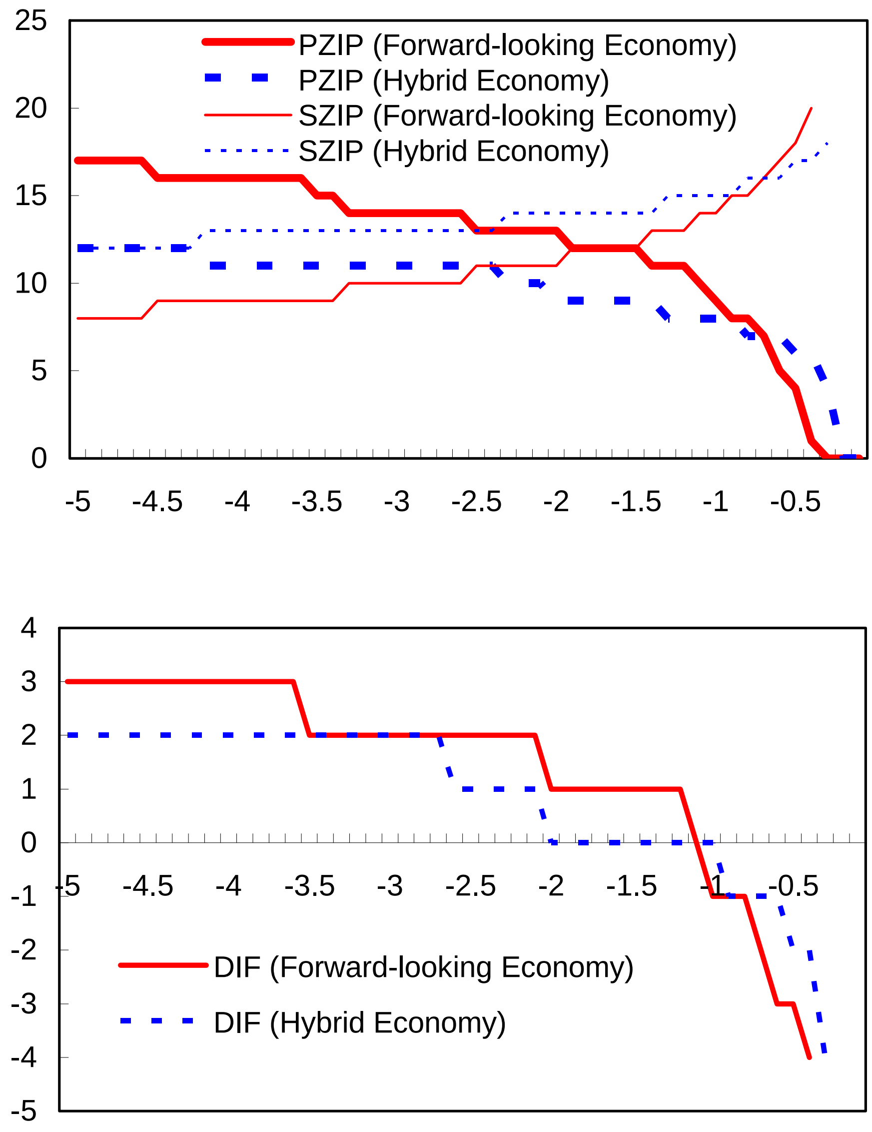


(%)
4
2
0
-2
-4
-6
0
5
10
15
20
25
Interest Rate (Forward-looking Economy)
Inflation (Forward-looking Economy)



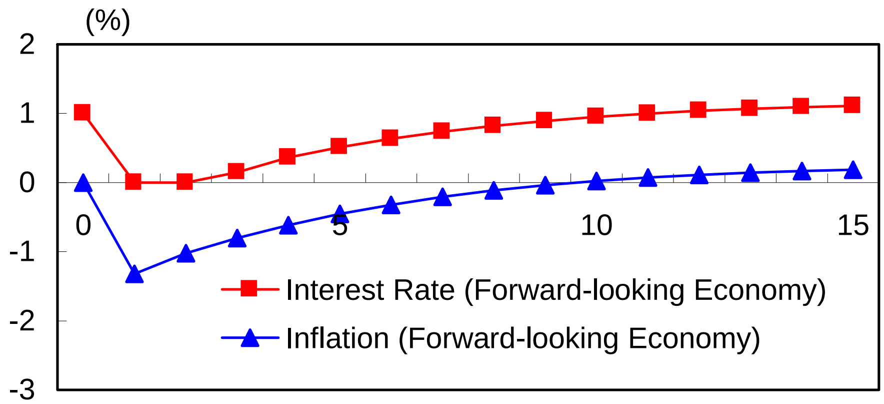
(%)
4
2
0
-2
-4
-6
0
5
10
15
20
25
Interest Rate (Hybrid Economy)
Inflation (Hybrid Economy)

Figure 5



PZIP denotes the duration of the ZIP period. SZIP denotes the time to start of the ZIP. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.

Figure 6


(%)
2
1
0
-1
-2
-3
0
5
10
15
Interest Rate (Forward-looking Economy)
Inflation (Forward-looking Economy)

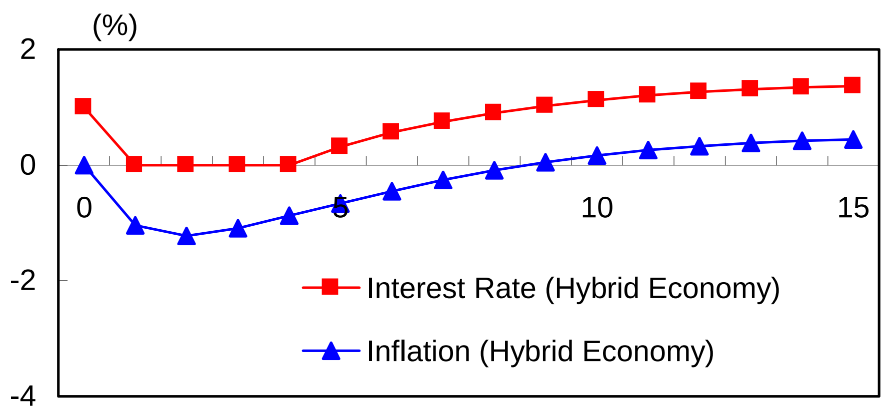
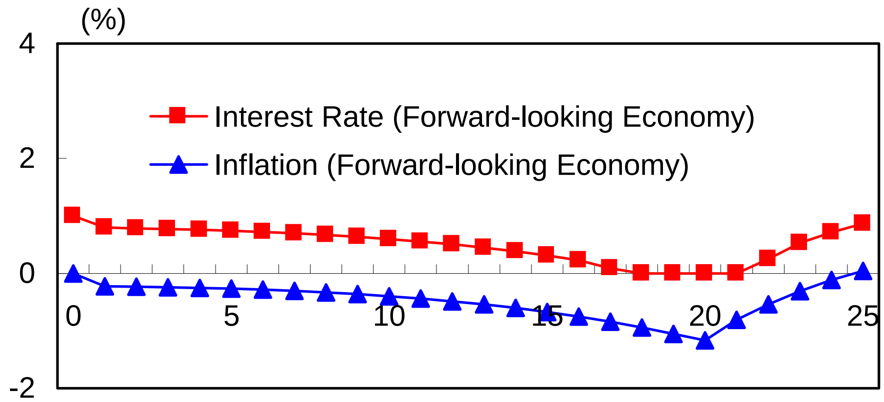

(%)
2
0
-2
-4
0
5
10
15
Interest Rate (Hybrid Economy)
Inflation (Hybrid Economy)

Figure 7


(%)
4
2
0
-2
Interest Rate (Forward-looking Economy)
Inflation (Forward-looking Economy)
0
5
10
15
20
25

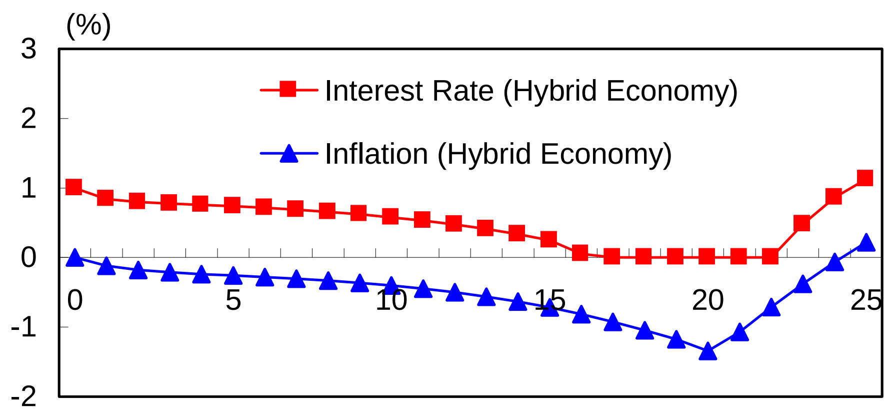
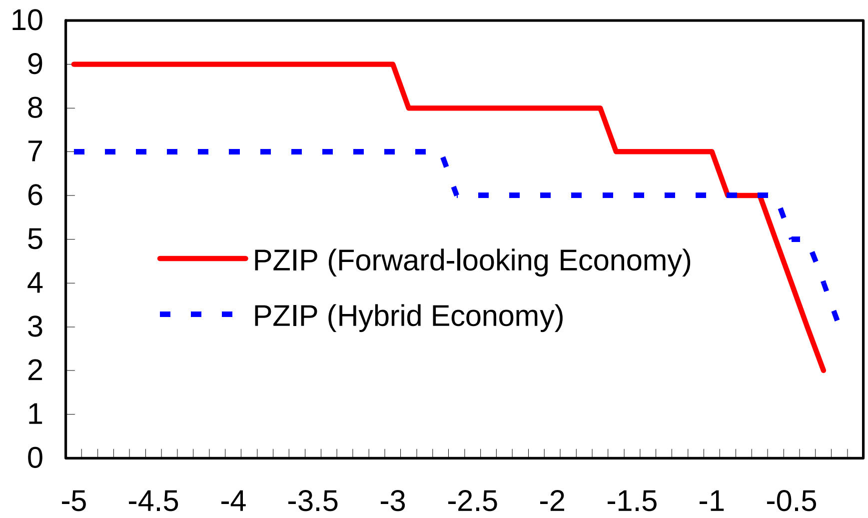
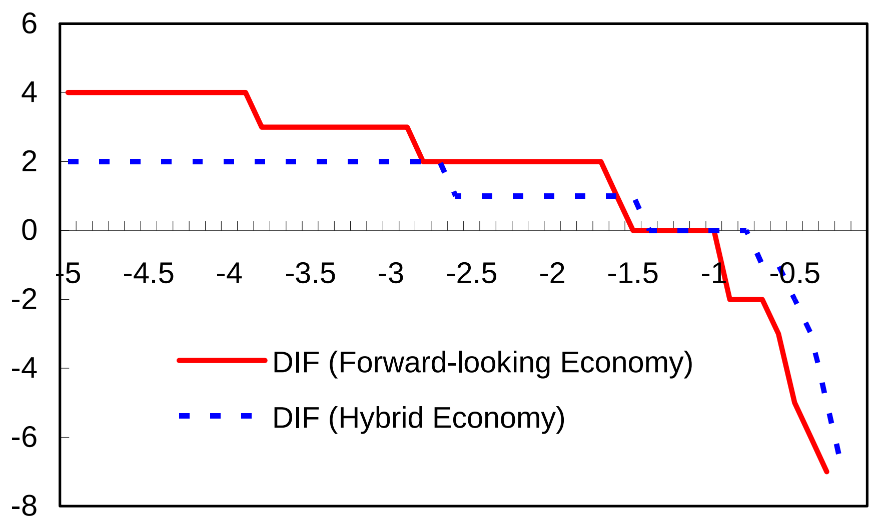

(%)
3
2
1
0
-1
-2
Interest Rate (Hybrid Economy)
Inflation (Hybrid Economy)
0
5
10
15
20
25

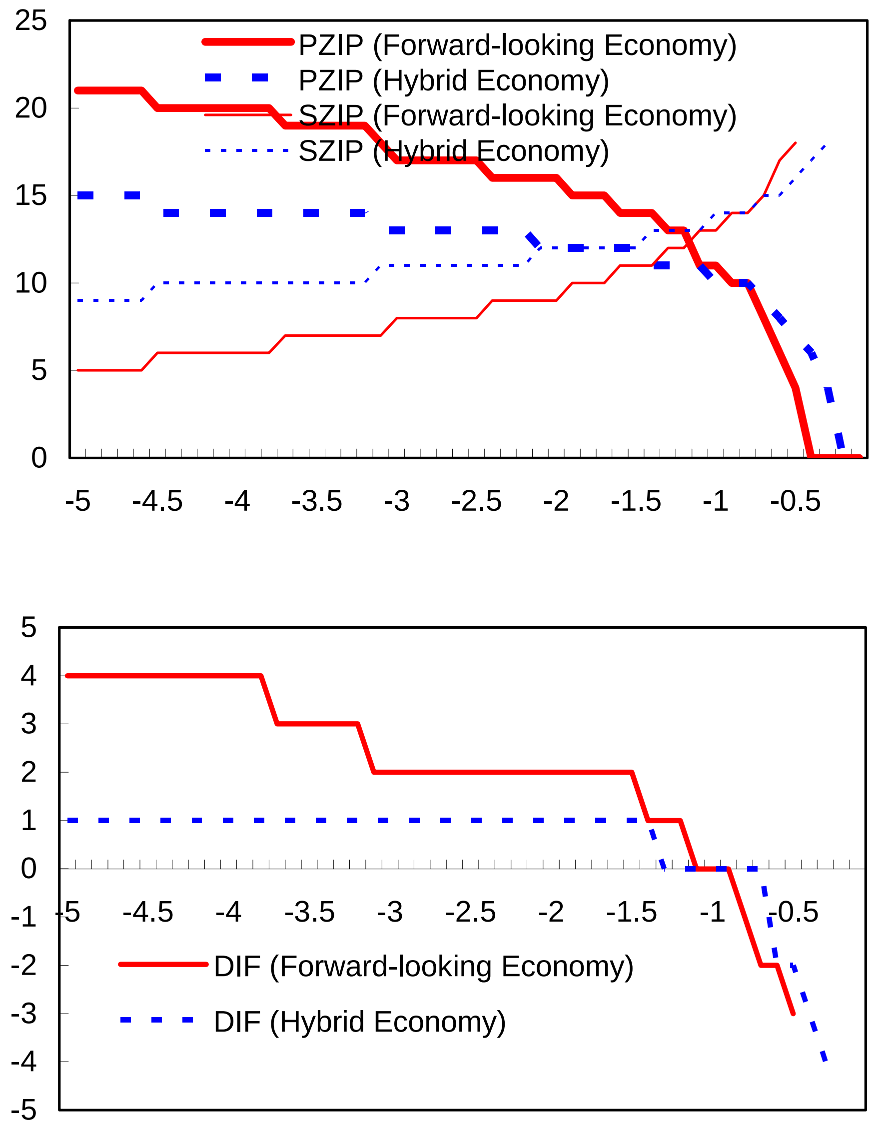
Figure 8





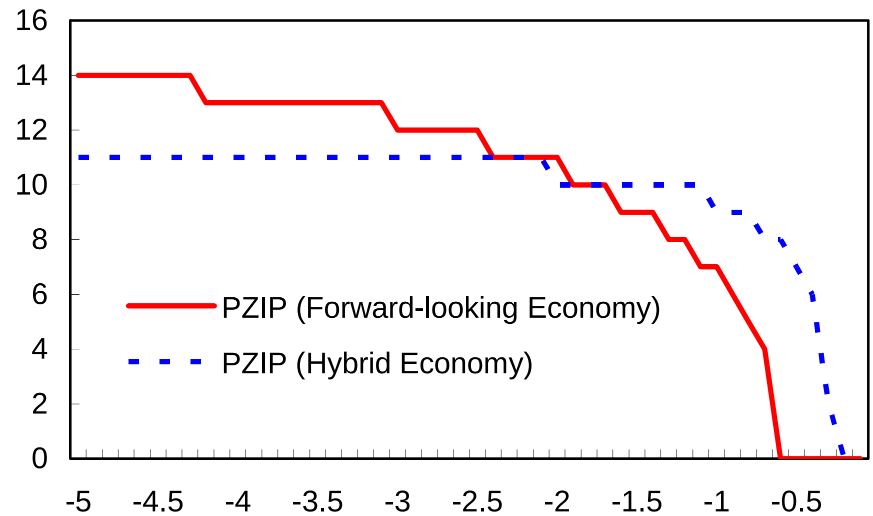
PZIP denotes the duration of the ZIP period. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.

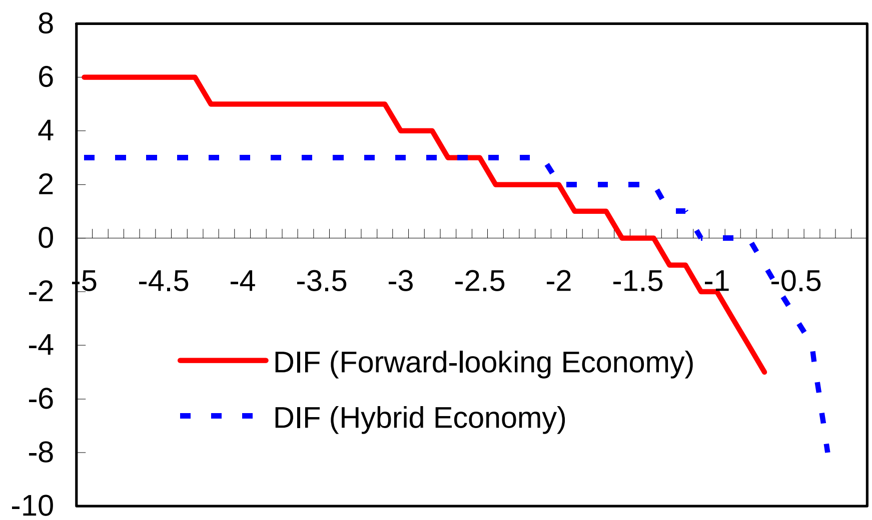
Figure 9



PZIP denotes the duration of the ZIP period. SZIP denotes the time to start of the ZIP. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.

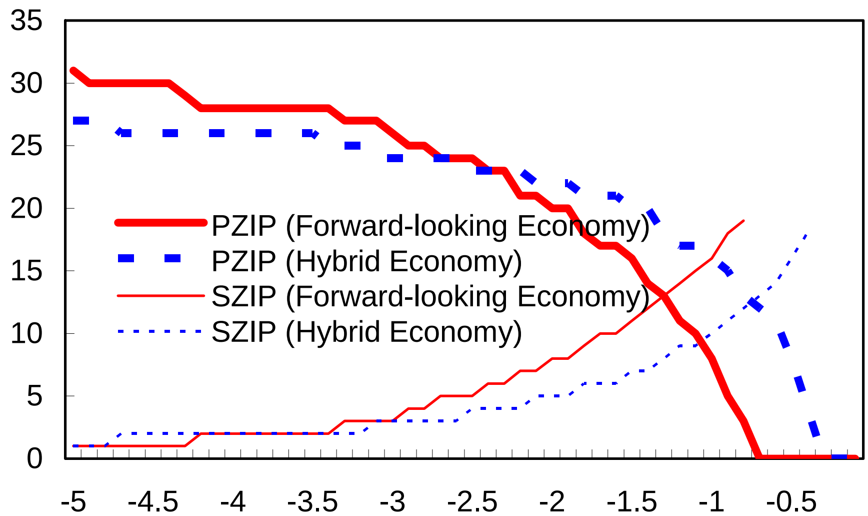
Figure 10

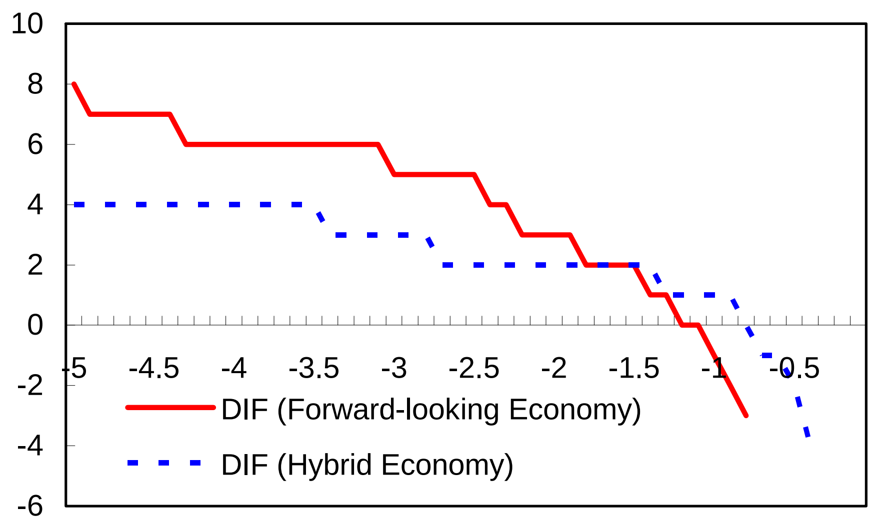




PZIP denotes the duration of the ZIP period. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.

Figure 11





PZIP denotes the duration of the ZIP period. SZIP denotes the time to start of the ZIP. DIF denotes the difference in length of time taken for inflation to become positive and for the policy interest rate to become positive.