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Abstract

This paper proposes a practical framework for the quantification of Liquidity-adjusted Value at Risk ("L-VaR") incorporating the market liquidity of financial products. This framework incorporates the mechanism of the market impact caused by the investor's own dealings through adjusting Value-at-Risk according to the level of market liquidity and the scale of the investor's position. Specifically, the optimal execution strategy for liquidating the investor's entire position is first calculated taking the market impact into account. Then the maximum loss that may be incurred by price fluctuations under optimal execution strategy is calculated as L-VaR.

This paper presents a specific model providing a closed-form solution for calculating L-VaR, and examines whether this framework can be applied to the practices of financial risk management by calculating numerical examples. This paper also demonstrates that this L-VaR calculation framework may be applied under more general conditions, such as (1) when the market impact is uncertain, (2) when the investor's portfolio consists of multiple financial assets, and (3) when there is a non-linear relationship between the market impact and the trading volume.

Key words: Liquidity risk; Value at risk; Market risk; Market impact; Optimal execution strategy; Optimal holding period

JEL classification: G20

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CONTENTS

1. Introduction

This paper proposes a practical framework for the quantification of the Liquidity-adjusted Value at Risk ("L-VaR") incorporating the market liquidity of financial products. This model incorporates the mechanism of the market impact caused by the investor's own dealings through adjusting Value-at-Risk according to the level of market liquidity and the scale of the investor's position. Specifically, the optimal execution strategy that should be adopted for liquidating the investor's position is derived taking the market impact into account, and the maximum loss that may be incurred by price fluctuations while implementing this optimal execution strategy is calculated as L-VaR. This paper presents a specific model based on the line of thought discussed in Oda, Hisata, and Yamai (1999).

The framework presented in this paper does not incorporate all the aspects of market liquidity. However, this framework is effective as a method for evaluating financial risk when the influence from the market impact of specified financial products is significant.

Following this introduction, Chapter 2 presents a simple explanation of the framework for the calculation of L-VaR, and Chapter 3 introduces the prior research. Chapter 4 presents a model modified from the model presented by Almgren and Chriss (1999) for practical applications. In Chapter 5, the potential of applying this model is verified briefly by presenting specific numerical examples. Chapter 6 presents a more generalized model by relaxing the assumptions of the model presented in Chapter 4. Chapter 7 discusses the future research issues, and Chapter 8 summarizes the paper's conclusions.

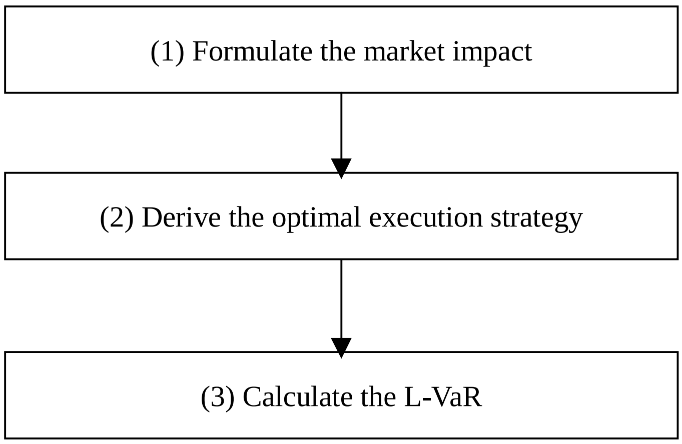
2. Basic Framework for Calculating the Liquidity-adjusted VaR Incorporating Market Liquidity

Before we present a detailed explanation of our framework, this chapter summarizes the basic concept for calculating L-VaR incorporating market liquidity.

Conventional Value at Risk (VaR) calculations assume that the investor's position can be closed at a fixed market price within a fixed period of time (typically one day), regardless of the size of the position. In other words, the measurement of financial risk in the conventional calculations (1) does not consider the influence from the investor's own dealings on price changes (this influence is called the market impact); (2) assumes that the investor's position can be liquidated within a short period of time; and (3) does not consider the influence from fluctuations in the bid-asked spread. It is difficult to claim that these assumptions are realistic during market stress periods, or even under normal market conditions.

While various VaR calculation methods that relax these assumptions have been proposed,[1] this paper develops an approach to VaR calculation that explicitly incorporates the market impact. The optimal execution strategy is derived incorporating the market impact, and L-VaR is then calculated based on this strategy. Thus, this approach calculates L-VaR in three steps, as summarized in Figure 1.

Figure 1: Framework for Quantifying the Market Risk Incorporating the Liquidity Risk



To begin with, in order to formulate the optimal execution strategy, it is first necessary to formulate the market impact. There is presently no consensus regarding the formulation of market impact, while various approaches have been attempted. In the model developed in this paper, the market impact is divided into the temporary portion and the permanent portion, both of which are assumed to be functions of the sales volume.

Next, the optimal execution strategy is derived using an optimization method. In general, the investor's utility function is assumed, and the optimal execution strategy is derived to maximize the investor's utility. In this paper, the cost of liquidating the

[1] In this paper, we focused on the market impact incorporating the liquidity risk into financial risk measurement. Another approach to the liquidity risk recognizes the fluctuations in the bid-asked spread as the market risk in modifying the conventional VaR. For an example of this approach, see Bangia, et al. (1999).

investor's position is formulated (we call this the "liquidation cost"), and the optimal execution strategy is derived to minimize this cost. After deriving the optimal execution strategy, L-VaR is calculated as the maximum loss that may be incurred by price fluctuations while the positions are liquidated according to the optimal execution strategy.

The framework presented in this paper may be applied to the trading activities of diverse financial assets, such as those related to stocks, bonds, and foreign exchange. However, because the impact on the relevant financial asset markets must be formulated for L-VaR calculations, there must be sufficient market data for estimating the market impact. Accordingly, products with low market liquidity, such as bank loans and privately-placed bonds, lie outside the scope of this paper.

3. Introduction of Prior Research

Prior research which investigated the optimal execution strategy for liquidating investors' portfolios includes Jarrow and Subramanian (1997), Bertsimas and Lo (1998), Lawrence and Robinson (1995), Almgren and Chriss (1999), and Konishi (1999).

Jarrow and Subramanian (1997) derive the optimal execution strategy by determining the sales schedule that will maximize the expected total sales value, assuming that the period until the liquidation (the sales period) is given as an exogenous factor. However, they do not take the market risk into account. Moreover, because they accept that the sales period is determined externally, there is a practical problem regarding how this period should be objectively set. Similarly, Bertsimas and Lo (1998) utilize dynamic programming techniques to derive the optimal execution strategy that maximizes the expected total sales value assuming that the sales period is determined externally. They conclude that sales at a constant speed are optimal when the market impact has a linear relationship with the sales volume and the asset price process is a random walk process. In terms of practical application of L-VaR, however, like Jarrow and Subramanian (1997), they do not incorporate the market risk of the position, and their model requires that the sales period be externally determined. On the other hand, Lawrence and Robinson (1995) provide a framework for calculating L-VaR by deriving the optimal execution strategy incorporating the market risk using a mean–standard deviation approach. Nevertheless, their derivation and calculation procedures are not specified, so there are difficulties in the practical application of their research as it is presented.

Almgren and Chriss (1999) present a concrete framework for deriving the optimal execution strategy using a mean–variance approach, and show a specific calculation methodology. Their framework has a high potential for practical application. Unfortunately, this framework still requires that the period until the sales completion be determined externally, like Jarrow and Subramanian (1997) and the others, so setting the sales period remains an outstanding issue for practical application. With an orientation toward practical application, Konishi (1999) presents a framework for deriving the optimal execution strategy using a mean–standard deviation approach and a continuous-time model that makes the period into an endogenous variable.

4. Method of Calculating L-VaR

This paper proposes a new method of calculating L-VaR based on the prior research. The basic idea is to modify the framework presented by Almgren and Chriss (1999), which has the merits of simplicity and specificity, to turn the sales period into an endogenous variable.[2] The following sections present explanations of setting discrete-time and continuous-time models, deriving the optimal execution strategy, and calculating L-VaR. While the setting of the models is entirely dependent on Almgren and Chriss (1999), from a viewpoint of practical application, the derivation of the optimal execution strategy is changed to incorporate the sales period as an endogenous variable, with the additional assumption of sales at a constant speed. Sections 4.1 – 4.3 present a model for sales on a discrete-time basis, and sections 4.4 – 4.6 present a model for sales on a continuous time-basis. This chapter assumes that the investor's portfolio consists of a single financial asset. Sections 6.3 and 6.4 consider the situation when the investor's portfolio contains more than one financial asset.

4.1. Discrete-Time Model

While this model may be applied to the trading of diverse financial assets including stocks, foreign exchange and bonds, for simplicity the explanation here assumes the trading of stocks.

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[2] While Konishi (1999) presents a method of deriving the optimal execution strategy in a continuous-time framework that turns the sales period into an endogenous variable, unlike the method presented in this paper, this still does not derive L-VaR.

When selling X shares of a given stock, if X is a significantly large figure, selling all of the shares at once will result in a substantial price decline due to the market impact. One possible strategy is to sell the shares sequentially to minimize the price decline.

Specifically, the sales may be implemented as follows. The sales period is equally divided into N periods, and the times partitioning the periods are $t_0, t_1, \cdots, t_N$. When the present time is $t_0 = 0$, the sales completion time $t_N = T$, and the interval between each instant (the sales interval) τ, then the position is sold at $t_k = k\tau$ for $k = 0, \cdots, N$, and the conditions that must be satisfied at the final instant are as shown in Equation 4.1.

$$t_N = T = N\tau \tag{4.1}$$

Equation 4.1 may also be viewed as the time required to sell the initial number of shares held X, that is to say, the holding period.[3]

Next, the number of shares held at each instant is defined as $x_0, x_1, \cdots, x_N$. The initial number of shares held is $x_0 = X$, and the final number of shares held is $x_N = 0$.

Additionally, the number of shares sold in each period is defined as $n_1, \cdots, n_N$ and the number of shares sold per unit time in the k^{th} period as shown in Equation 4.2.

$$v_k = \frac{n_k}{\tau} \tag{4.2}$$

Holthausen, Leftwich, and Mayers (1987) assume that the market impact can be separated into the permanent market impact that decreases the equilibrium price and the temporary market impact that temporarily pushes down the price (Figure 2). In other words, they assume a mechanism whereby immediately after the sales are completed the price decline from the permanent market impact[4] and the price decline from the temporary market impact occur simultaneously, and the price subsequently recovers only the portion of the decline from the temporary market impact. Almgren and Chriss (1999) also assume this mechanism in preparing their market impact model, and this mechanism is assumed in this paper as well.

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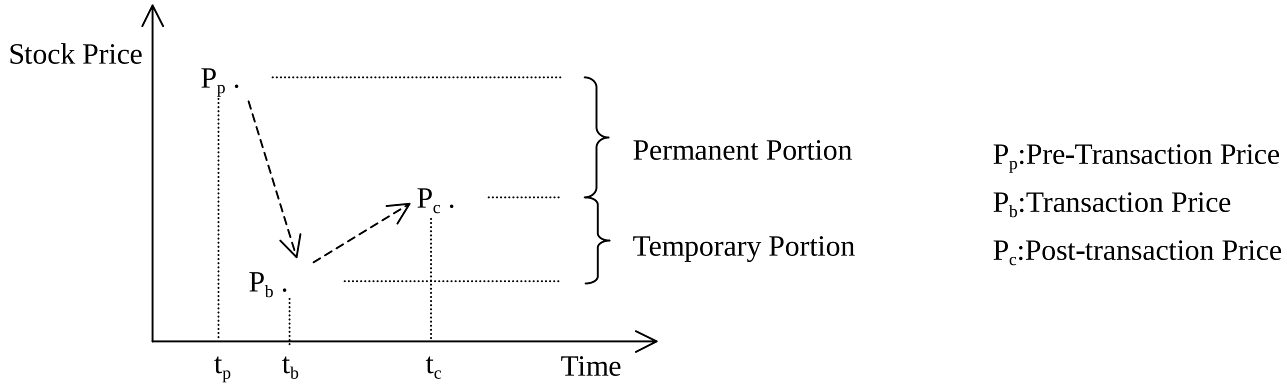
[3] In this paper, L-VaR is derived for the holding period, which is the time required from when the sales are initiated until they are completed.

[4] When estimating the market impact from actual market data, the portion of the price decline that is recovered after the sales are completed is defined as the temporary market impact, and the remaining portion is defined as the permanent market impact.

Figure 2: Permanent and Temporary Market Impact from Sale



This paper also adopts the same interpretation of the sales interval τ as Holthausen, Leftwich and Mayers (1987).[5] The sales interval τ is defined as the period from when the sales are initiated until the temporary market impact effect disappears ($t_c - t_b$). Under this interpretation, the sales interval τ is dependent upon the speed of the post-sales price convergence. This is apparently determined by the characteristics of the individual financial products and financial markets. Therefore, it becomes possible to derive the optimal holding period by adopting the sales interval as an exogenous parameter. In fact, there is a potential to make the execution strategy more optimal by utilizing the sales interval as a strategic variable. However, in this case, it becomes necessary to formulate the dynamic behavior of the convergence of the temporary market impact, so the optimization issue becomes more complicated. Accordingly, this paper leaves the sales interval as a strategic variable for the investor as an outstanding issue, and proceeds its arguments based on the simpler interpretation presented above.

Following Almgren and Chriss (1999), this paper assumes that the market impact has a linear relationship with the stock sales volume.[6] First, for the permanent market impact, we assume that a stock sales volume n_k over period k may be expressed as γn_k, and γ is called the permanent market impact coefficient.

Almgren and Chriss (1999) assume that the price changes are caused by three factors: drift, volatility, and market impact. Among these, they assume that drift and volatility are fluctuation factors that are not related to the investor's own dealings (they are influenced mainly by news regarding the fundamentals of the stock), while the market

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[5] Almgren and Chriss (1999) also develop their model using fixed sales intervals, but their justification of the assumption is unclear.

[6] See Chapter 6 for a discussion of the situation when the market impact has a non-linear relationship with the sales volume.

impact is related to the investor's own dealings. Moreover, they assume that the overall market fluctuations that are not related to the investor's own dealings can be expressed as an arithmetic random walk using the drift and volatility. Given these assumptions, the "market price" S_k taking the permanent market impact into account may be defined as shown in Equation 4.3.

$$\begin{aligned} S_k &= S_{k-1} + \sigma\tau^{\frac{1}{2}}\xi_k + \mu\tau - \gamma n_k \\ &= S_0 + \sigma\sum_{j=1}^{k}\tau^{\frac{1}{2}}\xi_j + \mu t_k - \gamma(X - x_k) \end{aligned} \tag{4.3}$$

Here, μ is the stock price drift, and σ is the stock price volatility.[7] ξ_j is a random variable that independently follows a standard normal distribution. Equation 4.3 is comprised of the first three terms which express the arithmetic random walk and the fourth term which expresses the permanent market impact. In financial theory, price fluctuations are often expressed as a geometric random walk, but the model proposed by Almgren and Chriss (1999) adopts an arithmetic random walk, for which the calculations are comparatively simple. They justify this assumption by claiming that the difference between the arithmetic and geometric figures can essentially be ignored if the holding period is relatively short. This paper follows the same approach.

Furthermore, it is assumed that the price at which the investor can sell the stock ("sales price") is calculated by decreasing this "market price" by the temporary market impact. In other words, if the temporary market impact from the sales volume per unit time v_k is expressed as $\varepsilon + \eta v_k$ (where ε is the bid-asked spread and η is the temporary market impact coefficient), the investor's "sales price" $\tilde{S}_k$ may be expressed as shown in Equation 4.4.[8]

$$\tilde{S}_k = S_k - \varepsilon - \eta v_k \tag{4.4}$$

The permanent market impact and the temporary market impact can be incorporated simultaneously by combining Equation 4.3 and Equation 4.4 as expressed in Equation

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[7] In the arithmetic random walk used in this paper, the unit of volatility used is yen/share. To convert the volatility expressed as yen/share into the volatility expressed as a percentage, it should be divided by some reference stock price such as the initial price or the current price.

[8] Almgren and Chriss (1999) calculate the sales price at time *k* by adding the temporary market impact to the sales price at time *k*-1, whereby $\tilde{S}_k = S_{k-1} - \varepsilon - \eta v_k$. This is inconsistent with the market impact formulation proposed by Holthausen, Leftwich and Mayers (1987). Accordingly, this paper formulates the temporary market impact as shown in Equation 4.4 following the approach adopted by Holthausen, Leftwich and Mayers (1987).

4.5.

$$\underbrace{\tilde{S}_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{\frac{1}{2}} \xi_j + \mu t_k}_{(1)} \underbrace{- \gamma (X - x_k)}_{(2)} \underbrace{- \varepsilon - \eta v_k}_{(3)} \tag{4.5}$$

In Equation 4.5, the "sales price" consists of (1) the arithmetic random walk, (2) the price decline from the permanent market impact, and (3) the price decline from the temporary market impact.

Because the sales price at time k becomes $n_k \tilde{S}_k$, when the number of shares initially held X are all sold, the total sales value $X\overline{S}$ can be calculated from Equation 4.5 as shown in Equation 4.6.[9]

$$\begin{aligned} X\overline{S} &= \sum_{k=1}^{N} n_k \tilde{S}_k \\ &= XS_0 + \sigma \sum_{k=1}^{N} \tau^{\frac{1}{2}} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \gamma \sum_{k=1}^{N} n_k (X - x_k) - \varepsilon X - \eta \sum_{k=1}^{N} \tau v_k^2 \\ &= XS_0 + \sigma \sum_{k=1}^{N} \tau^{\frac{1}{2}} x_k \xi_k + \mu \sum_{k=1}^{N} \tau x_k - \frac{1}{2} \gamma X^2 - \varepsilon X - (\eta + \frac{1}{2} \gamma \tau) \sum_{k=1}^{N} \tau v_k^2 \end{aligned} \tag{4.6}$$

If the market value of the position at the initial time is XS_0 this value becomes $X\overline{S}$ through the actual sale, and the differential between these two figures may be considered as the transaction cost C as shown in Equation 4.7.[10]

$$\begin{aligned} C &= XS_0 - X\overline{S} \\ &= -\sigma \sum_{k=1}^{N} \tau^{\frac{1}{2}} x_k \xi_k - \mu \sum_{k=1}^{N} \tau x_k + \frac{1}{2} \gamma X^2 + \varepsilon X + (\eta + \frac{1}{2} \gamma \tau) \sum_{k=1}^{N} \tau v_k^2 \end{aligned} \tag{4.7}$$

In Equation 4.7, ξ_k is a random variable following a standard normal distribution, so C, which incorporates the sum of ξ_k as a term, is also a random variable following a normal distribution. The characteristics of normal distributions can be described only with the first and the second moment, and the mean and variance of the transaction cost can be calculated as shown in Equations 4.8 and 4.9. $E[\bullet]$ and $V[\bullet]$ represent operators that take the mean and variance of the random variable, respectively.

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[9] Because this assumes that the sales can be completed within a relatively short period of time (within around 1-20 days), to simplify the calculations it is assumed that the present value of the proceeds from sales and the future value are essentially equal, and the proceeds from sales are not discounted.

[10] The transaction cost C may be negative.

$$E[C] = -\mu \sum_{k=1}^{N} \tau x_k + \frac{1}{2}\gamma X^2 + \varepsilon X + (\eta + \frac{1}{2}\gamma\tau)\sum_{k=1}^{N} \tau v_k^2 \tag{4.8}$$

$$V[C] = \sigma^2 \sum_{k=1}^{N} \tau x_k^2 \tag{4.9}$$

4.2. Optimal Execution Strategy Under the Discrete-Time Model

This section derives the optimal execution strategy using the model presented in section 4.1. The essence of the strategy is to minimize the cost incurred from liquidating the investor's position (the liquidation cost). While Almgren and Chriss (1999) sought the optimal execution strategy given the holding period as an exogenous variable, the approach adopted in this paper seeks the optimal holding period assuming sales at a constant speed in order to ensure an objective determination of the optimum holding period. Additionally, our approach assumes that the investor does not change the initially derived optimal execution strategy in response to changes in market conditions.

This paper assumes that the optimal execution strategy is determined by minimizing the cost of liquidating the investor's position. This cost is viewed as the sum of the mean value of the transaction cost and the cost of bearing market risk (standard deviation), and the objective function for determining the optimal execution strategy is formulated as shown in Equation 4.10.

$$L = E[C] + rZ_\alpha \sqrt{V[C]} \tag{4.10}$$

In this equation, r is the cost of capital, and Z_α is the upper 100α percentile of the standard normal distribution.

The first term on the right-hand side of the equation is the mean value of the transaction cost C. As demonstrated in Equation 4.8, this comprises the average price decline accompanying the bid-asked spread and the market impact. The second term on the right-hand side of the equation is calculated by multiplying the standard deviation of the transaction cost C by the cost of capital r and the upper 100α percentile of the standard normal distribution Z_α. This expresses, in value, the total costs derived from the market risk incurred while liquidating the position. Because the market value of the initial position XS_0 is deterministic, the standard deviation of transaction cost ($\sqrt{V[C]}$) is equivalent to the standard deviation of the total sales value $X\overline{S}$, and thus indicates the market risk of the investor's position from the start of the sales until the sales are completed. Thus, multiplying $\sqrt{V[C]}$ by the upper 100α

percentile of the standard normal distribution Z_α [11] provides the VaR with a 100α% confidence interval. When the cost of capital is r, multiplying the VaR value $Z_\alpha\sqrt{V[C]}$ by the cost of capital r yields $rZ_\alpha\sqrt{V[C]}$, which expresses the costs derived from the market risk. This paper adopts the optimal execution strategy under which this liquidation cost is minimized.[12]

Assuming sales at a constant speed, Equations 4.8 and 4.9 may be revised as shown in Equations 4.11 and 4.12.[13]

$$E[C] = -\frac{1}{2}\mu\tau X(N-1) + \frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\eta X^2}{\tau N} + \frac{\gamma X^2}{2N} \quad (4.11)$$

$$V[C] = \frac{1}{3}\sigma^2\tau X^2 N(1-\frac{1}{N})(1-\frac{1}{2N}) \quad (4.12)$$

The liquidation costs from Equations 4.10, 4.11, and 4.12 can then be expressed as shown in Equation 4.13.

$$\begin{aligned} L &= E[C] + rZ_\alpha\sqrt{V[C]} \\ &= -\frac{\mu\tau X(N-1)}{2} + \frac{\gamma X^2}{2} + \varepsilon X + \frac{\eta X^2}{\tau N} + \frac{\gamma X^2}{2N} + rZ_\alpha\sqrt{\frac{\sigma^2\tau X^2}{3}N(1-\frac{1}{N})(1-\frac{1}{2N})} \end{aligned} \quad (4.13)$$

The conditions for the optimal number of sales N to minimize this liquidation cost can be expressed as shown in Equation 4.14.

$$\frac{\partial L}{\partial N} = -\frac{\mu\tau X}{2} - \frac{\eta X^2}{\tau N^2} - \frac{\gamma X^2}{2N^2} + \frac{rZ_\alpha\sqrt{\frac{\sigma^2\tau X^2}{3}}(1-\frac{1}{2N^2})}{2\sqrt{N-\frac{3}{2}+\frac{1}{2N}}} = 0 \quad (4.14)$$

This condition can be arranged into a six-degree polynomial equation. However, the

[11] Z_α is determined by the level of the investor's risk aversion.

[12] This paper follows Lawrence and Robinson (1995) in formulating the objective function using the standard deviation. However, different methods of setting the objective function may also be considered. For example, when using the variance as a substitute for the standard deviation, the objective function could be formulated in a manner consistent with the expected utility theory (see Appendix A). However, in a formulation using the variance, the parameter expressing the level of the investor's risk aversion must be estimated separately. In contrast, with the formulation using the standard deviation adopted in this paper, the level of the investor's risk aversion can be incorporated into parameter Z_α. Additionally, as the units in Equation 4.10 are on a monetary basis, the significance of this formulation for practical applications is straightforward.

[13] With constant sales, $v_k = X/T = X/\tau N$, $x_k = (1-k/N)X$, and substituting these into Equations 4.8 and 4.9 results in Equations 4.11 and 4.12.

solution must be sought through numerical calculations because it is difficult to obtain a closed-form solution for six-degree polynomial equations. For the moment, if it is assumed that the drift term can be ignored in the short time period until completion of sales $(\mu = 0)$, the conditions may be simplified as expressed in Equation 4.15.

$$\frac{(N^2 - \frac{1}{2})}{2\sqrt{N - \frac{3}{2} + \frac{1}{2N}}} = (\frac{\eta}{\tau} + \frac{\gamma}{2})\frac{\sqrt{3}X}{rZ_\alpha \sigma \sqrt{\tau}} \tag{4.15}$$

The optimal holding period can then be sought by substituting the optimal number of sales derived in Equation 4.15 into Equation 4.1

4.3. L-VaR under the Discrete-Time Model

As mentioned in section 4.2, the expression $\sqrt{V[C]}$ in Equation 4.10 indicates the market risk of the investor's position, and multiplying $\sqrt{V[C]}$ by the upper 100α percentile of the standard normal distribution Z_α provides the VaR with a $100\alpha\%$ confidence interval. Consequently, L-VaR is calculated as shown in Equation 4.16 when the sales are executed according to the optimal execution strategy.

$$L - VaR = Z_\alpha \sqrt{V[C]} \tag{4.16}$$

4.4. Continuous-Time Model

A closed-form solution cannot be obtained when using the discrete-time model described in sections 4.1 through 4.3. Considering the practical application of the model, there may be cases where it is preferable to adopt methods that can provide a closed-form solution, even if these methods are approximate. Accordingly, this section considers a continuous-time model that takes the continuous limit of the discrete-time model presented in the previous section. As demonstrated below, with the continuous-time model, it is possible to obtain a closed-form solution for the optimal holding period and L-VaR.

The framework for the continuous-time model adopts a similar framework and the same notation used for the discrete-time model. In this section, the values of each variable at time t are expressed with the symbol t in parentheses,[14] and $z(t)$ denotes the standard Brownian motion.

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[14] For example the price $\tilde{S}_t$ at time t is expressed as $\tilde{S}(t)$.

The “sales price” process incorporating the market impact may be expressed as shown in Equation 4.17 by setting each of the parameters and taking the continuous limit in Equation 4.5 $(\tau \to 0\ ,\ N \to \infty)$.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \eta v(t) - \gamma \int_0^t v(s)ds \tag{4.17}$$

When executing the sales, dx has a negative value, so the total sales value is $-\int_0^T \tilde{S}(t)dx$. When $v(t) = v$ (fixed), or when the speed of sales is constant, $dx = -\ vdt$, resulting in Equation 4.18.[15]

$$\begin{aligned} -\int_0^T \tilde{S}(t)dx &= v\int_0^T \tilde{S}(t)dt \\ &= v\int_0^T \{S(0) + \mu t + \sigma z(t) - \varepsilon - \eta v - \gamma \int_0^t vds\}dt \\ &= XS(0) + \frac{1}{2}\mu v T^2 + v\sigma \int_0^T z(t)dt - \varepsilon vT - \eta v^2 T - \frac{1}{2}\gamma v^2 T^2 \end{aligned} \tag{4.18}$$

Consequently, the transaction cost under the continuous-time model may be expressed as shown in Equation 4.19.

$$\begin{aligned} C &= XS(0) - (-\int_0^T \tilde{S}(t)dx) \\ &= -\frac{1}{2}\mu v T^2 - v\sigma \int_0^T z(t)dt + \varepsilon vT + \eta v^2 T + \frac{1}{2}\gamma v^2 T^2 \end{aligned} \tag{4.19}$$

The mean and the variance of the transaction cost can then be determined as shown in Equations 4.20 and 4.21[16] (see Appendix B).

$$E[C] = -\frac{1}{2}\mu v T^2 + \varepsilon vT + \eta v^2 T + \frac{1}{2}\gamma v^2 T^2 = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^2}{T} + \frac{1}{2}\gamma X^2 \tag{4.20}$$

$$V[C] = v^2\sigma^2 V[\int_0^T z(t)dt] = \frac{1}{3}v^2\sigma^2 T^3 = \frac{1}{3}T\sigma^2 X^2 \tag{4.21}$$

4.5. Optimal Execution Strategy under the Continuous-Time Model

The investor’s liquidation cost is as shown in Equation 4.22 adopting the same approach used for the discrete-time model.

$$L = E[C] + rZ_\alpha \sqrt{V[C]} = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^2}{T} + \frac{1}{2}\gamma X^2 + rZ_\alpha \sqrt{\frac{1}{3}T}\sigma X \tag{4.22}$$

The conditions for minimizing the liquidation cost are as shown in Equation 4.23.

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15 $X = \int_0^T vdt$ is used for the derivation of the first term.

16 $X = vT$ is used to derive Equations 4.20 and 4.21.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta X^2}{T^2} + \frac{rZ_\alpha}{2}\sqrt{\frac{1}{3T}}\sigma X = 0 \tag{4.23}$$

Thus, if it is assumed that the drift term can be ignored in the short time period until completion of sales ($\mu = 0$), the optimal holding period can be obtained as shown in Equation 4.24.

$$T = \left(\frac{2\sqrt{3}\eta X}{rZ_\alpha \sigma}\right)^{\frac{2}{3}} \tag{4.24}$$

This demonstrates that when $\mu = 0$, the optimal holding period is proportionate to the two-thirds power of the investor's position X and the temporary market impact coefficient η.

4.6. L-VaR under the Continuous-Time Model

L-VaR can be obtained by substituting the optimal holding period from Equation 4.24 into Equations 4.16 and 4.21, as shown in Equation 4.25.

$$L - VaR = \left(\frac{2\eta \sigma^2 Z_\alpha^2 X^4}{3r}\right)^{\frac{1}{3}} \tag{4.25}$$

This demonstrates that when $\mu = 0$, the optimal holding period is proportionate to the four-thirds power of the investor's position X and to the one-third power of the temporary market impact coefficient η.

5. Numerical Examples

This chapter calculates numerical examples of L-VaR using the framework proposed in Chapter 4, and considers the potential for the practical application of this framework. This chapter does not propose an empirical method for measuring the market impact, but rather indicates how L-VaR can be calculated with a given assumption regarding formulation of the market impact. In this chapter, as a simple means of measuring the market impact, a method of estimating the market impact coefficient from the stock market tick data[17] is adopted for the calculations of the numerical examples. See Appendix C and Appendix D for the specific figures used and the methods of estimating

[17] The tick data of stocks listed on the Tokyo Stock Exchange for estimating the market impact coefficients are obtained from Bloomberg L.P.

the parameters.

This chapter considers the following three issues with relevant numerical examples. First, section 5.1 presents a comparison of the conventional VaR and L-VaR using the continuous-time model, and provides a simple examination of the characteristics of L-VaR. Next, section 5.2 considers the extent to which fluctuations in the market impact coefficient influence L-VaR. Finally, placing the continuous-time model as an approximation of the discrete-time model, section 5.3 considers the discrepancy between the continuous-time model and the discrete-time model by examining the difference of L-VaR figures generated.

5.1. Comparison between the Conventional VaR and L-VaR

This section calculates numerical examples of the conventional VaR and L-VaR, and explains the characteristics of L-VaR in comparison with the conventional VaR. Specifically, the section calculates the conventional VaR, L-VaR and the optimal holding periods for two different investor's positions for two companies: Company A, for which the market impact is relatively small, and Company B, for which the market impact is relatively large. The calculation results are presented in Table 1. The holding period for the conventional VaR is assumed to be one day, as this approach assumes sales within a short period of time.

Table 1: Calculation Results for the Conventional VaR and L-VaR (Continuous-time Model)

	Investor's Position (¥1,000)	Conventional VaR		L-VaR		(b)/(a)
		Holding Period (days)	VaR (¥1,000) (a)	Holding Period (days)	VaR (¥1,000) (b)	
Company A Stocks (1999.9.29)	165,500	1.00	8,567	0.09	1,472	0.17
	1,655,000	1.00	85,669	0.41	31,714	0.37
Company B Stocks (1999.9.29)	165,500	1.00	11,846	4.32	14,208	1.20
	1,655,000	1.00	118,464	20.03	306,105	2.58

First, as shown in Table 1, for L-VaR the holding period varies substantially depending upon the extent of the market impact and the size of the position. Especially, when the position is large, the holding period is prolonged accordingly. Thus, while the conventional VaR is linear with the position, L-VaR is non-linear with the position. For example, when the investor's position is increased tenfold, L-VaR increases by approximately 22 times.[18]

In comparing the results for Company A and Company B, the table also shows that in comparison with L-VaR the conventional VaR tends to overestimate (underestimate) the financial risk for stocks with relatively high (low) liquidity.

5.2. Influence on L-VaR from the Market Impact Coefficient Measurement Error

As stated in Chapter 4, there is presently no consensus on the method of calculating the market impact. This chapter adopts a simple method of estimating the market impact for the numerical examples from the stock market tick data. Nevertheless, this method is not necessarily satisfactory as a means of estimating the market impact, and there is a possibility that the measurement error may be significant. For this reason, this section considers the extent of the influence on L-VaR from the market impact measurement error by examining the change in L-VaR due to the change of the market impact coefficient.

Equation 4.25 shows that the temporary market impact coefficient influences L-VaR on the order of the one-third power. This is expressed numerically in Table 2 in terms of the percent change of the temporary market impact coefficient and the consequent change of L-VaR.

Table 2: The Change of L-VaR Compared with the Change of the Market Impact Coefficient

Market Impact Percent Change	L-VaR Percent Change
-90%	-54%
-50%	-21%
-25%	- 9%
-10%	- 3%
- 5%	- 2%
± 0%	± 0%
+ 5%	+ 2%
+10%	+ 3%
+25%	+ 8%
+50%	+14%
2 times	+26%
5 times	+71%
10 times	2.15 times

[18] Under the continuous-time model for L-VaR, from Equation 4.25 the optimal holding period is proportionate to the four-thirds power of the investor's initial position. Consequently, if the position is increased tenfold, L-VaR is increased by $10^{4/3} = 21.5$ times.

Table 2 shows that the change of L-VaR is not as large as that of the market impact coefficient. Particularly, when the market impact coefficient changes ±25%, the change in L-VaR remains within ±10%. Even when the market impact coefficient doubles, L-VaR increases only by +26%. This fact demonstrates that the model proposed in this paper has a certain amount of robustness to the coefficient measurement error.

5.3. Differences between the Continuous-Time Model and the Discrete-Time Model

Section 5.1 proposes the continuous-time model as a framework for the calculation of L-VaR, but in real markets it is not possible to execute sales continuously, so the discrete-time model is a more accurate model reflecting the reality of market practices. Nevertheless, the continuous-time model has the merit of providing a closed-form solution, and as the goal of this paper is to present a simpler framework for the calculation of L-VaR, the continuous-time model is therefore preferable to the discrete-time model. Thus, for the purpose of practical application, the continuous-time model is viewed as approximating the discrete-time model, and by evaluating the approximation error, the continuous-time model can then be utilized for the quantification of financial risk. From this perspective, this section examines the approximation error, which is represented by the differences in L-VaR values when calculated using the continuous-time model and the discrete-time model, respectively.

Table 3: Approximation Error of the Continuous-time Model
(Investor's Position = ¥170 million)

Company A Stocks

τ (Days*)	(Continuous – Discrete) /Discrete *100
0.000	0.000%
0.005	6.076%
0.010	13.169%
0.015	21.589%
0.020	31.803%
0.025	44.560%
0.030	61.191%

Company B Stocks

τ (Days*)	(Continuous – Discrete) /Discrete *100
0.000	0.000%
0.005	0.116%
0.010	0.232%
0.015	0.349%
0.020	0.466%
0.025	0.584%
0.030	0.701%

*0.02 days = approximately 5 minutes
(1 day = 4.5 hours <stock exchange operating hours> = 270 minutes)

Table 3 presents the approximation error of the continuous-time model for the case where the investor's initial position (Company A or Company B stocks) is ¥170 million, each of the parameters is fixed, and the sales interval τ $(0 \le \tau \le 0.03)$ varies. For example, Table 3 shows that when the sales interval τ = 0.02 days for selling Company A stocks, L-VaR error is greater than +30% in approximating the continuous-time model. In contrast, for selling Company B stocks L-VaR error is no more than +0.5%. As stated in section 4.1, the sales interval τ is apparently dependent on the price convergence speed for each financial product. Thus, the continuous-time model approximation error is small when the price convergence speed is fast (when the sales intervals are short) and the market impact is large, the approximation error is large when the price convergence speed is slow (when the sales intervals are long) and the market impact is small.

Therefore, the approximation error from using the continuous-time model is dependent on the value of τ. Regarding this point, Holthausen, Leftwich and Mayers (1990) concluded that the time required for post-sales price recovery is short,[19] and that τ may be set as a relatively short period of time. Thus, the use of the continuous-time model in approximating the discrete-time model may be deemed appropriate.

6. Generalization of the Model

This chapter relaxes the assumptions adopted for the model developed in Chapter 4 for its expanded application to more general cases. This expansion consists of the following three points: (1) introducing uncertainty into the market impact (sections 6.1 and 6.2); (2) handling portfolios composed of multiple financial assets (sections 6.3 and 6.4); and (3) introducing a non-linear relationship between the sales volume and the market impact (sections 6.5 and 6.6). For each of these points, the model is first set and then examinations are made using L-VaR numerical examples. The values of the parameters used for the numerical examples are the same as those in Chapter 5, unless otherwise noted. Additionally, to simplify the calculations, the discussions in this chapter are all based on the continuous-time model.

6.1. Stochastic Market Impact Model

Under the model developed in Chapter 4, it is assumed that the market impact

[19] By conducting empirical examinations on NYSE stock transaction tick data, they concluded that the recovery from the price reduction caused by the temporary market impact is fully completed within two subsequent sales.

parameters are all constants and that the market impact function is deterministic. In actual markets, however, the market impact parameters apparently change. Accordingly, this chapter proposes a continuous-time model introducing uncertainty in the temporary market impact.

Here, the uncertainty is introduced by assuming that the temporary market impact coefficient follows an arithmetic random walk.[20] The temporary market impact coefficient at time t is defined as shown in Equation 6.1, [21]

$$\eta_t = \eta_0 + \sigma_\eta z_\eta(t) \tag{6.1}$$

where η_0 is the temporary market impact at time 0, σ_η is the temporary market impact volatility, and $z_\eta(t)$ is the standard Brownian motion.

The "sales price" process can then be expressed as shown in Equation 6.2.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \{\eta_0 + \sigma_\eta z_\eta(t)\}v(t) - \gamma\int_0^t v(s)ds \tag{6.2}$$

Here, it is assumed that the market impact fluctuation and the stock price fluctuations have no mutual influence, and, consequently, that $z(t)$ and $z_\eta(t)$ are independent.[22] Under Equation 6.2, the total sales value is calculated as shown in Equation 6.3.

$$\begin{aligned} &-\int_0^T \tilde{S}(t)dx \\ &= XS(0) + \frac{1}{2}\mu vT^2 + v\sigma\int_0^T z(t)dt - \varepsilon vT - \eta_0 v^2 T - v^2\sigma_\eta\int_0^T z_\eta(t)dt - \frac{1}{2}\gamma v^2 T^2 \end{aligned} \tag{6.3}$$

The transaction cost is then calculated as shown in Equation 6.4.

$$C = -\frac{1}{2}\mu vT^2 - v\sigma\int_0^T z(t)dt + \varepsilon vT + \eta_0 v^2 T + v^2\sigma_\eta\int_0^T z_\eta(t)dt + \frac{1}{2}\gamma v^2 T^2 \tag{6.4}$$

Given the independence of $z(t)$ and $z_\eta(t)$, the mean and the variance of the transaction cost are calculated as shown in Equations 6.5 and 6.6.

$$E[C] = -\frac{1}{2}\mu vT^2 + \varepsilon vT + \eta_0 v^2 T + \frac{1}{2}\gamma v^2 T^2 = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2 \tag{6.5}$$

$$V[C] = \frac{1}{3}v^2\sigma^2 T^3 + \frac{1}{3}v^4\sigma_\eta^2 T^3 = \frac{1}{3}X^2(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T}) \tag{6.6}$$

---

[20] Another conceivable method of introducing uncertainty in the temporary market impact is the formulation whereby the initial value of the market impact coefficient η_0 follows a normal distribution and subsequently remains constant at the initial value until the execution is completed. See Appendix E for a discussion of this method.

[21] A similar approach may be adopted for introducing uncertainty to the permanent market impact (see Appendix F).

[22] Here, independence is assumed to simplify the calculations, but it is also possible to generalize a model where there is a correlation between $z(t)$ and $z_\eta(t)$ (see Appendix G).

Next, the liquidation cost is calculated as shown in Equation 6.7.

$$L = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2 + rZ_\alpha \sqrt{\frac{1}{3} X^2 (\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T})} \qquad (6.7)$$

The optimal holding period is calculated as shown in Equation 6.8. As there is no closed-form solution for this, the solution must be obtained using numerical calculations.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta_0 X^2}{T^2} + \frac{rZ_\alpha X^2}{3}(\sigma^2 - \frac{\sigma_\eta^2 X^2}{T^2})\left\{\frac{X^2}{3}(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T})\right\}^{-\frac{1}{2}} = 0 \; (6.8)$$

L-VaR can then be obtained by substituting the solution for T from Equation 6.8 into Equation 6.6, taking the square root, and multiplying this by the upper 100α percentile of the standard normal distribution Z_α.

6.2. Examinations using the Stochastic Market Impact Model: Numerical Examples

In this section, numerical calculations are conducted for the stochastic market impact model adopting the parameters used in Chapter 5 to examine the model presented in the previous section.

First, Table 4 presents the numerical examples for Company A, for which the market impact is relatively small, and for Company B, for which the market impact is relatively large. The proportionate volatility figures presented in the table show the standard deviation of the fluctuations in the temporary market impact coefficient over one year divided by the present temporary market impact coefficient.

Table 4: L-VaR under Stochastic Market Impact
Investor's Position = ¥1,655 million

Company A Stocks

Proportionate Volatility*	**Optimal Holding Period (Days)**	**L-VaR (¥1,000)**
0%	0.411	31,714
25%	0.411	31,714
50%	0.411	31,714
75%	0.411	31,714
100%	0.411	31,714
125%	0.411	31,715
150%	0.411	31,715
200%	0.411	31,716
500%	0.411	31,727

Company B Stocks

Proportionate Volatility*	**Optimal Holding Period (Days)**	**L-VaR (¥1,000)**
0%	20.03	306,105
25%	20.03	306,121
50%	20.03	306,168
75%	20.04	306,246
100%	20.05	306,355
125%	20.06	306,495
150%	20.07	306,665
200%	20.10	307,099
500%	20.43	312,146

* Proportionate Volatility = (Annual standard deviation of the temporary market impact) / (Temporary market impact)

As shown by these results, except when the uncertainty of the market impact is extremely high, the influence of the market impact uncertainty on L-VaR is limited. A possible reason for this is that compared with the fluctuations in the transaction cost due to stock price changes, the changes in the transaction cost due to market impact changes are extremely small. In other words, when dividing the variance of the transaction cost $V[C]$ into the portion from stock price changes $V_{price}[C]$ and the portion from market impact changes $V_{MI}[C]$ the ratio is as follows.

$$\frac{V_{MI}[C]}{V_{price}[C]} = \frac{\frac{1}{3}v^4\sigma_\eta^2 T^3}{\frac{1}{3}v^2\sigma^2 T^3} = \frac{\sigma_\eta^2}{\sigma^2}v^2 = \frac{\sigma_\eta^2}{\sigma^2}\frac{X^2}{T^2}$$

It can be safely said that this ratio is exceedingly small for conceivable stock issues. For example, if the annual volatility of the market impact for the stocks of Company B is set at 100%, the ratio is calculated as follows.

$$\frac{V_{MI}[C]}{V_{price}[C]} = \frac{\sigma_\eta^2}{\sigma^2}\frac{X^2}{T^2} = 0.081\%$$

Thus, the influence from the market impact changes is extremely small compared with the impact from the price changes.

Nevertheless, it is important to note that this model assumes that the investor does not change the optimal execution strategy during the holding period. When the market impact changes stochastically, it is quite possible that making appropriate adjustments to the execution strategy in response to changes in the market impact may result in lower transaction costs compared with those calculated by this model. In this case, there is a high likelihood that L-VaR will change substantially. However, this point is not considered here, and is left as an issue for further research.

6.3. Portfolio Model

In the model developed in Chapter 4, it is assumed that the investor's portfolio consists of a single stock. For practical application, however, it is preferable to provide a framework that can be applied to portfolios comprised of various types of stocks, to incorporate the correlation among securities prices. Accordingly, this section presents a framework to derive the optimal execution strategy for portfolios comprised of multiple stocks using the continuous-time model. The model setting is based on Almgren and Chriss (1999), revised for continuity in time, and then derives the optimal execution strategy assuming sales at a constant speed. When calculating L-VaR for portfolios comprised of multiple stocks, the stock price correlation must be considered not only when calculating the VaR but also when deriving the optimal execution strategy.

The portfolio contains m types of stocks, which are each numbered by j $(1 \le j \le m)$. Under the notation used, X_j is the initial number of stocks held in issue j, T_j is the holding period, x_{tj} is the number of stocks held at time t, and v_{tj} is the sales volume per unit time. The variance covariance matrix of the stock price is calculated as shown in Equation 6.9. [23]

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{pmatrix} \tag{6.9}$$

The lower triangular matrix presented as Equation 6.10 is obtained by conducting a Cholesky decomposition on this variance covariance matrix (that is, for $\Sigma = AA^T$ [24] when $j > i$, $\alpha_{ij} = 0$.)

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mm} \end{pmatrix} \tag{6.10}$$

The market price follows an arithmetic random walk, and the correlation among stock prices is given by the variance covariance matrix Σ. Moreover, it is assumed that the sale of a given stock does not influence the price of other stocks. In this case, the "sales price" of stock j at time t is calculated as shown in Equation 6.11 where $z_1(t), \cdots z_m(t)$ are mutually independent standard Brownian motions.

$$\tilde{S}_j(t) = S_j(0) + \mu_j t + (\alpha_{j1} \quad \cdots \quad \alpha_{jm}) \begin{pmatrix} z_1(t) \\ \vdots \\ z_m(t) \end{pmatrix} - \varepsilon_j - \eta_j v_j - \gamma_j \int_0^t v_j ds \tag{6.11}$$

With sales at a constant speed, $dx_j = - v_j dt$, and v_j is a constant, so the total sales value is calculated as shown in Equation 6.12.[25]

---

[23] If the investor hedges the systematic risk of the portfolio using stock futures or other instruments, this variance covariance matrix may be interpreted as the variance covariance matrix for the unsystematic (individual) risks.

[24] Here, A^T expresses the transposed matrix for matrix A.

[25] $v_j T_j = X_j$ is used for deriving Equation 6.12.

$$\sum_{j=1}^{m} X_j \bar{S}_j$$
$$= \sum_{j=1}^{m} \left\{ X_j S_j(0) + \frac{1}{2}\mu_j X_j T_j - \varepsilon_j X_j - \frac{\eta_j X_j^2}{T_j} - \frac{1}{2}\gamma_j X_j^2 + v_j \int_0^{T_j} \sum_{i=1}^{m} \alpha_{ji} z_i(t)dt \right\} \quad (6.12)$$

Consequently, the transaction cost is calculated as shown in Equation 6.13.

$$C = \sum_{j=1}^{m} \left\{ -\frac{1}{2}\mu_j X_j T_j + \varepsilon_j X_j + \frac{\eta_j X_j^2}{T_j} + \frac{1}{2}\gamma_j X_j^2 - v_j \int_0^{T_j} \sum_{i=1}^{m} \alpha_{ji} z_i(t)dt \right\} \quad (6.13)$$

Thus, the mean and the variance of the transaction cost may be calculated as shown in Equations 6.14 and 6.15 (see Appendix H).[26]

$$E[C] = \sum_{j=1}^{m} \left\{ -\frac{1}{2}\mu_j X_j T_j + \varepsilon_j X_j + \frac{\eta_j X_j^2}{T_j} + \frac{1}{2}\gamma_j X_j^2 \right\} \quad (6.14)$$

$$V[C] = V[\sum_{j=1}^{m} v_j \int_0^{T_j} \sum_{i=1}^{m} \alpha_{ji} z_i(t)dt] = V[\sum_{i,j=1}^{m} v_j \alpha_{ji} \int_0^{T_j} z_i(t)dt]$$
$$= \frac{1}{3}\sum_{j=1}^{m} \sigma_{jj} X_j^2 T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}} \sigma_{jk} X_j X_k \frac{T_j^2}{T_k} + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}} \sigma_{jk} X_j X_k \frac{T_k^2}{T_j} \quad (6.15)$$

Utilizing the fact that $\sigma_{jj} = \sum_{i=1}^{m} \alpha_{ji}^2$ and $\sigma_{jk} = \sum_{i=1}^{m} \alpha_{ji}\alpha_{ki}$, the investor's liquidation cost can then be calculated as shown in Equation 6.16.

$$L = E[C] + rZ_\alpha \sqrt{V[C]}$$
$$= \sum_{j=1}^{m} \left\{ -\frac{1}{2}\mu_j X_j T_j + \varepsilon_j X_j + \frac{\eta_j X_j^2}{T_j} + \frac{1}{2}\gamma_j X_j^2 \right\} \quad (6.16)$$
$$+ rZ_\alpha \sqrt{\begin{array}{l} \frac{1}{3}\sum_{j=1}^{m} \sigma_{jj} X_j^2 T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}} \sigma_{jk} X_j X_k \frac{T_j^2}{T_k} \\ + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}} \sigma_{jk} X_j X_k \frac{T_k^2}{T_j} \end{array}}$$

The conditions for minimizing the investor's liquidation cost for stock l $(1 \le l \le m)$ may then be expressed as shown in Equation 6.17.

$$\frac{\partial L}{\partial T_l} = -\frac{\mu_l X_l}{2} - \frac{\eta_l X_l^2}{T_l^2}$$

[26] The indicator function $I_{\{A\}}$ is 1 when A is true and 0 when A is false.

$$
+\frac{rZ_{\alpha}}{6}\left\{\begin{array}{l}
\sigma_{ll}X_l^2 \\
+4\sum_{k=l+1}^{m} I_{\{T_l \le T_k\}}\sigma_{lk}X_lX_k\frac{T_l}{T_k} - 2\sum_{n=1}^{l-1} I_{\{T_l \le T_n\}}\sigma_{nl}X_nX_l\frac{T_n^2}{T_l^2} \\
-2\sum_{k=l+1}^{m} I_{\{T_l > T_k\}}\sigma_{lk}X_lX_k\frac{T_k^2}{T_l^2} + 4\sum_{n=1}^{l-1} I_{\{T_l > T_n\}}\sigma_{nl}X_nX_l\frac{T_l}{T_n}
\end{array}\right\} \tag{6.17}
$$

$$
\times\left\{\begin{array}{l}
\frac{1}{3}\sum_{j=1}^{m}\sigma_{jj}X_j^2T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}}\sigma_{jk}X_jX_k\frac{T_j^2}{T_k} \\
+\frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}}\sigma_{jk}X_jX_k\frac{T_k^2}{T_j}
\end{array}\right\}^{-\frac{1}{2}} = 0
$$

The numerical calculations required to solve this equation for T_l are very complicated because of the presence of indicator functions concerning T_l. Even if there is no price correlation and $\sigma_{jk}=0$ ($j \neq k$), the conditions become as shown in Equation 6.18, and the resulting T_l (l=1,…,m) is a system of simultaneous equations.

$$
\frac{\partial L}{\partial T_l} = -\frac{\mu_l X_l}{2} - \frac{\eta_l X_l^2}{T_l^2} + \frac{1}{2\sqrt{3}} rZ_{\alpha}\sigma_{ll}X_l^2\left\{\sum_{j=1}^{m}\sigma_{jj}X_j^2T_j\right\}^{-\frac{1}{2}} = 0 \tag{6.18}
$$

Thus, it becomes clear that even when the price correlation is zero, unlike the situation when the investor's portfolio consists of a single asset, the optimal holding periods for each stock in portfolios containing multiple assets are influenced by the liquidity and volatility of the other assets.

L-VaR can be calculated by substituting the optimal holding period calculated in Equation 6.17 into Equation 6.19, as shown below.

$$
\begin{aligned}
L-VaR &= Z_{\alpha}\sqrt{V[C]} \\
&= Z_{\alpha}\sqrt{\begin{array}{l}
\frac{1}{3}\sum_{j=1}^{m}\sigma_{jj}X_j^2T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}}\sigma_{jk}X_jX_k\frac{T_j^2}{T_k} \\
+\frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}}\sigma_{jk}X_jX_k\frac{T_k^2}{T_j}
\end{array}}
\end{aligned} \tag{6.19}
$$

6.4. Examinations using the Portfolio Model: Numerical Examples

This section examines the portfolio model by calculating numerical examples under Equation 6.17 for the situation where the investor's portfolio contains two assets.

We have seen that the numerical calculations required to solve Equation 6.17 are quite complicated and this can be an obstacle to practical applications. As a basic approach to this difficulty, we suggest an approximation method in calculating L-VaR of portfolios of multiple stocks. First, Equation 4.24 is initially adopted for the derivation of the optimal holding period as if each of the stocks in the portfolio were independent. Second, the results are then substituted into Equation 6.19 to calculate the approximate portfolio L-VaR. If this type of approximation is possible, the portfolio L-VaR can then be calculated in a relatively simple manner. We verify the validity of this approximation method with a simple numerical example.

The numerical example here adopts the figures for Company A and Company B stocks used in Chapter 5. In order to examine L-VaR of a portfolio including stocks with a similar profile, a hypothetical Company C stock is created by slightly decreasing only the temporary market impact coefficient value of Company A stock ($\eta = 3.91 \times 10^{-6} \rightarrow 3.81 \times 10^{-6}$). Then L-VaR is calculated for two cases where the portfolio contains two assets: stocks of Company A and Company B, and stocks of Company A and Company C. Additionally, L-VaR is also calculated changing the correlation coefficient to examine the effect of varied correlation coefficients on approximation efficiency.

When the price correlation is zero, the optimal holding period calculated using Equation 6.17 is longer than that calculated using Equation 4.24. In other words, when the correlation is zero, L-VaR calculated using Equation 4.24 is underestimated compared with L-VaR calculated using Equation 6.17. The error is around 12% when the characteristics of the two stocks are similar (Table 6) and around 2% when the characteristics of the two stocks are different (Table 5).

On the other hand, when there is a correlation between the stock prices, it is difficult to make any generalization about whether the error increases or decreases compared with the case where the correlation is zero. Nevertheless, it should be noted, as demonstrated by the case of Company A and Company C stocks, that when the stock characteristics are similar and there is a high negative correlation between the stock prices, there may be cases where the error increases substantially.

Table 5: L-VaR for a Two-Asset Portfolio (Company A & Company B)

Investor's Position: Company A - ¥1,655 million, Company B - ¥1,655 million

	Correlation Coefficient	Optimal Holding Period (Days)		L-VaR
		Company A	Company B	(¥1,000)
Individual Optimization*	-1.00	0.41	20.03	307,651
	-0.75	0.41	20.03	307,674
	-0.50	0.41	20.03	307,697
	-0.25	0.41	20.03	307,721
	0.00	0.41	20.03	307,744
	0.25	0.41	20.03	307,767
	0.50	0.41	20.03	307,790
	0.75	0.41	20.03	307,813
	1.00	0.41	20.03	307,837
Portfolio Optimization	-1.00	1.83	20.11	312,091
	-0.75	1.57	20.17	312,343
	-0.50	1.44	20.20	312,544
	-0.25	1.36	20.23	312,718
	0.00	1.29	20.25	312,873
	0.25	1.24	20.27	313,016
	0.50	1.20	20.28	313,147
	0.75	1.16	20.30	313,271
	1.00	1.13	20.31	313,387

* The optimal holding periods are calculated for each stock, and the portfolio L-VaR is then calculated based on these.

Table 6: L-VaR for a Two-Asset Portfolio (Company A & Company C)

Investor's Position: Company A - ¥1,655 million, Company C - ¥1,655 million

	Correlation Coefficient	Optimal Holding Period (Days)		L-VaR
		Company C	Company A	(¥1,000)
Individual Optimization*	-1.00	0.40	0.41	7,146
	-0.75	0.40	0.41	23,171
	-0.50	0.40	0.41	31,980
	-0.25	0.40	0.41	38,840
	0.00	0.40	0.41	44,658
	0.25	0.40	0.41	49,801
	0.50	0.40	0.41	54,461
	0.75	0.40	0.41	58,752
	1.00	0.40	0.41	62,750
Portfolio Optimization	−0.75	0.82	0.82	31,579
	-0.50	0.65	0.65	39,786
	-0.25	0.57	0.57	45,544
	0.00	0.51	0.52	50,127
	0.25	0.41	0.60	52,933
	0.50	0.37	0.65	54,709
	0.75	0.34	0.69	56,079
	1.00	0.33	0.72	57,215

* The optimal holding periods are calculated for each stock, and the portfolio L-VaR is then calculated based on these.

6.5. Non-linear Market Impact Model

The model developed in Chapter 4 assumes a linear relationship between the market impact and the sales volume. However, as demonstrated by the empirical analyses presented by Nakatsuka (1998) and others, the actual market impact may not be a linear function. Accordingly, this section considers the case where the market impact function is a square root function ($y = b\sqrt{x}$) as one example of a non-linear market impact formulation. In addition to Nakatsuka (1998), other empirical analyses have been presented showing that square root functions are appropriate for the formulation of the market impact.

When formulating the market impact as a square root function, the continuous-time "sales price" process is as shown in Equation 6.20.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \eta\sqrt{v(t)} - \gamma\int_0^t \sqrt{v(s)}ds \tag{6.20}$$

Under this model, the total sales value is calculated as shown in Equation 6.21.

$$-\int_0^T \tilde{S}(t)dx = XS(0) + \frac{1}{2}\mu v T^2 + v\sigma\int_0^T z(t)dt - \varepsilon v T - \eta v^{\frac{3}{2}} T - \frac{1}{2}\gamma v^{\frac{3}{2}} T^2 \tag{6.21}$$

Next, the transaction cost is calculated as shown in Equation 6.22.

$$C = -\frac{1}{2}\mu v T^2 - v\sigma\int_0^T z(t)dt + \varepsilon v T + \eta v^{\frac{3}{2}} T + \frac{1}{2}\gamma v^{\frac{3}{2}} T^2 \tag{6.22}$$

Then, the mean and the variance of the transaction cost are calculated as shown in Equations 6.23 and 6.24.

$$\begin{aligned} E[C] &= -\frac{1}{2}\mu v T^2 + \varepsilon v T + \eta v^{\frac{3}{2}} T + \frac{1}{2}\gamma v^{\frac{3}{2}} T^2 \\ &= -\frac{1}{2}\mu X T + \varepsilon X + \eta X^{\frac{3}{2}} T^{-\frac{1}{2}} + \frac{1}{2}\gamma X^{\frac{3}{2}} T^{\frac{1}{2}} \end{aligned} \tag{6.23}$$

$$V[C] = v^2\sigma^2 V[\int_0^T z(t)dt] = \frac{1}{3}v^2\sigma^2 T^3 = \frac{1}{3}T\sigma^2 X^2 \tag{6.24}$$

The investor's liquidation cost, which should be minimized, is defined as shown in Equation 6.25.

$$L = -\frac{1}{2}\mu X T + \varepsilon X + \eta X^{\frac{3}{2}} T^{-\frac{1}{2}} + \frac{1}{2}\gamma X^{\frac{3}{2}} T^{\frac{1}{2}} + rZ_\alpha\sqrt{\frac{1}{3}} T^{\frac{1}{2}}\sigma X \tag{6.25}$$

The conditions for the optimal holding period are as shown in Equation 6.26.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{1}{2}\eta X^{\frac{3}{2}} T^{-\frac{3}{2}} + \frac{1}{4}\gamma X^{\frac{3}{2}} T^{-\frac{1}{2}} + \frac{rZ_\alpha}{2\sqrt{3}} T^{-\frac{1}{2}}\sigma X = 0 \tag{6.26}$$

When $\mu = 0$, the optimal holding period is as shown in Equation 6.27.

$$T = \frac{6\sqrt{X}\eta}{3\sqrt{X}\gamma + 2\sqrt{3}rZ_\alpha\sigma} \tag{6.27}$$

L-VaR can then be obtained as shown in Equation 6.28

$$L-VaR = Z_\alpha \sigma X \left(\frac{2\sqrt{X}\eta}{3\sqrt{X}\gamma + 2\sqrt{3}rZ_\alpha\sigma} \right)^{\frac{1}{2}} \tag{6.28}$$

6.6. Examinations using the Non-linear Market Impact Model: Numerical Examples

This section calculates numerical examples to compare the differences in L-VaR when the market impact is a non-linear function and a linear function. For comparison with the case where the function is linear, as in Chapter 5, the coefficients of the square root functions are estimated from the stock market tick data in a similar way to that in Chapter 5 (see Appendix D for the details). The resulting optimal holding periods and L-VaRs are presented in Table 7.

Table 7: L-VaR under Non-linear Market Impact

Company A Stocks

Linear function temporary market impact: 3.91×10^{-6} (yen•days)/share^2
Permanent market impact: 0
Non-linear function temporary market impact: 6.25×10^{-3}(yen•days$^{0.5}$)/share
Investor's position: ¥1,655 million

Market Impact	Optimal Holding Period (Days)	L-VaR (¥1,000)
Linear	0.411	31,714
Square Root	0.298	27,002

Company B Stocks

Linear function temporary market impact: 1.88×10^{-3} (yen•days)/share^2
Permanent market impact: 0
Non-linear function temporary market impact: 1.37×10^{-2}(yen•days$^{0.5}$)/share
Investor's position: ¥1,655 million

Market Impact	Optimal Holding Period (Days)	L-VaR (¥1,000)
Linear	20.03	306,105
Square Root	4.65	147,422

As shown in Table 7, compared with the case in which the market impact is assumed to be a linear function, the holding period and L-VaR are greatly reduced. Especially, for the case of Company B stock L-VaR is approximately halved when the market impact is assumed to be a square root function instead of a linear function. These results indicate that errors in the formulation of the market impact estimation may result in highly significant errors in L-VaR calculations.

7. Future Research Issues

This paper has presented a framework for calculating the VaR incorporating the market liquidity of financial products (L-VaR). This framework facilitates the relatively simple calculation of L-VaR under certain assumptions. Nevertheless, as stated above, careful attention must be paid to the following points when using this framework, and we expect that additional research will be conducted toward clarifying these issues.

The first point is the accuracy in estimating the market impact function. This paper assumes that the market impact can be divided into the temporary portion and the permanent portion and that both of them can be estimated in a stable manner. However, it is by no means simple to estimate accurately the market impact that arises from diverse factors, and there is presently no definitive estimation method. As a simplified method of calculating the market impact function, this paper uses a simplified approach of estimating from the stock market tick data. However, from the perspective of more accurate financial risk measurement, there may be cases where a more sophisticated estimation methodology is required. Additionally, as noted in section 6.6, errors in the assumptions regarding the shape of the market impact function may result in significant errors in the resulting L-VaR values. Accordingly, we expect that further theoretical and empirical research will be conducted on the market impact, and that more sophisticated estimation methodologies will be developed.

Next, regarding the determination of the optimal execution strategy, this paper assumes that the investor's initial strategy is not changed during the execution period.[27] In some cases, this assumption may result in major errors in L-VaR measurement. For example, when the market impact function is uncertain, if it is determined that the market impact is larger than initially projected, it may be optimal to revise the initial execution strategy and prolong the holding period. In this case, there may be substantial changes in the value of L-VaR accompanying the change in the execution strategy. Hypothetically, if we were to derive the optimal execution strategy incorporating changes in strategy during the execution period, dynamic optimization methods would have to be applied. However, the use of such methods would be

[27] This point is also noted by Almgren and Chriss (1999).

difficult in cases where the holding period is determined endogenously, as in this paper.

An additional problem with the model presented in this paper is that when the investor's portfolio contains multiple assets, the risk measurement and other calculations become very complex. In section 6.4, as one simple calculation approach, we present a method whereby the optimal holding periods are calculated independently for each asset and then consider the appropriateness using numerical calculation examples. However, this paper does not necessarily include comprehensive examinations of the error that may result from the use of this simple calculation method. Therefore, future research is needed both to verify whether this method is appropriate and to determine if there may be other simple methods of calculating the portfolio L-VaR for practical application.

Finally, it should be noted that it might not be possible to apply the framework presented in this paper during periods of market stress. The approach presented in this paper assumes that the formulated market impact function is stable and that the optimal execution strategy is always possible. However, during market stress periods when liquidity dries up, these assumptions may collapse. Therefore, in addition to the model for normal periods, a separate model may be required to quantify the market liquidity risk during market stress periods.

8. Conclusions

This paper presents a framework for deriving the optimal execution strategy incorporating the market impact of the investor's own dealings using a mean–standard deviation approach under certain assumptions. With this optimal execution strategy, we can calculate the VaR incorporating the market liquidity (L-VaR). The paper demonstrates that this framework has a high potential for practical application as it can also be applied (1) when the market impact is uncertain, (2) when the investor's portfolio consists of multiple financial assets, and (3) when there is a non-linear relationship between the market impact and the sales volume.

However, as noted in Chapter 7, with this framework there are still several outstanding issues including the accurate measurement of the market impact. Moreover, as stated at the beginning, the framework presented here focuses on the market impact, which represents just one aspect of market liquidity. We hope that additional research will be conducted on market liquidity, and that in the future a comprehensive method of evaluating market liquidity risk will be developed based on the framework presented in this paper.

Appendix A: Derivation of the Optimal Execution Strategy under the Mean - Variance Approach

In Chapter 4, the optimal execution strategy is derived by formulating the objective function with the mean and the standard deviation of the transaction cost. However, the objective function could also be set in a different manner. That is, the objective function could also be formulated with the mean and the variance of the transaction cost. This appendix considers this alternative method.

First, the merit of formulation using the mean and the variance is that the objective function can be formulated in a manner consistent with the expected utility hypothesis, which is dominant in finance and economic literature. When the amount of wealth gained by the investor is W and the investor's utility function U is an exponential utility function, the expected utility ($E[U(W)]$) may be expressed by the mean and the variance of W as follows.[28]

$$E[U(W)] = E[W] - \lambda V[W]$$

Here λ may be interpreted as a parameter expressing the level of the investor's risk aversion. This utility function may be reinterpreted in terms of the framework presented in the main body of this paper. The wealth gained is the total sales value ($X\overline{S}$) finally received by the investor and the market value of the investor's initial position (XS_0) is deterministic. Therefore, the expected utility can be reformulated as follows.

$$\begin{aligned} E[U(X\overline{S})] &= E[X\overline{S}] - \lambda V[X\overline{S}] \\ &= E[XS_0 - C] - \lambda V[XS_0 - C] \\ &= XS_0 - E[C] - \lambda V[C] \end{aligned}$$

Maximizing the above equation is equivalent to minimizing the objective function f, which is defined as follows.

$$f = E[C] + \lambda V[C]$$

In other words, formulating the objective function from the mean and the variance of the transaction cost makes it possible to derive the optimal execution strategy in a manner consistent with the expected utility theory.

The rest of this appendix presents considerations of the method of deriving L-VaR through formulation using the mean and the variance. For simplification, the considerations adopt the continuous-time model.

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[28] See Sakakibara, Aoyama, and Asano (1998), pp. 472-475.

The objective function under the mean–variance approach may be expressed as follows using Equations 4.20 and 4.21 from the main body of this paper.

$$f = E[C] + \lambda V[C] = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta X^2}{T} + \frac{1}{2}\gamma X^2 + \lambda \frac{1}{3} X^2 \sigma^2 T$$

The holding period to minimize this objective function is:

$$\frac{\partial f}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta X^2}{T^2} + \lambda \frac{1}{3} X^2 \sigma^2 = 0$$

When $\mu = 0$, this equation may be solved for the optimal holding period as follows.

$$T = \left(\frac{3\eta}{\lambda \sigma^2} \right)^{\frac{1}{2}}$$

From Equation 4.16, L-VaR becomes:

$$L - VaR = Z_\alpha X \left(\frac{\eta \sigma^2}{3\lambda} \right)^{\frac{1}{4}}$$

This demonstrates that under the mean–variance approach, when $\mu = 0$, (1) the holding period is independent of the amount of the investor's position, and (2) L-VaR has a linear relation with the amount of the investor's position.

Next, numerical examples of L-VaR are presented using the parameter values from Chapter 5 (see Appendix C). The examples are calculated for the stocks of Company A, for which the market impact is relatively small, and Company B, for which the market impact is relatively large, as shown in Table 8.[29]

Table 8: Calculation Results under the Mean–Variance Approach (Continuous-time Model)

	Investor's Position (¥1,000)	Conventional VaR		L-VaR	
		Holding Period (days)	VaR (¥1,000) (a)	Holding Period (days)	VaR (¥1,000) (b)
Company A Stock (1999.9.29)	165,500	1.00	8,567	0.28	2,595
	1,655,000	1.00	85,669	0.28	25,948
Company B Stock (1999.9.29)	165,500	1.00	11,846	4.32	14,209
	1,655,000	1.00	118,464	4.32	142,090

[29] Here, $\lambda = 2.9 \times 10^{-8}$ is adopted so that when the investor's position of Company B stock is ¥165 million, L-VaR presented in Table 1 and L-VaR under the mean-variance approach are equal.

Table 8 shows that L-VaR of Company A stock is smaller than that of Company B stock, in accordance with relative liquidity of those stocks. This indicates that the mean–variance approach may also be used to quantify the financial risk incorporating the market liquidity. Nevertheless, it is important to note that under the mean–variance approach, albeit only when $\mu = 0$, the holding period is not dependent upon the amount of the investor's position.

Here, the parameter expressing the level of the investor's risk aversion λ is set as explained in Footnote 29, but for accuracy the value of λ needs to be estimated separately. Considering this point, the mean–standard deviation approach, whereby the investor's risk aversion level is incorporated into parameter Z_α, is adopted in the main body of this paper.

Appendix B: Derivation of the Mean and the Variance of the Transaction Cost

This appendix explains the derivation method for Equations 4.20 and 4.21 in the main body of this paper.

Under Equation 4.20,

$$E[C] = E[-\frac{1}{2}\mu v T^2 - v\sigma\int_0^T z(t)dt + \varepsilon v T + \eta v^2 T + \frac{1}{2}\gamma v^2 T^2]$$
$$= -\frac{1}{2}\mu v T^2 + \varepsilon v T + \eta v^2 T + \frac{1}{2}\gamma v^2 T^2 - v\sigma E[\int_0^T z(t)dt]$$

so it is only necessary to prove that $E[\int_0^T z(t)dt] = 0$.

From Theorem 4.1.5 in Øksendal (1995), p. 46,

$$\int_0^T z(t)dt = [tz(t)]_0^T - \int_0^T tdz(t) = Tz(T) - \int_0^T tdz(t)$$
$$= T\int_0^T dz(t) - \int_0^T tdz(t) = \int_0^T (T-t)dz(t)$$

so using Fubini’s theorem,

$$E[\int_0^T z(t)dt] = E[\int_0^T (T-t)dz(t)] = \int_0^T (T-t)E[dz(t)] = 0$$

Q.E.D.

Under Equation 4.21

$$V[C] = V[-\frac{1}{2}\mu v T^2 - v\sigma\int_0^T z(t)dt + \varepsilon v T + \eta v^2 T + \frac{1}{2}\gamma v^2 T^2]$$
$$= v^2\sigma^2 V[\int_0^T z(t)dt]$$

so it is only necessary to prove that $V[\int_0^T z(t)dt] = \frac{T^3}{3}$.

From Corollary 3.1.7 in Øksendal (1995), p. 29,

$$V[\int_0^T z(t)dt] = E[(\int_0^T z(t)dt)^2] - E[\int_0^T z(t)dt]^2 = E[(\int_0^T z(t)dt)^2] = E[(\int_0^T (T-t)dz(t))^2]$$
$$= E[\int_0^T (T-t)^2 dt] = \int_0^T (T-t)^2 dt = [T^2 t - 2T\frac{t^2}{2} + \frac{t^3}{3}]_0^T = \frac{T^3}{3}$$

Q.E.D.

Appendix C: Parameter Values used for the Numerical Examples

Notation Explanation (units)	Company A[30]		Company B		Company C[31]
$S(0)$ Initial stock price (yen)	3,310		3,350		3,310
X Initial number of shares held (shares)	50,000	500,000	49,403	494,031	50,000
$XS(0)$ Investor's initial position (yen)	$1{,}655\times10^{6}$	$1{,}655\times10^{7}$	$1{,}655\times10^{6}$	$1{,}655\times10^{7}$	$1{,}655\times10^{6}$
μ Drift [(yen/share)/day][32]	0		0		0
σ Volatility [(yen/share)/day$^{1/2}$]	74		103		74
ε Bid-asked spread (yen/share)[33]	0		0		0
η Temporary market impact coefficient [(yen/share) / (shares/day)]	3.91×10^{-6}		1.88×10^{-3}		3.81×10^{-6}
γ Permanent market impact coefficient (yen/day^2)[34]	0		0		0
r Cost of Capital	0.15		0.15		0.15
Z_α upper 100α percentile of the standard normal distribution（α=0.99）	2.33		2.33		2.33
Observation date	1999.9.29		1999.9.29		

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30 The data on Companies A and B were obtained from Bloomberg L.P. Companies A and B were chosen from companies listing their stocks on the First Section of the Tokyo Stock Exchange.

31 Company C is a hypothetical company postulated for the purpose of examining the behavior of stocks with characteristics similar to those of Company A by slightly altering the market impact coefficient.

32 As the period until the sales are completed is relatively short, to simplify the calculations, the drift is set at 0.

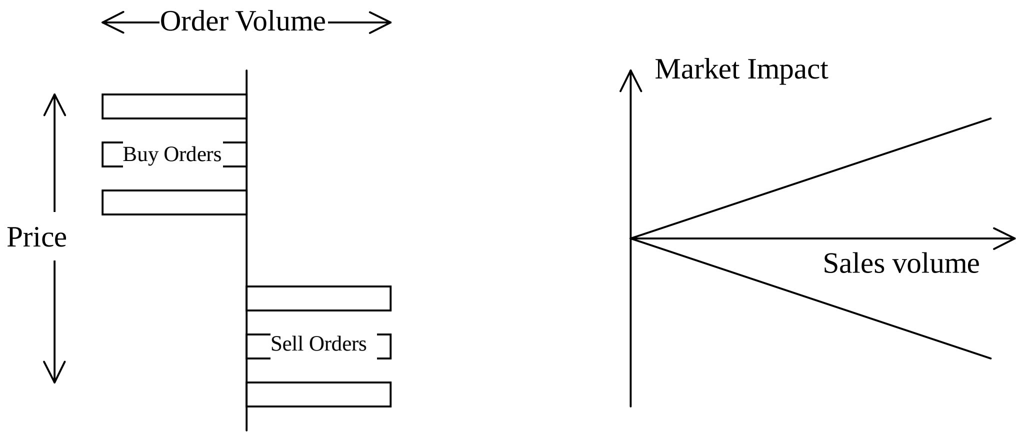
33 As the bid-ask spread does not influence L-VaR, it is set at 0.

34 To simplify the calculations, the permanent market impact is set at 0.

Appendix D: Market Impact Coefficient Estimation Method

This appendix explains the method used in the main body of this paper for estimating the market impact coefficient from the stock market tick data. We estimated market impact coefficients of stocks listed on the First Section of the Tokyo Stock Exchange. The Tokyo Stock Exchange adopts an "order-driven" market, where orders from traders are brought together on the order book of the Exchange and those orders are matched continuously according to specified rules. According to Nakatsuka (1998), in an order-driven market, the scale of the market impact changes in accordance with the volume of stocks sold, and the manner in which the market impact appears varies according to the shape of the order book. We can infer the shape of the order book with tick data by hypothetically assuming that the sales order volume (the bid order volume) is fixed with respect to varying bid prices. In that case, the scale of the resulting market impact becomes proportional to the sales volume regardless of the stock price (Figure 3).

Figure 3: Relationship between the Sales Volume and the Market Impact (Linear)



In this paper, the market impact is estimated assuming that the volume of the sales order remains constant regardless of the price level. The depth of orders is set as the time-weighted average of the best bid order volume on the date concerned, and the range of the price fluctuation is set as one tick of the concerned issue. Using these two variables, the price declines by the amount of one tick for each sales volume of the order depth. That is to say, the estimation is conducted in accordance with the following equation.

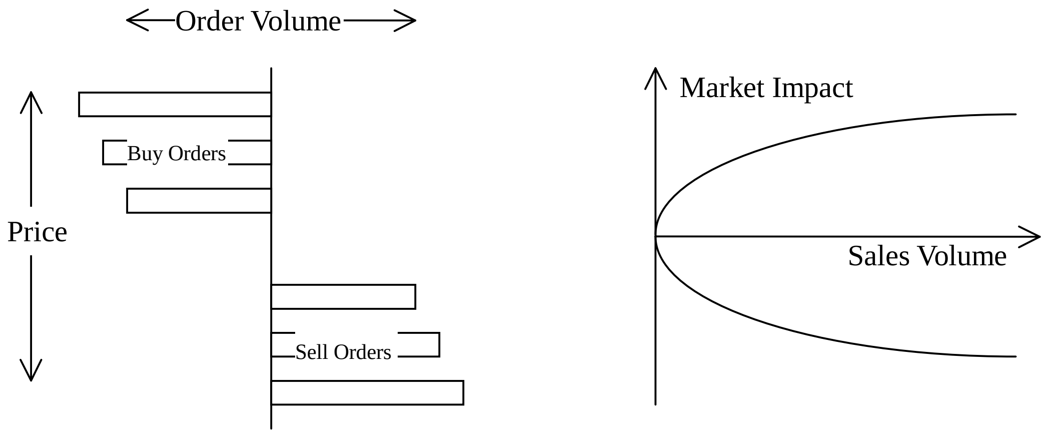
$$\text{Market impact coefficient} = \frac{\text{Change in price}}{\text{Order depth}}$$
$$= \frac{\text{One tick of the concerned stock}}{\text{Time}-\text{weighted average of the best bid order volume}}$$

Here, it is important to note that the temporary market impact coefficient η expresses the price change per unit time (the unit time is set as one day) when one stock is sold, and because of this definition the unit becomes ((yen/share)/(shares/day)). Because the unit of η includes a time component (days), the above-mentioned market impact value must be multiplied by the price recovery period[35] in order to convert the market impact value into the temporary market impact coefficient η.

The recovery period is mainly determined by the market and financial product characteristics, and strictly speaking appropriate values must be estimated for each issue. However, this paper adopts 0.02 days (approximately 5 minutes) as the recovery period for simplicity. Additionally, to simplify the calculations, the permanent market impact is assumed to be zero.

When the relationship between the sales volume and the market impact is assumed to be non-linear, the shape of the order book can be regarded as shown in Figure 4.

Figure 4 Relationship between Sales Volume and the Market Impact (Non-linear)

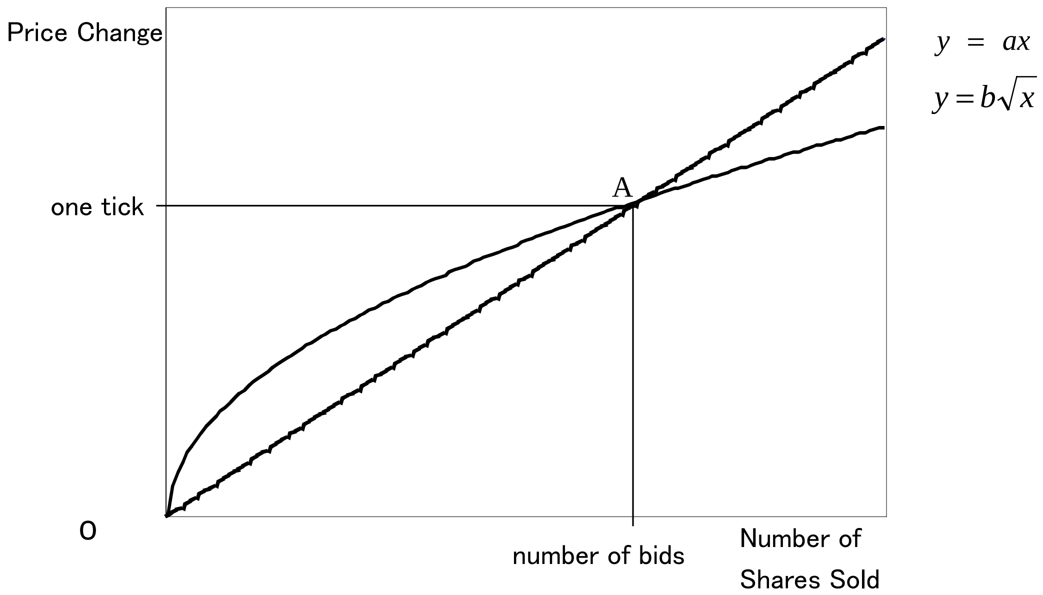


In section 6.6, L-VaR is derived when the market impact is formulated as the square root function $y = b\sqrt{x}$. The method used for estimating b here is the same as that

[35] Here, "the period from when the sales are initiated until the temporary market impact effect disappears" mentioned in section 4.1 is referred to as the "price recovery period."

adopted for estimating the linear function. Like the linear function, the coefficient of the square root function (b) is estimated from the single tick value and the number of bids on the order book. Figure 5 shows the plane surface defined by the number of stocks sold and the change in stock price. b is determined so that the $y = b\sqrt{x}$ passes through point A in Figure 5. The value of b per unit time is calculated using a recovery period of 0.02 days. To simplify the calculations, the permanent market impact is assumed to be zero.

Figure 5 Method of Determining the Non-linear Market Impact



Appendix E: Model for Introducing Uncertainty to the Initial Value of the Temporary Market Impact

In the main body of this paper, the uncertainty in market impact is introduced using an arithmetic random walk. This appendix presents a model whereby uncertainty is introduced to the initial value of the temporary market impact coefficient η and this initial value then remains constant until the end of the holding period.

First, the initial value of the temporary market impact coefficient η is defined as uncertain at the time of the initial sale, as expressed by the following equation.

$$\eta_t = \eta_0 + \sigma_\eta \xi_\eta$$

Here, ξ_η is a random variable following a standard normal distribution which is defined at the time of the first sale and maintains the same value thereafter. σ_η is the standard deviation of the temporary market impact coefficient. In this case, the sales price process for the investor is as shown by the following equation.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \{\eta_0 + \sigma_\eta \xi_\eta\} v(t) - \gamma \int_0^t v(s)ds$$

Assuming sales at a constant speed ($v(u) = v$ <constant>), instantaneous change in the investor's position can be formulated as $dx = -vdt$. Therefore, the total sales value can be determined by the following equation.

$$-\int_0^T \tilde{S}(t)dx = v\int_0^T \tilde{S}(t)dt = v\int_0^T \{S(0) + \mu t + \sigma z(t) - \varepsilon - \{\eta_0 + \sigma_\eta \xi_\eta\} v - \gamma \int_0^t vds\} dt$$

$$= XS(0) + \frac{1}{2}\mu v T^2 + v\sigma \int_0^T z(t)dt - \varepsilon v T - \eta_0 v^2 T - v^2 \xi_\eta \sigma_\eta T - \frac{1}{2}\gamma v^2 T^2$$

Thus, the transaction cost is expressed as shown by the following equation.

$$C = XS(0) - (-\int_0^T \tilde{S}(t)dx)$$

$$= -\frac{1}{2}\mu v T^2 - v\sigma \int_0^T z(t)dt + \varepsilon v T + \eta_0 v^2 T + v^2 \sigma_\eta \xi_\eta T + \frac{1}{2}\gamma v^2 T^2$$

Then, the mean and the variance of the transaction cost are calculated as shown in the following two equations.

$$E[C] = -\frac{1}{2}\mu v T^2 + \varepsilon v T + \eta_0 v^2 T + \frac{1}{2}\gamma v^2 T^2 = -\frac{1}{2}\mu X T + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$$

$$V[C] = v^2 \sigma^2 V[\int_0^T z(t)dt] + v^4 \sigma_\eta^2 T^2 = \frac{1}{3} v^2 \sigma^2 T^3 + v^4 \sigma_\eta^2 T^2 = \frac{1}{3} X^2 \sigma^2 T + \frac{X^4 \sigma_\eta^2}{T^2}$$

The investor's liquidation cost, which should be minimized, is defined as shown in the following equation.

$$L = E[C] + rZ_{\alpha}\sqrt{V[C]} = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2 + rZ_{\alpha}\sqrt{\frac{1}{3}X^2\sigma^2 T + \frac{X^4\sigma_{\eta}^2}{T^2}}$$

The following first-order condition is applied to derive the optimal holding period.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta_0 X^2}{T^2} + \frac{rZ_{\alpha}X^2(\sigma^2 - \frac{6\sigma_{\eta}^2 X^2}{T^3})}{6\sqrt{\frac{1}{3}X^2\sigma^2 T + \frac{X^4\sigma_{\eta}^2}{T^2}}} = 0$$

Here, the numerical examples are calculated using the same parameters adopted in Chapter 5.

To begin with, the numerical examples are calculated for Company A, for which the market impact is relatively small, and Company B, for which the market impact is relatively large. The calculation results are presented in Table 9. The standard deviation in this table expresses the ratio of the standard deviation of the temporary market impact coefficient (σ_{η}) to the mean of the temporary market impact coefficient η_0. For example, if this value is 10%, this means that the standard deviation of the temporary market impact coefficient is 10% of the mean of the temporary market impact coefficient.

The results demonstrate that the market impact uncertainty incorporated into this model has a relatively limited effect on L-VaR.

Table 9: L-VaR Under Stochastic Market Impact

Company A Stocks Investor's Position: ¥1,655 million

Standard Deviation (%)	Optimal Holding Period (days)	L-VaR (¥1,000)
0%	0.411	31,714
25%	0.411	31,722
50%	0.412	31,747
100%	0.413	31,846
200%	0.418	32,231

Company B Stocks Investor's Position: ¥1,655 million

Standard Deviation (%)	Optimal Holding Period (days)	L-VaR (¥1,000)
0%	20.03	306,105
25%	20.09	306,878
50%	20.28	309,129
100%	20.96	317,263
200%	23.08	341,438

Appendix F: Model for Introducing Uncertainty to the Permanent Market Impact

Assume that γ and η follow the following arithmetic random walk.

$$\eta_t = \eta_0 + \sigma_\eta z_\eta(t)$$
$$\gamma_t = \gamma_0 + \sigma_\gamma z_\gamma(t)$$

Here, γ_0 is the temporary market impact at time 0, σ_γ is the volatility of the permanent market impact, $z_\gamma(t)$ is the standard Brownian motion, and $z(t)$, $z_\eta(t)$, and $z_\gamma(t)$ are all independent of each other. The sales price is expressed by the following equation.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \{\eta_0 + \sigma_\eta z_\eta(t)\}v(t) - \{\gamma_0 + \sigma_\gamma z_\gamma(t)\}\int_0^t v(s)ds$$

The total sales value is then calculated by the following equation.

$$-\int_0^T \tilde{S}(t)dx = v\int_0^T \tilde{S}(t)dt$$

$$= XS(0) + \frac{1}{2}\mu v T^2 + v\sigma\int_0^T z(t)dt - \varepsilon v T - \eta_0 v^2 T - v^2\sigma_\eta\int_0^T z_\eta(t)dt$$
$$-\frac{1}{2}\gamma_0 v^2 T^2 - v^2\sigma_\gamma\int_0^T t z_\gamma(t)dt$$

Accordingly, the transaction cost is as shown by the following equation.

$$C = -\frac{1}{2}\mu v T^2 - v\sigma\int_0^T z(t)dt + \varepsilon v T + \eta_0 v^2 T$$
$$+ v^2\sigma_\eta\int_0^T z_\eta(t)dt + \frac{1}{2}\gamma_0 v^2 T^2 + v^2\sigma_\gamma\int_0^T t z_\gamma(t)dt$$

The mean and the variance of the transaction cost can then be determined through simple calculations as shown by the following equations.[36]

$$E[C] = -\frac{1}{2}\mu v T^2 + \varepsilon v T + \eta_0 v^2 T + \frac{1}{2}\gamma_0 v^2 T^2 = -\frac{1}{2}\mu X T + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$$

$$V[C] = \frac{1}{3}v^2\sigma^2 T^3 + \frac{1}{3}v^4\sigma_\eta^2 T^3 + \frac{2}{15}v^4\sigma_\gamma^2 T^5 = \frac{1}{15}X^2(5\sigma^2 T + 2\sigma_\gamma^2 X^2 T + \frac{5\sigma_\eta^2 X^2}{T})$$

The investor's liquidation cost, which should be minimized, is defined as shown in the following equation.

---

[36] The solution can easily be obtained as under the continuous-time model, as $E[\int_0^T t z(t)dt] = 0$, $V[\int_0^T t z(t)dt] = \frac{2}{15}T^5$ from Ito's lemma, and $\int_0^T t z(t)dt = \{\frac{1}{2}T^2 z(T) - \int_0^T t^2 dz(t)\}$.

$$L = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2 + rZ_\alpha \sqrt{\frac{1}{15} X^2 (5\sigma^2 T + 2\sigma_\gamma^2 X^2 T + \frac{5\sigma_\eta^2 X^2}{T})}$$

The following conditions are applied to the holding period, but as there is no closed-form solution to this, numerical calculations must be conducted to determine the optimal holding period.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta_0 X^2}{T^2} + \frac{\frac{rZ_\alpha X^2}{15}(5\sigma^2 + 2\sigma_\gamma^2 X^2 - \frac{5\sigma_\eta^2 X^2}{T^2})}{\sqrt{\frac{1}{15} X^2 (5\sigma^2 T + 2\sigma_\gamma^2 X^2 T + \frac{5\sigma_\eta^2 X^2}{T})}} = 0$$

Appendix G: Model Incorporating the Correlation between the Market Impact and Stock Price Fluctuations

This appendix presents a model incorporating the correlation between the fluctuations in the temporary market impact coefficient and the stock price fluctuations, as well as numerical examples.

As in the main body of this paper, the temporary market impact at time t is defined by the following equation.

$$\eta_t = \eta_0 + \sigma_\eta z_\eta(t)$$

Here, η_0 is the temporary market impact at time 0, σ_η is the volatility of the temporary market impact, and $z_\eta(t)$ is the standard Brownian motion. The sales price is expressed by the following equation.

$$\tilde{S}(t) = S(0) + \mu t + \sigma z(t) - \varepsilon - \{\eta_0 + \sigma_\eta z_\eta(t)\}v(t) - \gamma \int_0^t v(s)ds$$

Here, it is assumed that there is a correlation between the fluctuations in the temporary market impact and the stock price fluctuations, and that $z(t)$ and $z_\eta(t)$ have a correlation coefficient ρ as expressed by the following equation.[37]

$$E[dz(u)dz_\eta(v)] = \begin{cases} \rho du, & u = v \\ 0, & u \neq v \end{cases}$$

Thus under Brownian motions with a correlation, the following expression is established.

$$E[z(t)z_\eta(s)] = \int_0^s \int_0^t E[dz(u)dz_\eta(v)] = \int_0^s \rho du = \rho s \quad \text{Where, } s < t$$

The total sales value is then calculated by the following equation.

$$\begin{aligned} &-\int_0^T \tilde{S}(t)dx \\ &= XS(0) + \frac{1}{2}\mu v T^2 + v\sigma \int_0^T z(t)dt - \varepsilon v T - \eta_0 v^2 T - v^2 \sigma_\eta \int_0^T z_\eta(t)dt - \frac{1}{2}\gamma v^2 T^2 \end{aligned}$$

Accordingly, the transaction cost is as shown by the following equation.

$$C = -\frac{1}{2}\mu v T^2 - v\sigma \int_0^T z(t)dt + \varepsilon v T + \eta_0 v^2 T + v^2 \sigma_\eta \int_0^T z_\eta(t)dt + \frac{1}{2}\gamma v^2 T^2$$

The mean and the variance of the transaction cost can then be determined through

[37] See Kijima (1999), pp. 13-14.

simple calculations as shown by the following equations.[38]

$$E[C] = -\frac{1}{2}\mu vT^2 + \varepsilon vT + \eta_0 v^2 T + \frac{1}{2}\gamma v^2 T^2 = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2$$

$$V[C] = \frac{1}{3}v^2\sigma^2 T^3 + \frac{1}{3}v^4\sigma_\eta^2 T^3 - \frac{2}{3}v^3\sigma\sigma_\eta\rho T^3 = \frac{1}{3}X^2(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T} - 2\sigma\sigma_\eta\rho X)$$

The investor's liquidation cost, which should be minimized, is defined as shown in the following equation.

$$L = -\frac{1}{2}\mu XT + \varepsilon X + \frac{\eta_0 X^2}{T} + \frac{1}{2}\gamma X^2 + rZ_\alpha\sqrt{\frac{1}{3}X^2(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T} - 2\sigma\sigma_\eta\rho X)}$$

The following conditions are then applied to the holding period.

---

[38]First, show that

$$V[C] = \frac{1}{3}v^2\sigma^2 T^3 + \frac{1}{3}v^4\sigma_\eta^2 T^3 - \frac{2}{3}v^3\sigma\sigma_\eta\rho T^3 = \frac{1}{3}X^2(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T} - 2\sigma\sigma_\eta\rho X).$$

$$V[C] = V[-\frac{1}{2}\mu vT^2 - v\sigma\int_0^T z(t)dt + \varepsilon vT + \eta_0 v^2 T + v^2\sigma_\eta\int_0^T z_\eta(t)dt + \frac{1}{2}\gamma v^2 T^2]$$

$$= V[-v\sigma\int_0^T z(t)dt + v^2\sigma_\eta\int_0^T z_\eta(t)dt]$$

$$= v^2\sigma^2 V[\int_0^T z(t)dt] + v^4\sigma_\eta^2 V[\int_0^T z_\eta(t)dt] - 2v^3\sigma\sigma_\eta Cov[\int_0^T z(t)dt, \int_0^T z_\eta(t)dt]$$

$$= v^2\sigma^2\frac{T^3}{3} + v^4\sigma_\eta^2\frac{T^3}{3} - 2v^3\sigma\sigma_\eta E[\{\int_0^T z(t)dt - E[\int_0^T z(t)dt]\}\{\int_0^T z_\eta(t)dt - E[\int_0^T z_\eta(t)dt]\}]$$

Because $E[\int_0^T z(t)dt] = 0$, $E[\int_0^T z_\eta(t)dt] = 0$, and

$$= v^2\sigma^2\frac{T^3}{3} + v^4\sigma_\eta^2\frac{T^3}{3} - 2v^3\sigma\sigma_\eta E[\int_0^T z(t)dt\int_0^T z_\eta(t)dt]$$

it is only necessary to prove that $E[\int_0^T z(t)dt\int_0^T z_\eta(t)dt] = \frac{\rho T^3}{3}$.

Because $E[\int_0^T z(t)dt\int_0^T z_\eta(t)dt] = E[\int_0^T z(t)dt\int_0^T z_\eta(s)ds]$,

using Fubini's theorem

$$= E[\int_0^T\int_0^T z(t)z_\eta(s)dtds] = \int_0^T\int_0^T E[z(t)z_\eta(s)]dtds$$

Here, when $s < t$,

$$E[z(t)z_\eta(s)] = \int_0^s\int_0^t E[dz(u)dz_\eta(v)] = \int_0^s \rho du = \rho s$$

so,

$$\int_0^T\int_0^T E[z(t)z_\eta(s)]dtds = \int_0^T\int_0^s \rho tdtds + \int_0^T\int_s^T \rho sdtds = \frac{\rho T^3}{3}$$

Q.E.D.

$$\frac{\partial L}{\partial T} = -\frac{1}{2}\mu X - \frac{\eta_0 X^2}{T^2} + \frac{rZ_\alpha X^2}{3}(\sigma^2 - \frac{\sigma_\eta^2 X^2}{T^2})\left\{\frac{X^2}{3}(\sigma^2 T + \frac{\sigma_\eta^2 X^2}{T} - 2\sigma\sigma_\eta \rho X)\right\}^{-\frac{1}{2}} = 0$$

Next, numerical examples are calculated using the parameters from Chapter 5. The calculation results for Company A, for which the market impact is relatively small, and Company B, for which the market impact is relatively large, are presented in Table 10. Here, to clarify the effect from the correlation, the fluctuation in the market impact is assumed to be large, whereby $\sigma_\eta = 2\eta_0$.

Table 10: L-VaR Incorporating the Correlation Between the Market Impact and Stock Prices

▽Company A Stocks Investor's Position: ¥1,655 million

Correlation Coefficient	Optimal Holding Period (days)	L-VaR (¥1,000)
-1	0.413	32,059
-0.75	0.413	31,974
-0.5	0.412	31,888
-0.25	0.412	31,802
0	0.411	31,716
0.25	0.411	31,629
0.5	0.410	31,542
0.75	0.409	31,455
1	0.409	31,367

▽Company B Stocks Investor's Position: ¥1,655 million

Correlation Coefficient	Optimal Holding Period (days)	L-VaR (¥1,000)
-1	20.80	329,090
-0.75	20.63	323,794
-0.5	20.46	318,371
-0.25	20.28	312,810
0	20.10	307,099
0.25	19.90	301,224
0.5	19.70	295,172
0.75	19.49	288,922
1	19.27	282,455

This table shows that L-VaR increases when the correlation is negative and that L-VaR decreases when the correlation is positive. This is because when the correlation is negative, the fluctuations in the market impact result in a wider price variation.

When the correlation is negative and the price declines (rises), the market impact increases (decreases). Especially, when the correlation is negative and sales orders are executed, the market impact exerts downward pressure on the price and the price fluctuation becomes greater than that when there is no correlation. Conversely, when the correlation is positive and the price decreases (increases), because the market impact becomes small (large), the price fluctuation becomes smaller than that when there is no correlation.

Appendix H: Derivation of the Variance of the Transaction Cost under the Portfolio Model

The derivation begins with the following equation.

$$V[C] = V[\sum_{i,j=1}^{m} v_j \alpha_{ji} \int_0^{T_j} Z_i(t)dt]$$

$$= \frac{1}{3}\sum_{j=1}^{m} \sigma_{jj} X_j^2 T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}} \sigma_{jk} X_j X_k \frac{T_j^2}{T_k} + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}} \sigma_{jk} X_j X_k \frac{T_k^2}{T_j}$$

$$V[C] = V[\sum_{i,j=1}^{m} v_j \alpha_{ji} \int_0^{T_j} Z_i(t)dt]$$

The $Z_i(t)$ are all independent, so the following equation can then be used.

$$= \sum_{i=1}^{m} V[\sum_{j=1}^{m} v_j \alpha_{ji} \int_0^{T_j} Z_i(t)dt]$$

$$= \sum_{i=1}^{m} \{\sum_{j=1}^{m} V[v_j \alpha_{ji} \int_0^{T_j} Z_i(t)dt] + 2\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} Cov[v_j \alpha_{ji} \int_0^{T_j} Z_i(t)dt, \quad v_k \alpha_{ki} \int_0^{T_k} Z_i(t)dt]\}$$

$V[\int_0^{T_j} Z_i(t)dt] = \frac{T_j^3}{3}$ and $Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_0^{T_k} Z_i(t)dt] = I_{\{T_j \le T_k\}} \frac{T_j^3}{3} + I_{\{T_j > T_k\}} \frac{T_k^3}{3}$ [39],

$$= \sum_{i=1}^{m} \{\sum_{j=1}^{m} \alpha_{ji}^2 v_j^2 \frac{T_j^3}{3} + 2\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}} v_j v_k \alpha_{ji} \alpha_{ki} \frac{T_j^3}{3} + 2\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}} v_j v_k \alpha_{ji} \alpha_{ki} \frac{T_k^3}{3}\}$$

Using the fact that $v_j T_j = X_j$ results in the following equation.

---

[39] Because it has already been proved under the continuous-time model for a single asset that $V[\int_0^{T_j} Z_i(t)dt] = \frac{T_j^3}{3}$, the following presents $\frac{T_j^3}{3}$ for $Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_0^{T_k} Z_i(t)dt]$ when $T_j \le T_k$.

From the Markov property of $Z_i(t)$, $\int_0^{T_j} Z_i(t)dt$ and $\int_{T_j}^{T_k} Z_i(t)dt$ are independent. Assuming that $T_j \le T_k$ results in the following equation,

$$Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_0^{T_k} Z_i(t)dt] = Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_0^{T_j} Z_i(t)dt + \int_{T_j}^{T_k} Z_i(t)dt]$$

$$= Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_0^{T_j} Z_i(t)dt] + Cov[\int_0^{T_j} Z_i(t)dt, \quad \int_{T_j}^{T_k} Z_i(t)dt] = \frac{T_j^3}{3}$$

and a similar approach may be used for the case in which $T_j > T_k$.

$$
\begin{aligned}
&= \frac{1}{3}\sum_{j=1}^{m}\Big(\sum_{i=1}^{m}\alpha_{ji}^{2}\Big)X_j^{2}T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}}\Big(\sum_{i=1}^{m}\alpha_{ji}\alpha_{ki}\Big)X_j X_k \frac{T_j^2}{T_k} \\
&\qquad + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}}\Big(\sum_{i=1}^{m}\alpha_{ji}\alpha_{ki}\Big)X_j X_k \frac{T_k^2}{T_j}
\end{aligned}
$$

$$
= \frac{1}{3}\sum_{j=1}^{m}\sigma_{jj}X_j^{2}T_j + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j \le T_k\}}\sigma_{jk}X_j X_k \frac{T_j^2}{T_k} + \frac{2}{3}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} I_{\{T_j > T_k\}}\sigma_{jk}X_j X_k \frac{T_k^2}{T_j}
$$

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