IMES DISCUSSION PAPER SERIES

A Time-Invariant Duration Policy under the Zero Lower Bound

Kozo Ueda

Discussion Paper No. 2010-E-12

IMES

INSTITUTE FOR MONETARY AND ECONOMIC STUDIES

BANK OF JAPAN

2-1-1 NIHONBASHI-HONGOKUCHO
CHUO-KU, TOKYO 103-8660
JAPAN

IMES Discussion Paper Series 2010-E-12
July 2010

A Time-Invariant Duration Policy under the Zero Lower Bound

Kozo Ueda*

Abstract

Optimal commitment policy under the zero lower bound entails a high degree of complexity and time-inconsistency in a stochastic economy. This paper proposes a time-invariant duration policy that mitigates those problems and facilitates policy implementation and communication while retaining effectiveness in inflation stabilization. Under the time-invariant duration policy, a central bank commits itself to maintaining low interest rates for some duration even after adverse shocks disappear, but unlike the optimal commitment policy, the duration is independent of the *ex post* spell of the adverse shocks. Consequently, the time-inconsistency problem does not increase even if the *ex post* spell of the adverse shocks lengthens, and policy rates are expressed in an extremely simple, explicit form. Simulation results suggest that the time-invariant duration policy performs virtually as effectively as the optimal commitment policy in stabilizing inflation, and far better than a discretionary policy and simple interest rate rules with or without inertia.

Keywords: Zero lower bound on nominal interest rates; optimal monetary policy; liquidity trap; time-inconsistency

JEL classification: E31, E52

***Director and Senior Economist, Institute for Monetary and Economic Studies, Bank of Japan (E-mail: kouzou.ueda@ boj.or.jp)

The author would like to thank Klaus Adam, Kosuke Aoki, Teruyoshi Kobayashi, Tomoyuki Nakajima, Matthias Paustian, Tsutomu Watanabe, seminar participants at the Bank of England, Winter Institute of Macroeconomics, and the staff of the Institute for Monetary and Economic Studies (IMES), the Bank of Japan, for their useful comments. Views expressed in this paper are those of the author and do not necessarily reflect the official views of the Bank of Japan.

1 Introduction

In this paper, I propose a time-invariant duration policy under the zero lower bound (ZLB) on nominal interest rates as a practical policy using a theoretical framework. A number of theoretical papers have studied optimal commitment policy to enhance the effectiveness of monetary policy under the ZLB.[1] Under optimal commitment policy, a central bank commits itself to maintaining low interest rates for some duration even after adverse shocks disappear. Such policy has a policy duration effect on inflation expectations, mitigating deflation while the adverse shocks hit the economy.[2] This paper follows the lines of previous research, but I propose a policy that is easier to implement and communicate while retaining effectiveness in inflation stabilization.

To this end, I begin by pointing out that the optimal commitment policy derived from a micro-founded model is highly complex and time-inconsistent. This is because the optimal duration to maintain accommodative policy is time-variant, depending on the *ex post* spell of an adverse shock. To illustrate this, consider the price-level targeting policy that is known to be a close approximation to the optimal commitment policy.[3] Under the price-level targeting policy, as the *ex post* spell of the adverse shock lengthens, the price deviation from the target level expands. It yields a need for higher inflation after the adverse shock disappears. A central bank should thus commit itself to a more accommodative policy by maintaining low interest rates for a longer duration. In a stochastic economy, such a commitment policy with variant duration worsens the problems of time-inconsistency and complexity. The central bank may announce, "If the economy normalizes half a year from now, we will maintain the accommodative policy for the duration of one year. If the economy normalizes a year from now, we will maintain the accommodative policy for the duration of two years, and so on." Clearly, such an announcement entails complexity, and the central bank has an incentive to renegotiate its commitment after half a year if the adverse shock continues to affect the economy, because it entails high inflation after the adverse shock disappears.

To mitigate the aforementioned problems, I propose a time-invariant duration policy. Under this policy, the duration of maintaining low interest rates is time-invariant, independent of the *ex post* spell of the adverse shock. Policy is accordingly simplified; even in a stochastic economy, the time-invariant duration policy is expressed in an extremely simple, explicit form. A central

[1] See Reifschneider and Williams (2000), Eggertsson and Woodford (2003a, b), Jung, Teranishi, and Watanabe (2005), Kato and Nishiyama (2005), Sugo and Teranishi (2005), Adam and Billi (2006, 2007), Nakov (2008), Fujiwara, Sudo, and Teranishi (2010), and Levin, Lopez-Salido, Nelson, and Yun (2009). Other policy measures are also proposed such as quantitative easing, credit easing, and capital injection policy, but the interest rate continues to be the main policy instrument for central banks.

[2] See Fujiki and Shiratsuka (2002).

[3] See Eggertsson and Woodford (2003a, b).

bank's announcement is expressed as follows: "After the economy normalizes, we will maintain the accommodative policy for the duration of one year." Practically, the time-invariant duration policy resembles actual policy conducted by central banks in Japan, the United States, and Canada.[4]

Mathematically, the time-invariant duration policy is described as the optimal policy under which the Lagrange multiplier for the ZLB constraint is restricted to be non-zero and constant. Because the Lagrange multiplier is time-invariant, the duration of maintaining low interest rates after the adverse shock disappears is also time-invariant, and an incentive to deviate from the policy does not increase with the *ex post* spell of the adverse shock. The Lagrange multiplier under the commitment policy, on the other hand, begins with zero and increases over time. This suggests that the duration of maintaining low interest rates after the adverse shock disappears is time-variant, and an incentive to deviate from the policy increases with the *ex post* spell of the adverse shock. Under discretionary policy, the predetermined Lagrange multiplier is zero. In other words, the discretionary policy is not constrained by the history.

Using a simple stochastic New Keynesian model, I simulate the model and compare the degree of inflation stability among various policies. Simulation results reveal that *ex ante*, the time-invariant duration policy achieves virtually the same degree of inflation stability as the commitment policy. On the contrary, if the *ex post* spell of the adverse shock is sufficiently long, the time-invariant policy achieves greater inflation stability than the commitment policy. This is because unlike the commitment policy, even if the *ex post* spell of the adverse shock lengthens, the time-invariant duration policy does not require increasingly high inflation after the adverse shock disappears. Moreover, simulation results suggest that the time-invariant duration policy performs virtually as effectively as the optimal commitment policy in stabilizing inflation, and far better than other policies. Three other policies, that is, discretionary policy, a non-inertial interest rate rule, and an inertial interest rate rule, are nearly the same, in terms of inflation stability.

As for related literature, the idea of the time-invariant duration policy is analogous to that of the fully timeless commitment policy as analyzed by Jensen and McCallum (2002), McCallum (2005), and Damjonovic, Damjonovic, and Nolan (2008). Under fully timeless commitment policy, a central bank minimizes the unconditional expectation of welfare loss across stochastic steady states, rendering endogenous variables such as the Lagrange multiplier constant. Unlike this policy, however, the time-invariant duration policy guarantees the time-consistency only while the adverse shock hits the economy and thus retains the time-inconsistency when the adverse shock disappears.

[4] For details of the policy, see Section 3.6.

The concept of the time-invariant duration policy is first discussed in Fujiwara, Sudo, and Teranishi (2010). The policy in their paper is described as a special form of the time-invariant duration policy, committing to a low policy rate only during the period when adverse shocks disappear. In this paper, I touch upon the properties of their policy, what I call the one-period duration policy, by comparing them with those of the time-invariant duration policy.

This paper is structured as follows. In Section 2, I present a model. In Section 3, I consider various policies, and derive the discretionary and time-invariant duration policy in a simple, explicit form, in addition to the commitment policy and simple interest rate rules. In Section 4, I simulate my model to compare a welfare loss among various policies, and check the sensitivity of the results. Section 5 concludes.

2 Model Setup

2.1 Basic Equations

The Phillips curve and IS curve are described as

$$\pi_t = \gamma x_t + \beta \mathrm{E}_t \pi_{t+1}, \tag{2.1}$$

$$i_t = \mathrm{E}_t \left\{ \sigma \left(x_{t+1} - x_t \right) + \pi_{t+1} + r_t^n \right\}, \tag{2.2}$$

where π_t, x_t, and i_t are an inflation rate, output gap, and a nominal interest rate at period t, respectively. r_t^n is a real interest rate shock. β is a discount factor, γ represents the elasticity of inflation to output and σ represents the inverse of the elasticity of output to real interest rates. Eliminating x_t, I can simplify these two equations as

$$i_t = \mathrm{E}_t \left\{ -\frac{1}{\gamma/\sigma} \left(\beta \pi_{t+2} - (1 + \beta + \gamma/\sigma) \pi_{t+1} + \pi_t \right) \right\} + r_t^n. \tag{2.3}$$

2.2 Markov Process

The economy is stochastic with respect to the real interest rate shock r_t^n. The shock obeys an absorbing Markov process as in Eggertsson and Woodford (2003a, b).[5] Defining the current state at t as $s_t = \{z, n\}$, the real interest rate shock r_t^n takes the value of $\underline{r} < 0$ at state z and

[5]See Adam and Billi (2006, 2007) and Nakov (2008) for more general shock processes. Basic properties of commitment policy under ZLB, such as history dependence, hold true in their general model. Their model, however, enables us to consider preemptive policy, which is beyond the scope of this paper.

$\bar{r} > 0$ at state n. The conditional probabilities are represented as

$$\Pr(s_t = z | s_{t-1} = z) = p, \tag{2.4}$$

$$\Pr(s_t = z | s_{t-1} = n) = 0. \tag{2.5}$$

The adverse shock does not reoccur, once it disappears.

A stochastic model is the key to the time-invariant duration policy in relation to the commitment policy. In a deterministic model,[6] the spell of the adverse shock is known, and the duration of maintaining low interest rates is uniquely specified. That makes the time-invariant duration policy the same as the commitment policy.

2.3 Welfare

The loss of utility U_t at period t is described as follows:

$$U_t = \lambda x_t^2 + \pi_t^2, \tag{2.6}$$

where λ is the weight of output variability to inflation variability. A welfare loss L_t becomes

$$L_t = \sum \beta^{t+i} \left(\lambda x_{t+i}^2 + \pi_{t+i}^2 \right). \tag{2.7}$$

2.4 State

The whole state at t that includes the past state is defined as $S_t = \{s_0, \cdots, s_{t-1}, s_t\}$. Because of equation (2.5), the whole state at t is expressed as the set of

$$S_t^{(k)} \equiv \{\overbrace{\underbrace{z, z, \cdots, z}_{t+1-k}, \underbrace{n, n, \cdots, n}_{k}}^{t+1}\}, \tag{2.8}$$

where $k = 0, 1, 2, \cdots$.

k represents the length of periods after the real interest rate shock turns positive. The spell of the negative shock equals $t - k + 1$. For example, $k = 0$ means that the shock remains negative. Corresponding to $S_t^{(k)}$, I define the values of other endogenous variables X_t with the superscript of (k), $X_t^{(k)}$, such as $i_t^{(k)}$, $\pi_t^{(k)}$, $x_t^{(k)}$.

With the notation, I describe my model as

$$i_t^{(k)} = \mathrm{E}_t \left\{ -\frac{1}{\gamma/\sigma} \left(\beta \pi_{t+2}^{(j2)} - (1 + \beta + \gamma/\sigma) \pi_{t+1}^{(j1)} + \pi_t^{(k)} \right) + r_t^n \right\}, \tag{2.9}$$

[6]See, for example, Jung, Teranishi, and Watanabe (2005) and Levin, Lopez-Salido, Nelson, and Yun (2009).

where $j1$ and $j2$ are determined by the Markov process.

If $k=0$, the above equation becomes

$$
\begin{aligned}
0 \quad = \quad & -\frac{1}{\gamma/\sigma}\beta\left\{p^2\pi^{(0)}_{t+2+m}+p(1-p)\pi^{(1)}_{t+2+m}+(1-p)\pi^{(2)}_{t+2+m}\right\} \\
& +\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+1+m}+(1-p)\pi^{(1)}_{t+1+m}\} \\
& -\frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m}+\underline{r}. \qquad (2.10)
\end{aligned}
$$

The left-hand side of the equation is zero because $i^{(0)}_t=0$. The first term in the first row of the right-hand side represents the value when the state z continues until $t+2$, whose probability is given by p^2. The second term in the first row represents the value when the state stays z until $t+1$, and changes to n at $t+2$, whose probability is given by $p(1-p)$. The third term in the first row represents the value when the state z continues until t, and at $t+1$ the state changes to n, whose probability is given by $1-p$.

For $k\geq 1$, equation (2.9) is given by

$$
i^{(k)}_t=\left\{-\frac{1}{\gamma/\sigma}\left(\beta\pi^{(k+2)}_{t+2}-(1+\beta+\gamma/\sigma)\pi^{(k+1)}_{t+1}+\pi^{(k)}_t\right)+\overline{r}\right\}\ . \qquad (2.11)
$$

2.5 Assumptions

For simplicity, I assume the following.

Assumption 1: A central bank aims to minimize the following welfare loss:

$$
L_t=\sum\beta^{t+i}\pi^2_{t+i}.
$$

The loss of utility is given by $\lambda=0$ in equation (2.6). Utility does not depend on output variations, but only on inflation variations. That assumption is introduced to make my analysis easier. Owing to Assumption 1, I do not need two separate equations, the IS and the Phillips equations. Possible defences of the assumption are that calibrated λ is as small as 0.05 (Woodford [2003]) and that the assumption does not necessarily lose generality. With respect to the latter point, in general, a supply shock in the Phillips curve yields trade-off between output variations and inflation variations, so the parameter λ plays an important role. However, in this paper I do not consider the effects of the supply shock. I consider only the effects of the real interest rate shock in the IS equation, and in response to this type of the shock, both inflation and output typically move in the same direction, so the trade-off between output variations and inflation variations is small. Instead, in the wake of a negative real interest rate shock, the lack of intertemporal optimization due to the ZLB becomes more important.

Assumption 2: Parameters satisfy

$$p^2\beta - p(1+\beta+\gamma/\sigma) + 1 > 0. \tag{2.12}$$

This assumption yields negative output and inflation when the current state is z under discretionary policy.

Assumption 3: All of the policies are credible.

Although I consider policies that differ in simplicity and time-inconsistency, I assume that all the policies are credible. In this sense, my focus is not on how credible the policy should be or how the credibility can be improved.[7] Instead, I focus on how simple and less time-inconsistent the policy can be without sacrificing too much stability.

3 Analytical Solutions under Various Policies

In this section, I consider various kinds of monetary policy under the ZLB and their effects on the economy. The policies I consider are commitment policy, discretionary policy, and simple interest rate rules as well as time-invariant duration policy. Simple interest rate rules include a non-inertial interest rate rule and an inertial interest rate rule.

3.1 Commitment Policy

First, I derive optimal commitment policy. The policy has a variant duration of maintaining low interest rates dependent on the spell of the adverse shock.

A central bank minimizes the welfare loss subject to the IS and Phillips curves and the ZLB condition. A welfare loss is described as

$$\begin{aligned} L_t^{(0)} &= (\pi_t^{(0)})^2 + \beta\{pL_{t+1}^{(0)} + (1-p)L_{t+1}^{(1)}\} \\ &= \textstyle\sum_{k=0}^{\infty} p^k(1-p)\left[\sum_{j=0}^{k}\beta^j(\pi_{t+j}^{(0)})^2 + \sum_{j=k+1}^{\infty}\beta^j(\pi_{t+j}^{(j-k)})^2\right] \\ &= \textstyle\sum_{m=0}^{\infty} p^m\beta^m(\pi_{t+m}^{(0)})^2 + \sum_{m=3}^{\infty}(1-p)\beta^m\{p^{m-3}(\pi_{t+m}^{(3)})^2 \\ &\quad + p^{m-2}(\pi_{t+m}^{(2)})^2 + p^{m-1}(\pi_{t+m}^{(1)})^2\} + (1-p)\beta^2\{(\pi_{t+2}^{(2)})^2 + p(\pi_{t+2}^{(1)})^2\} \\ &\quad + (1-p)\beta(\pi_{t+1}^{(1)})^2 + \Delta(\pi_{t+l}^{(k)}), \end{aligned} \tag{3.1}$$

[7] See, for example, Jeanne and Svensson (2007) and Schaumburg and Tambalotti (2007). The former considers a way to make the commitment policy credible taking account of the central bank's balance sheet. The latter analyzes the welfare effect of monetary policy under different degrees of credibility, calling it a quasi-commitment.

where $\Delta(\pi_{t+l}^{(k)})$ is a function for $k \geq 4$ and $l \geq 3$.

Using the constraint given by equation (2.10), I can define the Lagrangian as

$$\begin{aligned}
\pounds_t &= \textstyle\sum_{m=0}^{\infty} p^m \beta^m (\pi_{t+m}^{(0)})^2 \\
&+ \textstyle\sum_{m=3}^{\infty} (1-p)\beta^m \{p^{m-3}(\pi_{t+m}^{(3)})^2 + p^{m-2}(\pi_{t+m}^{(2)})^2 + p^{m-1}(\pi_{t+m}^{(1)})^2\} \\
&+ (1-p)\beta^2\{(\pi_{t+2}^{(2)})^2 + p(\pi_{t+2}^{(1)})^2\} + (1-p)\beta(\pi_{t+1}^{(1)})^2 + \Delta(\pi_{t+l}^{(k)}) \\
&- 2\textstyle\sum_{m=0}^{\infty} p^m \beta^m \lambda_{t+m}\left[-\frac{1}{\gamma/\sigma}\beta\{p^2\pi_{t+2+m}^{(0)}\right. \\
&+ p(1-p)\pi_{t+2+m}^{(1)} + (1-p)\pi_{t+2+m}^{(2)}\} \\
&\left. + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi_{t+1+m}^{(0)} + (1-p)\pi_{t+1+m}^{(1)}\} - \frac{1}{\gamma/\sigma}\pi_{t+m}^{(0)} + \underline{r}\right].
\end{aligned} \tag{3.2}$$

λ_{t+m} represents the Lagrange multiplier, where $\lambda_{t-1} = 0$.

As Appendix A.1 demonstrates, provided $i_t^{(1)}$ is non-negative, I obtain the following first-order conditions:

$$0 = \pi_{t+m}^{(0)} + \lambda_{t+m-2}\frac{1}{\gamma/\sigma}\beta^{-1} - \lambda_{t+m-1}\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\beta^{-1} + \lambda_{t+m}\frac{1}{\gamma/\sigma}. \tag{3.3}$$

$$0 = \pi_{t+m}^{(2)} + \frac{1}{\gamma/\sigma}\beta^{-1}\lambda_{t+m-2}. \tag{3.4}$$

$$\begin{aligned}
0 &= \pi_{t+m}^{(1)} + \frac{1}{\gamma/\sigma}\beta^{-1}\lambda_{t+m-2} \\
&- \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\beta^{-1}\lambda_{t+m-1}.
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
0 &= -\frac{1}{\gamma/\sigma}\beta\{p^2\pi_{t+2+m}^{(0)} + p(1-p)\pi_{t+2+m}^{(1)} + (1-p)\pi_{t+2+m}^{(2)}\} \\
&+ \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi_{t+1+m}^{(0)} + (1-p)\pi_{t+1+m}^{(1)}\} \\
&- \frac{1}{\gamma/\sigma}\pi_{t+m}^{(0)} + \underline{r}.
\end{aligned} \tag{3.6}$$

For $n \geq 3$, I obtain

$$\pi_{t+m}^{(n)} = 0. \tag{3.7}$$

The policy for $n \geq 3$ is the same as the discretionary policy below. In other words, optimal commitment policy does not need to optimize over the infinite time horizon after the adverse shock disappears. If the adverse shock is not too big, a two-period commitment is sufficient. See Appendix A.1 for the case in which $i_t^{(1)}$ is negative and longer commitment is required.

From the four equations, equations (3.3), (3.4), (3.5), and (3.6), I can calculate the optimal commitment solution with respect to $\{\pi^{(0)}_{t+m}, \pi^{(1)}_{t+m}, \pi^{(2)}_{t+m}, \lambda_{t+m}\}$.

3.2 Discretionary Policy

I turn now to discretionary policy. This is the policy that optimizes equation (3.2) by imposing an additional condition on the predetermined Lagrange multipliers

$$\lambda_{t+m-1} = \lambda_{t+m-2} = 0, \tag{3.8}$$

at $t+m$. The discretionary policy is not constrained by the past Lagrange multipliers. The previous first-order conditions (3.3), (3.4), (3.5), and (3.6) are valid.

From these conditions, I can obtain the following properties of the discretionary policy. Note that, for the same $k = 0, 1, ...$, all endogenous variables become constant: $X^{(k)}_t = X^{(k)}_{t+1}$. So I can omit the subscript of t and express variables only with k. For $k \geq 1$, that is, when the current state is n, discretionary policy yields

$$\pi^{(k)} = x^{(k)} = 0. \tag{3.9}$$

Equation (2.9) yields

$$i^{(k)} = \overline{r}. \tag{3.10}$$

The loss of utility at this case is zero.

For $k = 0$, that is, when the current state is z, the nominal interest rate is

$$i^{(0)} = 0. \tag{3.11}$$

From equation (3.6), the inflation rate becomes

$$\begin{aligned} \pi^{(0)} &= \{p^2\beta - p(1+\beta+\gamma/\sigma) + 1\}^{-1}\gamma/\sigma\underline{r} && (3.12) \\ &< 0. && (3.13) \end{aligned}$$

The inequality arises from Assumption 2.

The welfare loss when the current state is z is described as

$$\begin{aligned} L^D &= U^0 + \beta p L^D = \frac{1}{1-p\beta}(\pi^{(0)})^2 \\ &= \frac{1}{1-p\beta}\left\{\frac{\gamma/\sigma\underline{r}}{p^2\beta - p(1+\beta+\gamma/\sigma)+1}\right\}^2. \end{aligned} \tag{3.14}$$

Such a policy is equivalent to the policy that minimizes only the present loss function

$$U_t = \pi_t^2, \tag{3.15}$$

subject to equation (2.10) and $i_t \geq 0$, with π_{t+2} and π_{t+1} given.

3.3 Time-Invariant Duration Policy

I next consider a time-invariant duration policy. Under the policy, the duration of low interest rates after the adverse shock disappears is independent of the spell of the adverse shock.

Mathematically, the time-invariant duration policy is expressed as the one that optimizes equation (3.2) by imposing an additional condition:[8]

$$\pi_{t+m+1}^{(k)} = \pi_{t+m}^{(k)}. \tag{3.16}$$

This condition suggests that inflation rates are independent of time t as long as k is the same. In particular, inflation rates are constant during $k = 0$, that is, while the adverse shock hits the economy. Because the *ex post* spell of the adverse shock is described as $t + 1 - k$, for a given k, inflation rates are independent of the *ex post* spell of the adverse shock. As is shown just below, it implies that the optimal duration to maintain low interest rates and the Lagrange multiplier are independent of the *ex post* spell of the adverse shock.

As Appendix A.2 demonstrates, provided $i_{t+m+1}^{(1)}$ is non-negative, I can obtain the following first-order conditions

$$0 = \pi_{t+m}^{(0)} - C\lambda_{t+m}, \tag{3.17}$$

$$0 = (1-p)\beta^2 \pi_{t+m+2}^{(2)} - A\lambda_{t+m}, \tag{3.18}$$

$$0 = (1-p)\beta \pi_{t+m+1}^{(1)} - B\lambda_{t+m}, \tag{3.19}$$

$$0 = A\pi_{t+m+2}^{(2)} + B\pi_{t+m+1}^{(1)} + C\pi_{t+m}^{(0)} + \underline{r}, \tag{3.20}$$

[8] In theory, the time-invariant duration policy is analogous to the fully timeless or unconditional commitment policy analyzed by Jensen and McCallum (2002), McCallum (2005), and Damjonovic, Damjonovic, and Nolan (2008). Fully timeless commitment policy minimizes the unconditional expectation of welfare loss across stochastic steady states. Like equation (3.16), the policy is characterized by the unconditional expectations of $\mathrm{E}(X_{t+m+1}) = \mathrm{E}(X_{t+m})$. Unconditional expectations are thus timeless, reducing the time-inconsistency problem. A difference from the unconditional commitment policy is, however, that the time-invariant duration policy is not entirely timeless. When the state shifts from z to n, a central bank has an incentive not to keep the time-invariant duration policy but to conduct the discretionary policy.

where

$$
\begin{aligned}
A &= -\frac{(1-p)\beta}{\gamma/\sigma} < 0 \\
B &= \frac{(1-p)\{1+\gamma/\sigma+\beta(1-p)\}}{\gamma/\sigma} > 0 \qquad (3.21) \\
C &= -\frac{p^2\beta - p(1+\beta+\gamma/\sigma)+1}{\gamma/\sigma} < 0.
\end{aligned}
$$

These equations imply that, for the same $k = 0, 1, ...$, all endogenous variables, $\pi_{t+m}^{(k)}$, $x_{t+m}^{(k)}$, λ_{t+m}, and $i_{t+m}^{(k)}$, become constant. Policy rates are independent of the spell of the adverse shock. The Lagrange multiplier is constant and positive as

$$
\lambda_{t+m} = \lambda = -\left(\frac{A^2}{(1-p)\beta^2} + \frac{B^2}{(1-p)\beta} + C^2\right)^{-1} \underline{r} > 0. \qquad (3.22)
$$

Nominal interest rates as well as inflation rates can be written in an extremely simple explicit form. Properties of the policy can be simply examined analytically. Provided $i^{(1)} \geq 0$, which is defined below, the invariant duration policy yields

$$
\pi^{(3)} = \pi^{(4)} = \cdots = 0, \qquad (3.23)
$$

$$
\pi^{(2)} = \frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)\pi^{(0)} < 0, \qquad (3.24)
$$

$$
\pi^{(1)} = \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)\pi^{(0)} > 0, \qquad (3.25)
$$

$$
\pi^{(0)} = -\left(\frac{A^2}{(1-p)\beta^2} + \frac{B^2}{(1-p)\beta} + C^2\right)^{-1} C\underline{r} < 0, \qquad (3.26)
$$

The optimal set of nominal interest rates is given by

$$
i^{(3)} = i^{(4)} = \cdots = \overline{r}, \qquad (3.27)
$$

$$
i^{(2)} = \overline{r} - \frac{1}{\gamma/\sigma}\pi^{(2)} > \overline{r}, \qquad (3.28)
$$

$$
\begin{aligned}
i^{(1)} &= \overline{r} - \frac{1}{\gamma/\sigma}\{-(1+\beta+\gamma/\sigma)\pi^{(2)} + \pi^{(1)}\} < \overline{r} \\
i^{(0)} &= 0. \qquad (3.29)
\end{aligned}
$$

The welfare loss when the current state is z is

$$L^{(0)} = \frac{1}{1-p\beta}\left[(\pi^{(0)})^2 + \beta(1-p)\{(\pi^{(1)})^2 + \beta(\pi^{(2)})^2\}\right]. \tag{3.30}$$

3.4 One-Period Duration Policy

I showed above that unless the adverse shock is too big or persistent, it is sufficient to make a commitment of up to two periods under the commitment and the time-invariant duration policy after the adverse shock disappears.

In this subsection, as a special case of the time-invariant duration policy, I consider a one-period duration policy.[9] This policy commits up to only one period after the state becomes n. Afterward, a central bank conducts the discretionary policy. Because it is closer to the discretionary policy, it is less time-inconsistent than the commitment and the time-invariant duration policy. A drawback is, however, an increase in welfare losses when the adverse shock is big or persistent. In such a case, a longer commitment is required to mitigate the large-scale deflation during the adverse shock. The one-period duration policy may not be sufficiently effective in stabilizing inflation.

As Appendix A.3 shows, the one-period duration policy is expressed as follows:

$$\pi^{(2)} = \pi^{(3)} = \cdots = 0, \tag{3.31}$$

$$\pi^{(1)} = -\left\{1 + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)^2\right\}^{-1} \frac{B/C^2\underline{r}}{\beta(1-p)} > 0, \tag{3.32}$$

$$\pi^{(0)} = \left\{1 + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)^2\right\}^{-1} \left(-\frac{\underline{r}}{C}\right) < 0, \tag{3.33}$$

$$i^{(1)} = -\frac{1}{\gamma/\sigma}\pi^{(1)} + \overline{r} < \overline{r}, \tag{3.34}$$

$$i^{(2)} = i^{(3)} = \cdots = \overline{r}. \tag{3.35}$$

The welfare loss when the current state is z is

$$L^{(0)} = \frac{1}{1-p\beta}\left[(\pi^{(0)})^2 + \beta(1-p)(\pi^{(1)})^2\right]. \tag{3.36}$$

[9] One-period duration policy is first introduced in Fujiwara, Sudo, and Teranishi (2010). This paper supports their approach by showing that the welfare loss from using the policy compared with commitment policy is not large quantitatively.

3.5 Simple Interest Rate Rules

Finally, I consider simple interest rate rules. These are described as

$$i_t = \max(i_t^*, 0), \tag{3.37}$$

where imaginary interest rates i_t^* that can be negative are

$$i_t^* = \rho i_{t-1}^* + (1-\rho)(\overline{r} + \phi_\pi \pi_t). \tag{3.38}$$

This policy rule is motivated by Reifschneider and Williams (2000), who propose that a policymaker delays a rise or reduces an increase in nominal interest rates by the amount that the policymaker should have reduced if there were no ZLB. Its law of motion is analyzed in Appendix A.4.

Alternatively, I also consider the policy rule of the following form:

$$i_t^* = \rho i_{t-1} + (1-\rho)(\overline{r} + \phi_\pi \pi_t). \tag{3.39}$$

I call this inertial rule 2. This rule is similar to the above rule, but differs in policy responses at the exit period. This policy does not care about i^* at state z.

3.6 Discussion

Before simulating the model, I discuss the properties of the policies, in particular, focusing on the commitment policy and the time-invariant duration policy.

3.6.1 Commitment Policy

The optimal interest rates under the commitment policy depend on the spell of the adverse shock. At the beginning of optimal commitment policy t, the predetermined Lagrange multiplier, λ_{t-1}, is zero. It then changes over time. For different λ_{t+m}, subsequent policy rates differ, so the optimal interest rates under the commitment policy depend on the timing when the state changes from z to n. Thus, the optimal durations of low interest rates after the shock turns to n vary.

Such time-variant duration policy increases a time-inconsistency problem. Over time, the Lagrange multiplier λ_{t+m} deviates increasingly from zero. Since the optimal commitment policy solution is derived conditional on $\lambda_{t-1} = 0$, the deviation of λ_{t+m} loses the benefits from conducting the commitment policy. Over time, the incentive to deviate from the commitment policy increases.

Moreover, the commitment policy becomes complex. That makes policy implementation and communication with the public difficult. Suppose, for example, the current economy remains at state z in 2009:Q1. Under the commitment policy, a central bank needs to contemplate all the future timing of recovery and determine the optimal subsequent path of interest rates. The central bank may announce that if the economy normalizes in 2010:Q1, it will maintain an accommodative policy for one year until 2010:Q4, and raise interest rates above a natural rate at 2011:Q1. It also needs to announce all other possibilities: if the economy normalizes in 2011:Q1, it will maintain an accommodative policy for one and a half year until 2012:Q2, and raise interest rates above a natural rate at 2012:Q3; and if the economy normalizes in 2012:Q1, it will maintain an accommodative policy for two years until 2013:Q4, and raise interest rates above a natural rate at 2014:Q1. All the interest paths are not parallel. Considering the stochastic nature of the economy, therefore, policy implementation and communication become extremely difficult.

3.6.2 Time-Invariant Duration Policy

Under the time-invariant duration policy, interest rates and inflation rates are independent of time t as long as k is the same. It implies that the optimal duration to maintain low interest rates is independent of the *ex post* spell of the adverse shock.

Compared with the commitment policy, the time-invariant duration policy is less complex and time-inconsistent. First, regarding the time-inconsistency, the Lagrange multiplier under the time-invariant duration policy remains constant. An incentive to deviate from the promised policy does not increase, even if the economy is trapped in an adverse state for a long spell. It helps increase the credibility of the policy, relative to the commitment policy.

Second, the time-invariant duration policy is simple. It is expressed in an extremely simple, explicit form. Policy implementation and communication with the public are easier than under the commitment policy. For example, a central bank's announcement under the ZLB is described as follows: "After the economy normalizes, we will maintain an accommodative policy for the duration of one year. In the following period, we will raise policy rates above a natural rate."

Actual policy measures adopted by several central banks can be compared with the time-invariant duration policy. In Japan, the Bank of Japan announced in March 2001 under the ZLB that quantitative easing policy was to "continue to be in place until the consumer price index (excluding perishables, on a nationwide statistics) registers stably a zero percent or an increase year on year." In October 2005, when the recovery of the Japanese economy began to support the end the quantitative easing policy in the near future,[10] the Bank announced that "a change of the policy framework itself does not imply an abrupt change in terms of effects of policy. Conceptually, the course of monetary policy after the change of the framework will be a period of very low short-term interest rates followed by a gradual adjustment to a level consistent with economic activity and price developments." Policy measures adopted in the United States and Canada are also relevant.[11]

Those statements resemble the time-invariant duration policy. In them, time-variant duration does not appear. Duration to maintain accommodative policy depends on the state of the economy (economic activity and inflation in Japan and the outlook for inflation in Canada), similar to the time-invariant duration policy that depends on the state of the real interest rate shock. The duration is, however, not specified so as to depend on the spell of the adverse shock. These policies are not discretionary policy, either. The central banks commit to maintaining an accommodative policy for some duration even after the adverse shock disappears.

3.6.3 Other Properties

Although slightly less important, there are other consequential properties. The above analysis suggests that, provided $i^{(1)} \geq 0$, the time-invariant duration policy is sufficient to make a commitment up to two periods after the adverse shock disappears. That property is not unique to the the time-invariant duration policy; it is observed also in the commitment policy. Such a property arises mainly for two reasons. The first is Assumption 1, which neglects welfare losses from output deviations. The second is a condition that guarantees $i^{(1)} \geq 0$. If an adverse shock is large or persistent, the optimal level of $i^{(1)}$ becomes negative, which requires a longer accommodative policy duration than two periods. See Appendix A.2 for the solution in such a case.

Because the time-invariant duration policy is simple, it clarifies the basic pattern of optimal policy rate adjustments. When the state shifts from z to n, the optimal interest rate $i^{(1)}$ should be below $\bar{r}$. That helps increase $\pi^{(0)}$ under z, close to the optimal zero inflation, but a sacrifice is that $\pi^{(1)}$ increases above zero. During the next period, the optimal interest rate $i^{(2)}$ should overshoot so as to be above $\bar{r}$. That helps constrain the overshooting of inflation $\pi^{(1)}$ during the previous period, but a sacrifice is that $\pi^{(2)}$ falls below zero. Such a pattern of interest rates

[10] It was in March 2006 that ended the quantitative easing policy.

[11] In March 2009, the U.S. Federal Reserve announced "The Committee will maintain the target range for the federal funds rate at 0 to 1/4 percent and anticipates that economic conditions are likely to warrant exceptionally low levels of the federal funds rate for an extended period." In April 2009, Bank of Canada introduced a conditional commitment, stating "Conditional on the outlook for inflation, the target overnight rate can be expected to remain at its current level until the end of the second quarter of 2010 in order to achieve the inflation target. The Bank will continue to provide such guidance in its scheduled interest rate announcements as long as the overnight rate is at the effective lower bound."

is similar to that shown by Eggertsson and Woodford (2003a, b). After the following periods, the optimal interest rates can be $\overline{r}$, which makes inflation rates at the optimal rate zero. Such a policy is optimal to achieve a balance between deflation during the adverse shock and inflation afterward.

Because I obtain the analytical solution, I can prove the following properties:

$$\frac{\partial}{\partial \overline{r}} i^{(1)} > 0, \frac{\partial}{\partial \overline{r}} i^{(2)} > 0 \tag{3.40}$$

$$\frac{\partial}{\partial \underline{r}} i^{(1)} > 0, \frac{\partial}{\partial \underline{r}} i^{(2)} < 0 \tag{3.41}$$

$$\frac{\partial}{\partial p} i^{(1)} < 0, \frac{\partial}{\partial p} i^{(2)} > 0 \text{ for } p \ll 1 \tag{3.42}$$

$$\frac{\partial}{\partial (\gamma/\sigma)} i^{(1)} > 0, \frac{\partial}{\partial (\gamma/\sigma)} i^{(2)} < 0 \text{ for } p \ll 1. \tag{3.43}$$

Those inequalities suggest two things. First, if the persistence of the adverse shock increases too much or the size of the shock becomes too big, then a commitment duration longer than two periods is needed. This is because the optimal interest rate when the real interest rate shock becomes positive, $i^{(1)}$, should be lower as $\underline{r}$ is lower or p is higher. Similar conditions are applied to the case where either $\overline{r}$ or γ/σ is too low. Second, the adjustments of interest rates are required to be more volatile, as the persistence of the adverse shock is higher, the size of the shock is bigger, the elasticity of inflation to output is lower, and the elasticity of output to interest rates is lower.[12]

4 Numerical Comparison of Policies

4.1 Parameters for Simulation

In this section, I simulate my model to compare economic behaviors and welfare losses among various policies. I use standard parameter values. See Table 1 for details. The discount factor β is 0.99, the elasticity of inflation to output γ is 0.024, and the inverse of the elasticity of output to real interest rates σ is one. The size and persistence of the adverse real interest rate shock are modest. That allows policymakers to set a positive but lower interest rate than a natural rate when the state becomes n (i.e., $k = 1$). The persistence of the adverse shock, p, is 0.5. A positive real interest rate $\overline{r}$ is 0.01 (4 percent annually) and a negative real interest rate $\underline{r}$ is -0.0025 (-1 percent annually). Later I will check the sensitivity of the results by changing these parameter values.

[12] This finding is consistent with Levin, Lopez-Salido, Nelson, and Yun (2009), who argue that the effectiveness of a commitment policy is sensitive to the real interest rate elasticity of aggregate demand.

4.2 Simulation Results

Commitment Policy Figure 1 demonstrates the economic behaviors under the commitment policy. The pattern of interest rates resembles that found in the analysis of the time-invariant duration policy. Interest rates when the state becomes n are set lower than the natural rate of 4 percent. It leads to milder deflation during the adverse shock. During the following period, the interest rate surpasses the natural rate. It prevents inflation during the previous period from overshooting.

Different from the time-invariant duration policy, the paths of interest rates, inflation, and the output gap vary, depending on the timing of the recovery. When the state becomes n at $t = 2$, the interest rate at $t = 2$ becomes 2.6 percent, lower than the natural rate. When the state becomes n at $t = 3$, the interest rate at $t = 3$ becomes even lower, 1.9 percent. Over time, the interest rate at the timing of the recovery converges to a certain level around 1.6 percent.[13]

Because the commitment policy maintains low interest rates when the state becomes n (i.e., $k = 1$), inflation overshoots during the period. Furthermore, because the interest rate during the period decreases over time, the overshooting of inflation grows as the spell of the adverse shock lengthens. The deviation of the output gap during the period also becomes bigger.

Such responses of interest rates, inflation, and the output gap enhance the time-inconsistency problem. The top panel of Figure 2 demonstrates the paths of the Lagrange multiplier λ_t. The solid line represents the paths under the commitment policy. The thick solid line with dots represents the paths under the time-invariant duration policy, which will be discussed below. Under the commitment policy, starting from zero, the Lagrange multiplier increases over time. Since the optimal commitment policy solution is derived conditional $\lambda_1 = 0$, the deviation of λ_t from zero means a loss of benefits from conducting commitment policy. The loss of benefits is captured by the bottom panel of Figure 2. The figure demonstrates the path of expected welfare losses at t conditional on that the state at t is still z. The expected welfare losses increase from $t = 1$ to $t = 3$, suggesting that an incentive to deviate from the commitment policy increases. From $t = 4$, expected welfare losses decrease, indicating that the time-inconsistency problem is mitigated. Such mitigation is, however, modest, not sufficient to dominate the time-invariant duration policy.[14]

---

[13] Equation (3.3) suggests that the Lagrange multiplier λ_{t+m} converges to a certain positive level. Inflation rates during the adverse shock, $\pi^{(0)}_{t+m}$, also converge to a certain level, and the level is positive. Because the price level does not continue to fall, the positiveness relaxes a need for higher inflation when the adverse shock disappears. For this reason, changes in inflation rates and interest rates are not monotonous.

[14] Although the expected welfare losses increase under the commitment policy, they are still lower than those under the discretionary policy. In this sense, the commitment policy is sustainable, as long as the state is z (see Kurozumi [2008]).

Dynamics under Various Policies Figures 3 and 4 demonstrate the dynamic paths of the economy under various policies. Except for the commitment policy, dynamics under all the policies are independent of when the state becomes n. In the graphs, therefore, I specify the timing when the adverse shock disappears to be $t = 5$. Note, however, that it is not predicted *ex ante* because the shock is stochastic.

I consider discretionary policy. As the dotted lines in Figure 3 indicate, under the discretionary policy, the interest rate has no history dependence. There is no policy duration. The interest rate quickly reverts to the steady state level, 4 percent. The size of deflation at state z is relatively big, as low as -0.1 percent annually. The drop in the output gap is also relatively big.

Under the time-invariant duration policy, the interest rate falls below the steady-state level at the time of exit ($t = 5$) and surpasses it at the next period ($t = 6$). The inflation rate at state z is close to zero, which is optimal. Its path is extremely close to the path under the commitment policy.

Under the one-period duration policy, there is no overshooting in the interest rate during the following period of the exit ($t = 6$). It causes a higher inflation rate at the time of exit ($t = 5$), even though the rise in the interest rate is higher than that under commitment policy. However, at state z, one-period duration policy yields a similar inflation rate to the commitment policy and the time-invariant duration policy.

Figure 4 demonstrates the economy under simple interest rate rules, compared with that under time-invariant duration policy. I consider three policy rules. These are a non-inertial policy rule with $\rho = 0$, the inertial rule of equation (3.38), and inertial rule 2 given by equation (3.39). The latter two policy rules have the inertia of $\rho = 0.8$. I find that the non-inertial policy rule yields exactly the same outcome as the discretionary policy.

Regarding inertial rules, the (first) inertial policy rule yields a slight improvement of inflation rates relative to the discretionary policy and the non-inertial policy rule. It is counterintuitive that the path of interest rates is almost the same. This is because, under the inertia rule, inflation rates are higher than those under the non-inertial rule. Responding to high inflation rates, the inertia rule sets high interest rates and this offsets a persistence of interest rate adjustments. This inertial rule is inferior to the time-invariant duration policy in terms of inflation stability. The time-invariant duration policy sets a lower interest rate during the exit period ($t = 5$) than the non-inertial and inertial rules, which helps mitigate deflation at state z.

In terms of inflation and output stability, inertial rule 2 appears to destabilize the economy. Although I do not show it in Figure 4, both inflation and the output gap at state z become highly positive. The reason for this is understood as follows. As is shown in the top panel of Figure 4, at state n, nominal interest rates are raised slowly due to the policy inertia. It increases inflation and output at state z. If this goes too far, the imaginary interest rate i^* becomes positive. But since the zero interest rates are maintained at state z and inertial rule 2 does not raise interest rates sufficiently during the exit period ($t = 6$) by reacting to the past i^*, inertial rule 2 cannot stop inflation and output from being destabilized.

Welfare Loss Comparison Table 2 reports the comparison of welfare losses under various policies. The time-invariant duration policy can achieve almost the same level of inflation stability as the commitment policy. The discretionary policy and the simple interest rate rules are worse by far than the commitment policy. Clearly, the commitment policy achieves the lowest welfare loss. This amounts to a permanent deviation of inflation rates by 0.0024 percent. Among other policies, the time-invariant duration policy and the one-period duration policy achieve almost the same welfare as the commitment policy. The welfare losses are 0.0026 percent and 0.0031 percent, respectively. The fourth-best policy is the inertial rule, but there is a big gap. The welfare loss is 0.0091 percent annually, almost three times as big. The welfare losses under the discretionary policy and the non-inertial rule are both 0.0140 percent annually. Inertial rule 2 has by far the worst welfare.

Although the commitment policy achieves the lowest welfare loss at $t = 1$, this does not mean it is so at later dates. Reexamining Figure 2, I find that the commitment policy is dominated by the time-invariant duration policy after $t = 2$. The invariant duration policy has less time-inconsistency than the commitment policy. However, the commitment policy is still far better than the discretionary policy.

4.3 Sensitivity

I examine the sensitivity of my results to various parameters.[15] Figures 5 to 8 demonstrate sensitivity to the real interest rate at state z ($\underline{r}$), the persistence of the adverse shock (p), the real interest rate at state z ($\underline{r}$), and the elasticity of inflation to real interest rates (γ/σ), respectively. In each figure, welfare loss, interest rates during the exit period ($i^{(1)}$), and during the next period ($i^{(2)}$) are presented. Since the commitment policy, the time-invariant duration policy, and the one-period duration policy yield similar welfare loss, I magnify them in the upper-right panel.

Regarding the real interest rate at state z ($\underline{r}$), according to Figure 5, welfare losses are almost the same under the commitment policy, the time-invariant duration policy, and one-

[15] Regarding sensitivity to models, I confirm the good performance of the time-invariant duration policy in a model with capital and with hybrid Phillips and IS curves.

period duration policy, while the discretionary policy and the inertial rule yield larger welfare losses. As $\underline{r}$ falls, the interest rate during the exit period ($i^{(1)}$) needs to fall.

When the adverse shock is large, the performance of the one-period duration policy worsens compared with the commitment policy and the time-invariant duration policy. For a sufficiently low $\underline{r}$ that makes $i^{(1)}$ zero under one-period duration policy, welfare losses from conducting the one-period duration policy becomes nonlinearly larger than that from the commitment or the time-invariant duration policy. Such a result arises because the one-period duration policy increases the interest rate too quickly, even though a longer period commitment is needed. Both the commitment policy and the time-invariant duration policy lower the interest rate during the next period ($i^{(2)}$). Such longer-duration commitment policy helps soften the deflation at the ZLB. Under the one-period duration policy, however, a central bank commits only up to one period, setting the interest rate during the next period ($i^{(2)}$) equal to the natural rate. It brings serious deflation at the ZLB.

Regarding the persistence of the adverse shock (p), Figure 6 suggests that welfare loss is almost the same under the commitment policy, the time-invariant duration policy, and the one-period duration policy. Compared with them, the discretionary policy and the inertial rule yield a larger welfare loss. Its deviation becomes wider as p becomes larger. Under the commitment policy and the time-invariant duration policy, the interest rate during the exit period ($i^{(1)}$) needs to fall as p increases to mitigate deflation at state z. When p exceeds a certain threshold, $i^{(1)}$ needs to be negative, but because of the ZLB it becomes zero. In such a case, both the commitment policy and the time-invariant duration policy lower the interest rate during the next period ($i^{(2)}$). As Figures 7 and 8 show, similar results can be obtained regarding the real interest rate at state z ($\underline{r}$) and the elasticity of inflation to real interest rates (γ/σ).

I next examine the effect of Assumption 1 that neglects output volatility. The bottom panels of Figures 1, 3, and 4 demonstrate the path of the output gap. The graph shows that the output gap is stabilized when inflation is stabilized. I also calculate the welfare loss when λ is not zero but 0.05. Under the commitment policy, the discretionary policy, the time-invariant duration policy, and the one-period duration policy, the welfare loss amounts to 0.0361, 0.0675, 0.0386, and 0.0360 percent (annualized) measured by permanent changes in inflation, respectively. Regarding simple interest rate rules, the welfare loss from conducting the non-inertial rule, the inertial rule, and inertial rule 2 amounts to 0.0675, 0.0563, and 0.5732. The order of welfare loss among the four policies does not change except that the one-period duration policy is shown to be superior.

5 Conclusion

In this paper, I have explored optimal interest rate policy under the ZLB. In particular, I have proposed the time-invariant duration policy, which is simpler and less time-inconsistent than the commitment policy. The duration to maintain accommodative policy is time-invariant, independent of the *ex post* spell of the adverse shock. Even if the *ex post* spell of the adverse shock lengthens, it does not increase the incentive to deviate from the policy, unlike the commitment policy. The time-invariant duration policy can be expressed in an extremely simple, explicit form. It facilitates policy implementation and communication in a stochastic economy. Under the policy, the central bank maintains a lower interest rate for some duration after the adverse shock disappears, and sets a higher interest rate during the following period. I report that the time-invariant duration policy performs virtually as effectively as the optimal commitment policy in stabilizing inflation, and far better than a discretionary policy and simple interest rate rules with or without inertia.

For the future research, first, it is important to relax the assumption of full credibility. Commitment policy including the time-invariant duration policy is not fully time-consistent, so the credibility is not easily guaranteed. For the success of the policy, communication to the public is crucial, and in this respect the time-invariant duration policy discussed in this paper has a virtue due to its simplicity and less time-inconsistency. Second, as a measure of central bank policy, I consider only interest rate policy. In the presence of the ZLB, other measures such as quantitative easing and credit easing, capital injection, and coordination with fiscal policy increase their importance. Interest rate policy, however, remains the key instrument for central banks, so the implementation of interest rate policy that is as good as possible never loses its importance despite the ZLB. Last but not least, regarding central banks' objective, I focus only on inflation stability, but other factors such as output stability and the stability of the financial markets and financial system cannot be underestimated. Investigation of those factors is an extremely important task.

References

[1] Adam, Klaus, and Roberto M. Billi (2006). "Optimal Monetary Policy under Commitment with a Zero Bound on Nominal Interest Rates." *Journal of Money, Credit and Banking*, 38(7), 1877-1905.

[2] Adam, Klaus, and Roberto M. Billi (2007). "Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates." *Journal of Monetary Economics*, 54(3), 728-752.

[3] Damjonovic, Tatiana, Vladislav Damjonovic, Charles Nolan (2008). "Unconditionally Optimal Monetary Policy." *Journal of Monetary Economics.* 55(3). 491-500.

[4] Eggertsson, Gauti B., and Michael Woodford (2003a). "Optimal Monetary Policy in a Liquidity Trap." NBER Working Papers 9968,

[5] Eggertsson, Gauti B., and Michael Woodford (2003b). "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity*, 34(2003-1), 139-235.

[6] Fujiki, Hiroshi and Shigenori Shiratsuka (2002). "Policy Duration Effect under the Zero Interest Rate Policy in 1999–2000: Evidence from Japan's Money Market Data." *Monetary and Economic Studies*, 20(1), 1–32.

[7] Fujiwara, Ippei, Nao Sudo, and Yuki Teranishi (2010). "The Zero Lower Bound and Monetary Policy in a Global Economy: A Simple Analytical Investigation." *International Journal of Central Banking*, 6(1), 103-134.

[8] Jeanne, Olivier and Lars E.O. Svensson (2007). "Credible Commitment to Optimal Escape from a Liquidity Trap: The Role of the Balance Sheet of an Independent Central Bank." *American Economic Review*, 97, 474-490.

[9] Jensen, Christian and Benett T. McCallum (2002). "The Non-Optimality of Proposed Monetary Policy Rules under Timeless Perspective Commitment." *Economics Letters*, 77(2), 163-168.

[10] Jung, Taehun, Yuki Teranishi, and Tsutomu Watanabe (2005). "Optimal Monetary Policy at the Zero-Interest-Rate Bound." *Journal of Money, Credit and Banking*, 37(5), 813-835.

[11] Kato, Ryo, and Shin-Ichi Nishiyama (2005). "Optimal Monetary Policy When Interest Rates Are Bounded at Zero." *Journal of Economic Dynamics and Control*, 29(1-2), 97-133.

[12] Kurozumi, Takushi (2008). "Optimal Sustainable Monetary Policy." *Journal of Monetary Economics*, 55, 1277-1289.

[13] Levin, Andrew, David Lopez-Salido, Edward Nelson, and Tack Yun (2009). "Limitations on the Effectiveness of Forward Guidance at the Zero Lower Bound." Draft.

[14] McCallum, Benett T. (2005). "What Is the Proper Perspective for Monetary Policy Optimality?" *Monetary and Economic Studies*, 23(S-1), 13-24.

[15] McCallum, Benett T. and Edward Nelson (2004). "Timeless Perspective vs. Discretionary Monetary Policy in Forward-Looking Models." FRB St. Louis Review, 86(2), 43-56.

[16] Nakov, Anton (2008). "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate." *International Journal of Central Banking*, 4(2), 73-127.

[17] Reifschneider, David, and John C. Williams (2000). "Three Lessons for Monetary Policy in a Low-Inflation Era." *Journal of Money, Credit and Banking*, 32, 936-966.

[18] Schaumburg, Ernst and Andrea Tambalotti (2007). "An Investigation of the Gains From Commitment in Monetary Policy." *Journal of Monetary Economics*, 54(2), 302-324.

[19] Sugo, Tomohiro, and Yuki Teranishi (2005). "The Optimal Monetary Policy Rule under the Non-Negativity Constraint on Nominal Interest Rates." *Economics Letters*, 89(1), 95-100.

[20] Woodford, Michael (2003). Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton: Princeton University Press.

A Appendix

A.1 Commitment Policy

Under commitment policy, I define the Lagrangian as

$$\begin{aligned}
\pounds_t \;=\; & \textstyle\sum_{m=0}^{\infty} p^m \beta^m (\pi_{t+m}^{(0)})^2 \\
& +\textstyle\sum_{m=3}^{\infty} (1-p)\beta^m \{p^{m-3}(\pi_{t+m}^{(3)})^2 + p^{m-2}(\pi_{t+m}^{(2)})^2 + p^{m-1}(\pi_{t+m}^{(1)})^2\} \\
& +(1-p)\beta^2\{(\pi_{t+2}^{(2)})^2 + p(\pi_{t+2}^{(1)})^2\} \\
& +(1-p)\beta(\pi_{t+1}^{(1)})^2 + \Delta(\pi_{t+l}^{(k)}) \\
& -2\textstyle\sum_{m=0}^{\infty} p^m \beta^m \lambda_{t+m}\left[-\frac{1}{\gamma/\sigma}\beta\{p^2\pi_{t+2+m}^{(0)} + p(1-p)\pi_{t+2+m}^{(1)} + (1-p)\pi_{t+2+m}^{(2)}\}\right. \\
& \left. +\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi_{t+1+m}^{(0)} + (1-p)\pi_{t+1+m}^{(1)}\} - \frac{1}{\gamma/\sigma}\pi_{t+m}^{(0)} + \underline{r}\right], \qquad \text{(A.1)}
\end{aligned}$$

where $\Delta\pi_{t+l}^{(k)}$ is a function of $\pi_{t+l}^{(k)}$ for $k \geq 4$ and $l \geq 3$. With respect to $\pi_{t+m}^{(n)}$ and λ_{t+m}, I write down the first-order conditions. Here, I introduce a timeless perspective following Woodford

(2003) and McCallum and Nelson (2004). The derivative of $\pi_{t+m}^{(0)}$ becomes

$$\begin{aligned} 0 &= p^m\beta^m\pi_{t+m}^{(0)} \\ &+p^{m-2}\beta^{m-2}\lambda_{t+m-2}\frac{1}{\gamma/\sigma}\beta p^2 \\ &-p^{m-1}\beta^{m-1}\lambda_{t+m-1}\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)p \\ &+p^m\beta^m\lambda_{t+m}\frac{1}{\gamma/\sigma}. \end{aligned}$$

This becomes

$$\begin{aligned} 0 &= \pi_{t+m}^{(0)}+\lambda_{t+m-2}\frac{1}{\gamma/\sigma}\beta^{-1} \\ &-\lambda_{t+m-1}\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\beta^{-1}+\lambda_{t+m}\frac{1}{\gamma/\sigma}. \end{aligned} \tag{A.2}$$

For $n \geq 3$, the derivative of $\pi_{t+m}^{(n)}$ is written by

$$0 = \pi_{t+m}^{(n)}. \tag{A.3}$$

The derivative of $\pi_{t+m}^{(2)}$ is written by

$$\begin{aligned} 0 &= (1-p)\beta^m p^{m-2}\pi_{t+m}^{(2)} \\ &+\frac{1}{\gamma/\sigma}\beta(1-p)p^{m-2}\beta^{m-2}\lambda_{t+m-2}. \end{aligned}$$

This becomes

$$0 = \pi_{t+m}^{(2)}+\frac{1}{\gamma/\sigma}\beta^{-1}\lambda_{t+m-2}. \tag{A.4}$$

The derivative of $\pi_{t+m}^{(1)}$ is written by

$$\begin{aligned} 0 &= (1-p)\beta^m p^{m-1}\pi_{t+m}^{(1)} \\ &+\frac{1}{\gamma/\sigma}\beta p(1-p)p^{m-2}\beta^{m-2}\lambda_{t+m-2} \\ &-\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)(1-p)p^{m-1}\beta^{m-1}\lambda_{t+m-1}. \end{aligned}$$

This becomes

$$\begin{aligned} 0 &= \pi_{t+m}^{(1)}+\frac{1}{\gamma/\sigma}\beta^{-1}\lambda_{t+m-2} \\ &-\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\beta^{-1}\lambda_{t+m-1}. \end{aligned} \tag{A.5}$$

The derivative of λ_{t+m} is written by

$$\begin{aligned} 0 &= -\frac{1}{\gamma/\sigma}\beta\{p^2\pi^{(0)}_{t+2+m} + p(1-p)\pi^{(1)}_{t+2+m} + (1-p)\pi^{(2)}_{t+2+m}\} \\ &\quad + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+1+m} + (1-p)\pi^{(1)}_{t+1+m}\} \\ &\quad - \frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m} + \underline{r}. \end{aligned} \tag{A.6}$$

The above four equations, the first-order conditions with respect to $\pi^{(0)}_{t+m}$, $\pi^{(1)}_{t+m}$, $\pi^{(2)}_{t+m}$, and λ_{t+m}, yield the optimal commitment policy.

Once I obtain $\{\pi^{(0)}_{t+m}, \pi^{(1)}_{t+m}, \pi^{(2)}_{t+m}, \lambda_{t+m}\}$, I can derive nominal interest rates at state n (or $k \geq 1$) using equation (2.9):

$$i^{(k)}_t = -\frac{1}{\gamma/\sigma}\left[\beta\pi^{(k+2)}_{t+2} - (1+\beta+\gamma/\sigma)\pi^{(k+1)}_{t+1} + \pi^{(k)}_t\right] + \overline{r}. \tag{A.7}$$

Since $\pi^{(n)}_{t+m} = 0$ for $n \geq 3$, I find $i^{(k)}_t = \overline{r}$ for $k \geq 3$.

Commitment Policy When $i^{(1)}$ Is Zero Consider the case when $i^{(1)}$ calculated above becomes negative. Due to $i^{(0)} = i^{(1)} = 0$, a minimization problem is restricted by two conditions represented by equation (2.9) for $k = 0$ and 1. Therefore, I define the Lagrangian as

$$\begin{aligned} \pounds_t &= \textstyle\sum_{m=0}^{\infty} p^m\beta^m(\pi^{(0)}_{t+m})^2 \\ &\quad + \textstyle\sum_{m=3}^{\infty}(1-p)\beta^m\{p^{m-3}(\pi^{(3)}_{t+m})^2 + p^{m-2}(\pi^{(2)}_{t+m})^2 + p^{m-1}(\pi^{(1)}_{t+m})^2\} \\ &\quad + (1-p)\beta^2\{(\pi^{(2)}_{t+2})^2 + p(\pi^{(1)}_{t+2})^2\} \\ &\quad + (1-p)\beta(\pi^{(1)}_{t+1})^2 + \Delta(\pi^{(k)}_{t+l}) \\ &\quad - 2\textstyle\sum_{m=0}^{\infty} p^m\beta^m\lambda^{(0)}_{t+m}\left[-\frac{1}{\gamma/\sigma}\beta\{p^2\pi^{(0)}_{t+2+m} + p(1-p)\pi^{(1)}_{t+2+m} + (1-p)\pi^{(2)}_{t+2+m}\}\right. \\ &\quad \left. + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+1+m} + (1-p)\pi^{(1)}_{t+1+m}\} - \frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m} + \underline{r}\right] \\ &\quad - 2\textstyle\sum_{m=1}^{\infty} p^m\beta^m\lambda^{(1)}_{t+m}\frac{1}{\gamma/\sigma}\left[-\beta\pi^{(3)}_{t+2+m} + (1+\beta+\gamma/\sigma)\pi^{(2)}_{t+1+m} - \pi^{(1)}_{t+m} + \gamma/\sigma\overline{r}\right] \end{aligned} \tag{A.8}$$

With respect to $\pi^{(n)}_{t+m}$, $\lambda^{(0)}_{t+m}$, and $\lambda^{(1)}_{t+m}$, I write down the first-order conditions. Again, I introduce a timeless perspective following Woodford (2003) and McCallum and Nelson (2004).

The derivative of $\pi_{t+m}^{(0)}$ becomes

$$\begin{aligned} 0 &= \pi_{t+m}^{(0)} + \lambda_{t+m-2}^{(0)} \frac{1}{\gamma/\sigma} \beta^{-1} \\ &\quad - \lambda_{t+m-1}^{(0)} \frac{1}{\gamma/\sigma} (1+\beta+\gamma/\sigma) \beta^{-1} + \lambda_{t+m}^{(0)} \frac{1}{\gamma/\sigma}. \end{aligned} \tag{A.9}$$

For $n \geq 4$, the derivative of $\pi_{t+m}^{(n)}$ is written by

$$0 = \pi_{t+m}^{(n)}. \tag{A.10}$$

The derivative of $\pi_{t+m}^{(3)}$ becomes

$$0 = (1-p)\beta\pi_{t+m}^{(3)} + \frac{1}{\gamma/\sigma} p \lambda_{t+m-2}^{(1)}. \tag{A.11}$$

The derivative of $\pi_{t+m}^{(2)}$ become:

$$0 = (1-p)\beta\pi_{t+m}^{(2)} + \frac{1-p}{\gamma/\sigma} \lambda_{t+m-2}^{(0)} - \frac{p}{\gamma/\sigma} (1+\beta+\gamma/\sigma) \lambda_{t+m-1}^{(1)}. \tag{A.12}$$

The derivative of $\pi_{t+m}^{(1)}$ becomes

$$\begin{aligned} 0 &= (1-p)\beta\pi_{t+m}^{(1)} + \frac{1-p}{\gamma/\sigma} \lambda_{t+m-2}^{(0)} \\ &\quad - \frac{1-p}{\gamma/\sigma} (1+\beta+\gamma/\sigma) \lambda_{t+m-1}^{(0)} + \frac{p\beta}{\gamma/\sigma} \lambda_{t+m}^{(1)}. \end{aligned} \tag{A.13}$$

The derivative of $\lambda_{t+m}^{(0)}$ is written by

$$\begin{aligned} 0 &= -\frac{1}{\gamma/\sigma} \beta \{ p^2 \pi_{t+2+m}^{(0)} + p(1-p)\pi_{t+2+m}^{(1)} + (1-p)\pi_{t+2+m}^{(2)} \} \\ &\quad + \frac{1}{\gamma/\sigma} (1+\beta+\gamma/\sigma) \{ p\pi_{t+1+m}^{(0)} + (1-p)\pi_{t+1+m}^{(1)} \} \\ &\quad - \frac{1}{\gamma/\sigma} \pi_{t+m}^{(0)} + \underline{r}. \end{aligned} \tag{A.14}$$

The derivative of $\lambda_{t+m}^{(1)}$ is written by

$$0 = -\frac{1}{\gamma/\sigma} \beta \pi_{t+2+m}^{(3)} + \frac{1}{\gamma/\sigma} (1+\beta+\gamma/\sigma) \pi_{t+1+m}^{(2)} - \frac{1}{\gamma/\sigma} \pi_{t+m}^{(1)} + \overline{r}. \tag{A.15}$$

The above six equations, the first-order conditions with respect to $\pi_{t+m}^{(0)}$, $\pi_{t+m}^{(1)}$, $\pi_{t+m}^{(2)}$, $\pi_{t+m}^{(3)}$, $\lambda_{t+m}^{(0)}$, and $\lambda_{t+m}^{(1)}$ yield the optimal commitment policy subject to $i_{t+m}^{(0)} = i_{t+m}^{(1)} = 0$.

A.2 Time-Invariant Duration Policy

Under the time-invariant duration policy, I impose

$$\pi^{(k)}_{t+m+1} = \pi^{(k)}_{t+m}. \tag{A.16}$$

Then, the Lagrangian, defined as

$$\begin{aligned}
\pounds_t &= \textstyle\sum_{m=0}^{\infty} p^m \beta^m (\pi^{(0)}_{t+m})^2 \\
&\quad + \textstyle\sum_{m=3}^{\infty} (1-p)\beta^m \{p^{m-3}(\pi^{(3)}_{t+m})^2 + p^{m-2}(\pi^{(2)}_{t+m})^2 + p^{m-1}(\pi^{(1)}_{t+m})^2\} \\
&\quad + (1-p)\beta^2\{(\pi^{(2)}_{t+2})^2 + p(\pi^{(1)}_{t+2})^2\} \\
&\quad + (1-p)\beta(\pi^{(1)}_{t+1})^2 + \Delta(\pi^{(k)}_{t+l}) \\
&\quad - 2\textstyle\sum_{m=0}^{\infty} p^m \beta^m \lambda_{t+m} \Bigg[-\frac{1}{\gamma/\sigma}\beta\{p^2\pi^{(0)}_{t+2+m} + p(1-p)\pi^{(1)}_{t+2+m} + (1-p)\pi^{(2)}_{t+2+m}\} \\
&\quad + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+1+m} + (1-p)\pi^{(1)}_{t+1+m}\} - \frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m} + \underline{r} \Bigg],
\end{aligned} \tag{A.17}$$

is transformed into

$$\begin{aligned}
\pounds_t &= \textstyle\sum_{m=0}^{\infty} p^m \beta^m (\pi^{(0)}_{t+m})^2 \\
&\quad + \textstyle\sum_{m=3}^{\infty} (1-p)\beta^m \{p^{m-3}(\pi^{(3)}_{t+m})^2 + p^{m-2}(\pi^{(2)}_{t+m})^2 + p^{m-1}(\pi^{(1)}_{t+m})^2\} \\
&\quad + (1-p)\beta^2\{(\pi^{(2)}_{t+2})^2 + p(\pi^{(1)}_{t+2})^2\} \\
&\quad + (1-p)\beta(\pi^{(1)}_{t+1})^2 + \Delta(\pi^{(k)}_{t+l}) \\
&\quad - 2\textstyle\sum_{m=0}^{\infty} p^m \beta^m \lambda_{t+m} \Bigg[-\frac{1}{\gamma/\sigma}\beta\{p^2\pi^{(0)}_{t+m} + p(1-p)\pi^{(1)}_{t+1+m} + (1-p)\pi^{(2)}_{t+2+m}\} \\
&\quad + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+m} + (1-p)\pi^{(1)}_{t+1+m}\} - \frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m} + \underline{r} \Bigg].
\end{aligned} \tag{A.18}$$

With respect to $\pi^{(n)}_{t+m}$ and λ_{t+m}, I write down the first-order conditions. The derivative of $\pi^{(0)}_{t+m}$ becomes

$$\begin{aligned}
0 &= p^m\beta^m\pi^{(0)}_{t+m} + p^m\beta^m\lambda_{t+m}\frac{1}{\gamma/\sigma}\beta p^2 \\
&\quad - p^m\beta^m\lambda_{t+m}\frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)p + p^m\beta^m\lambda_{t+m}\frac{1}{\gamma/\sigma}.
\end{aligned}$$

This becomes

$$0 = \pi^{(0)}_{t+m} - C\lambda_{t+m}, \tag{A.19}$$

where

$$C = -\frac{p^2\beta - p(1+\beta+\gamma/\sigma)+1}{\gamma/\sigma} < 0. \tag{A.20}$$

For $n \geq 3$, the derivative of $\pi^{(n)}_{t+m}$ is written by

$$0 = \pi^{(n)}_{t+m}. \tag{A.21}$$

The derivative of $\pi^{(2)}_{t+m+2}$ is written by

$$\begin{aligned} 0 &= (1-p)\beta^{m+2}p^m\pi^{(2)}_{t+m+2} \\ &\quad + \frac{1}{\gamma/\sigma}p^m\beta^m\lambda_{t+m}(1-p)\beta. \end{aligned}$$

This becomes

$$0 = (1-p)\beta^2\pi^{(2)}_{t+m+2} - A\lambda_{t+m}, \tag{A.22}$$

where

$$A = -\frac{(1-p)\beta}{\gamma/\sigma} < 0. \tag{A.23}$$

The derivative of $\pi^{(1)}_{t+m+1}$ is written by

$$\begin{aligned} 0 &= (1-p)\beta^{m+1}p^m\pi^{(1)}_{t+m+1} \\ &\quad + \frac{1}{\gamma/\sigma}p^m\beta^m\lambda_{t+m}\beta p(1-p) \\ &\quad - \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)p^m\beta^m\lambda_{t+m}(1-p). \end{aligned}$$

This becomes

$$0 = (1-p)\beta\pi^{(1)}_{t+m+1} - B\lambda_{t+m}, \tag{A.24}$$

where

$$B = \frac{(1-p)\{1+\gamma/\sigma+\beta(1-p)\}}{\gamma/\sigma} > 0. \tag{A.25}$$

The derivative of λ_{t+m} is written by

$$\begin{aligned} 0 &= -\frac{1}{\gamma/\sigma}\beta\{p^2\pi^{(0)}_{t+m} + p(1-p)\pi^{(1)}_{t+1+m} + (1-p)\pi^{(2)}_{t+2+m}\} \\ &\quad + \frac{1}{\gamma/\sigma}(1+\beta+\gamma/\sigma)\{p\pi^{(0)}_{t+m} + (1-p)\pi^{(1)}_{t+1+m}\} \\ &\quad - \frac{1}{\gamma/\sigma}\pi^{(0)}_{t+m} + \underline{r} \\ &= A\pi^{(2)}_{t+2+m} + B\pi^{(1)}_{t+1+m} + C\pi^{(0)}_{t+m} + \underline{r}. \end{aligned} \tag{A.26}$$

The above four equations, the first-order conditions with respect to $\pi^{(0)}_{t+m}$, $\pi^{(1)}_{t+m}$, $\pi^{(2)}_{t+m}$, and λ_{t+m}, yield the solution for the time-invariant duration policy. Expressing $\pi^{(0)}_{t+m}$, $\pi^{(1)}_{t+m}$, and $\pi^{(2)}_{t+m}$ with λ_{t+m}, I can rewrite the first-order condition with respect to λ_{t+m} as

$$0 = \frac{A^2}{(1-p)\beta^2}\lambda_{t+m} + \frac{B^2}{(1-p)\beta}\lambda_{t+m} + C^2\lambda_{t+m} + \underline{r}.$$

Therefore, λ_{t+m} becomes

$$\begin{aligned} \lambda_{t+m} &= \lambda = -\left(\frac{A^2}{(1-p)\beta^2} + \frac{B^2}{(1-p)\beta} + C^2\right)^{-1} \underline{r} && \text{(A.27)} \\ &> 0. && \text{(A.28)} \end{aligned}$$

Clearly, λ_{t+m} is constant.

Inflation rates, $\pi^{(k)}_t$, are also constant given the same k, being described as

$$\pi^{(0)} = -\left(\frac{A^2}{(1-p)\beta^2} + \frac{B^2}{(1-p)\beta} + C^2\right)^{-1} C\underline{r} < 0, \tag{A.29}$$

$$\pi^{(2)} = \frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)\pi^{(0)} < 0, \tag{A.30}$$

$$\pi^{(1)} = \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)\pi^{(0)} > 0. \tag{A.31}$$

Once I obtain $\{\pi^{(0)}_{t+m}, \pi^{(1)}_{t+m}, \pi^{(2)}_{t+m}, \lambda_{t+m}\}$, I can derive nominal interest rates at state n (or $k \geq 1$) using equation (2.9):

$$i^{(k)}_t = -\frac{1}{\gamma/\sigma}\left[\beta\pi^{(k+2)}_{t+2} - (1+\beta+\gamma/\sigma)\pi^{(k+1)}_{t+1} + \pi^{(k)}_t\right] + \overline{r}.$$

Because all endogenous variables are constant, it becomes

$$i^{(k)}_t = i^{(k)} = -\frac{1}{\gamma/\sigma}\left[\beta\pi^{(k+2)} - (1+\beta+\gamma/\sigma)\pi^{(k+1)} + \pi^{(k)}\right] + \overline{r}. \tag{A.32}$$

Since $0 = \pi^{(k)}_{t+m}$ for $k \geq 3$, I have

$$i^{(3)} = i^{(4)} = \cdots = \overline{r}. \tag{A.33}$$

For $k = 2$ and 1, I have

$$\begin{aligned} i^{(2)} &= -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(4)} - (1+\beta+\gamma/\sigma)\pi^{(3)} + \pi^{(2)}\right) + \overline{r}, && \text{(A.34)} \\ i^{(1)} &= -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(3)} - (1+\beta+\gamma/\sigma)\pi^{(2)} + \pi^{(1)}\right) + \overline{r}. && \text{(A.35)} \end{aligned}$$

This becomes

$$i^{(2)} = \overline{r} - \frac{1}{\gamma/\sigma}\pi^{(2)} > \overline{r}, \tag{A.36}$$

$$i^{(1)} = \overline{r} - \frac{1}{\gamma/\sigma}\{-(1+\beta+\gamma/\sigma)\pi^{(2)} + \pi^{(1)}\} < \overline{r}. \tag{A.37}$$

Finally, I need to check if $i^{(1)}$ and $i^{(2)}$ are equal to zero or positive. Clearly, $i^{(2)}$ is positive. Regarding $i^{(1)}$, I explicitly write it down as

$$\begin{aligned} i^{(1)} &= \overline{r} - \frac{1}{\gamma/\sigma}\{-(1+\beta+\gamma/\sigma)\pi^{(2)} + \pi^{(1)}\} \\ &= \overline{r} - \frac{1}{\gamma/\sigma}\left\{-(1+\beta+\gamma/\sigma)\frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)\pi^{(0)} + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)\pi^{(0)}\right\} \\ &= .\overline{r} - \frac{1}{\gamma/\sigma}\left\{-(1+\beta+\gamma/\sigma)\frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right) + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)\right\} \\ &\quad \cdot \left\{1 + \frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)^2 + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)^2\right\}^{-1}\left(-\frac{1}{C}\underline{r}\right). \end{aligned} \tag{A.38}$$

From the IS equation, the output gap is described as

$$x^{(0)} = \frac{1}{\gamma}\{\pi^{(0)} - \beta(p\pi^{(0)} + (1-p)\pi^{(1)})\}, \tag{A.39}$$

$$x^{(k)} = \frac{1}{\gamma}(\pi^{(k)} - \beta\pi^{(k+1)}), \tag{A.40}$$

for $k \geq 1$. Since $\pi^{(2)} = \pi^{(3)} = \cdots = 0$, I obtain $x^{(k)} = 0$ for for $k \geq 2$.

Another Solution Method Because I impose $\pi^{(k)}_{t+m+1} = \pi^{(k)}_{t+m}$, I can easily infer that other endogenous variables are also constant. It simplifies the form of welfare losses and enables us to derive the invariant duration policy without explicitly defining the Lagrangian. This alternative solution method is convenient to apply to other policy valuations.

Consider how equation (2.9) can be expressed. For $k \geq 1$, equation (2.9) becomes

$$i^{(k)} = -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(k+2)} - (1+\beta+\gamma/\sigma)\pi^{(k+1)} + \pi^{(k)}\right) + \overline{r}. \tag{A.41}$$

For $k=0$, equation (2.9) becomes

$$\begin{aligned} 0 &= p^2\left[-\frac{1}{\gamma/\sigma}\left(\beta\pi^{(0)}-(1+\beta+\gamma/\sigma)\pi^{(0)}+\pi^{(0)}\right)\right] \\ &\quad +p(1-p)\left[-\frac{1}{\gamma/\sigma}\left(\beta\pi^{(1)}-(1+\beta+\gamma/\sigma)\pi^{(0)}+\pi^{(0)}\right)\right] \\ &\quad +(1-p)\left[-\frac{1}{\gamma/\sigma}\left(\beta\pi^{(2)}-(1+\beta+\gamma/\sigma)\pi^{(1)}+\pi^{(0)}\right)\right] \\ &\quad +\underline{r}. \end{aligned} \tag{A.42}$$

$$0 = A\pi^{(2)}+B\pi^{(1)}+C\pi^{(0)}+\underline{r}, \tag{A.43}$$

where

$$\begin{aligned} A &= -\frac{(1-p)\beta}{\gamma/\sigma}<0 \\ B &= \frac{(1-p)\{1+\gamma/\sigma+\beta(1-p)\}}{\gamma/\sigma}>0 \\ C &= -\frac{p^2\beta-p(1+\beta+\gamma/\sigma)+1}{\gamma/\sigma}<0. \end{aligned} \tag{A.44}$$

The welfare loss at $k\geq 1$ is described as

$$\begin{aligned} L^{(k)} &= U^{(k)}+\beta L^{(k+1)} \\ &= (\pi^{(k)})^2+\beta\{(\pi^{(k+1)})^2+\beta L^{(k+2)}\} \\ &= (\pi^{(k)})^2+\beta(\pi^{(k+1)})^2+\beta^2(\pi^{(k+2)})^2+\cdots. \end{aligned} \tag{A.45}$$

The welfare loss when the current state is z is described as

$$\begin{aligned} L^{(0)} &= U^{(0)}+\beta\{pL^{(0)}+(1-p)L^{(1)}\} \\ &= \frac{1}{1-p\beta}\left\{(\pi^{(0)})^2+\beta(1-p)L^{(1)}\right\} \\ &= \frac{1}{1-p\beta}\left[(\pi^{(0)})^2+\beta(1-p)\{(\pi^{(1)})^2+\beta(\pi^{(2)})^2+\beta^2(\pi^{(3)})^2+\cdots\}\right]. \end{aligned} \tag{A.46}$$

The time-invariant duration policy chooses the optimal set of nominal interest rates $\{i^{(0)}=0, i^{(1)}, i^{(2)}, \cdots\}$ to minimize the welfare loss $L^{(0)}$ subject to equations (A.41) and (A.43) and $i^{(k)}\geq 0$. This problem can be reduced to the problem of choosing the optimal set of nominal inflation rates $\{\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \cdots\}$ to minimize the welfare loss $L^{(0)}$ subject to equation (A.43) provided $i^{(k)} \geq 0$. Substituting equation (A.43) into equation (A.46), I obtain

$$\begin{aligned}
& \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min L^{(0)} \\
= \; & \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min \frac{1}{1-p\beta}\left[(\pi^{(0)})^2 + \beta(1-p)\{(\pi^{(1)})^2 + \beta(\pi^{(2)})^2 + \beta^2(\pi^{(3)})^2 + \cdots\}\right] \\
= \; & \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min \frac{1}{1-p\beta}\left[\left(-\frac{A\pi^{(2)} + B\pi^{(1)} + \underline{r}}{C}\right)^2\right. \\
& \left. + \beta(1-p)\{(\pi^{(1)})^2 + \beta(\pi^{(2)})^2 + \beta^2(\pi^{(3)})^2 + \cdots\}\right].
\end{aligned}$$

I then check if $i^{(k)} \geq 0$ is satisfied.

With respect to $\pi^{(k)}$ (for $k \geq 3$), the first-order condition is

$$0 = \frac{\partial}{\partial \pi^{(k)}} L^{(0)} = \pi^{(k)}. \tag{A.47}$$

For $\pi^{(2)}$ and $\pi^{(1)}$, the first-order conditions are

$$0 = \frac{\partial}{\partial \pi^{(2)}} L^{(0)} = -\frac{A}{C}\pi^{(0)} + \beta^2(1-p)\pi^{(2)}, \tag{A.48}$$

$$0 = \frac{\partial}{\partial \pi^{(1)}} L^{(0)} = -\frac{B}{C}\pi^{(0)} + \beta(1-p)\pi^{(1)}. \tag{A.49}$$

Substituting $\pi^{(2)}$ and $\pi^{(1)}$ in the above equations into equation (A.43), I obtain

$$\pi^{(0)} = \left\{1 + \frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)^2 + \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)^2\right\}^{-1}\left(-\frac{1}{C}\underline{r}\right) < 0. \tag{A.50}$$

Also, I obtain

$$\pi^{(2)} = \frac{1}{\beta^2(1-p)}\left(\frac{A}{C}\right)\pi^{(0)} < 0, \tag{A.51}$$

$$\pi^{(1)} = \frac{1}{\beta(1-p)}\left(\frac{B}{C}\right)\pi^{(0)} > 0. \tag{A.52}$$

Time-Invariant Duration Policy When $i^{(1)}$ Is Zero If $i^{(1)}$ calculated above becomes negative, $i^{(1)}$ is set at zero. The time-invariant duration policy therefore has longer persistence. Assume $i^{(2)}$ is positive. Due to $i^{(0)} = i^{(1)} = 0$, a minimization problem is restricted by two conditions. These are equation (A.43),

$$0 = A\pi^{(2)} + B\pi^{(1)} + C\pi^{(0)} + \underline{r}, \tag{A.53}$$

and equation (A.41),

$$0 = -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(3)} - (1+\beta+\gamma/\sigma)\pi^{(2)} + \pi^{(1)}\right) + \overline{r}. \tag{A.54}$$

It yields

$$\pi^{(1)} = -\beta\pi^{(3)} + (1+\beta+\gamma/\sigma)\pi^{(2)} + \gamma/\sigma\overline{r}, \tag{A.55}$$

$$\begin{aligned}
\pi^{(0)} &= -\frac{A\pi^{(2)} + B\pi^{(1)} + \underline{r}}{C} \\
&= -\frac{-B\beta\pi^{(3)} + \{A + B(1+\beta+\gamma/\sigma)\}\pi^{(2)} + B\gamma/\sigma\overline{r} + \underline{r}}{C}.
\end{aligned} \tag{A.56}$$

A minimization problem is therefore given by

$$\begin{aligned}
& \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min L^{(0)} \\
= \ & \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min[(\pi^{(0)})^2 + \beta(1-p)\{(\pi^{(1)})^2 + \beta(\pi^{(2)})^2 + \beta^2(\pi^{(3)})^2 + \cdots\}] \\
= \ & \arg_{\{\pi^{(0)},\pi^{(1)},\pi^{(2)},\cdots\}} \min \\
& \left[\left(-\frac{-B\beta\pi^{(3)} + \{A + B(1+\beta+\gamma/\sigma)\}\pi^{(2)} + B\gamma/\sigma\overline{r} + \underline{r}}{C}\right)^2\right. \\
& +\beta(1-p)\{(-\beta\pi^{(3)} + (1+\beta+\gamma/\sigma)\pi^{(2)} + \gamma/\sigma\overline{r})^2 \\
& \left.+\beta(\pi^{(2)})^2 + \beta^2(\pi^{(3)})^2 + \cdots\}\right].
\end{aligned}$$

With respect to $\pi^{(3)}$ and $\pi^{(2)}$, the first-order conditions are

$$0 = \frac{\partial}{\partial\pi^{(3)}}L^{(0)} = \frac{B\beta}{C}\pi^{(0)} - \beta^2(1-p)\pi^{(1)} + \beta^3(1-p)\pi^{(3)}, \tag{A.57}$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial\pi^{(2)}}L^{(0)} \\
&= -\frac{A + B(1+\beta+\gamma/\sigma)}{C}\pi^{(0)} + \beta(1-p)(1+\beta+\gamma/\sigma)\pi^{(1)} + \beta^2(1-p)\pi^{(2)}.
\end{aligned} \tag{A.58}$$

Equations (A.55) to (A.58) yield the solution of $\pi^{(3)}$, $\pi^{(2)}$ $\pi^{(1)}$, and $\pi^{(0)}$. With respect to $\pi^{(k)}$ (for $k \geq 4$), the first-order condition is

$$0 = \frac{\partial}{\partial\pi^{(k)}}L^{(0)} = \pi^{(k)}.$$

Nominal interest rates are derived from equation (A.41):

$$i^{(k)} = -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(k+2)} - (1+\beta+\gamma/\sigma)\pi^{(k+1)} + \pi^{(k)}\right) + \bar{r}. \tag{A.59}$$

Provided $i^{(2)} \geq 0$, the time-invariant duration policy is sufficient to make a commitment of up to three periods after the shock becomes normal.

A.3 One-Period Duration Policy

Under one-period duration policy, for $k \geq 2$, discretionary policy is conducted so that inflation is completely stabilized as

$$\pi^{(2)} = \pi^{(3)} = \cdots = 0. \tag{A.60}$$

Since $\pi^{(2)} = 0$, when the current state is z (for $k = 0$), equation (2.9) or more simply equation (A.43) becomes

$$0 = B\pi^{(1)} + C\pi^{(0)} + \underline{r}, \tag{A.61}$$

where

$$\begin{aligned} B &= \frac{(1-p)\{1+\gamma/\sigma+\beta(1-p)\}}{\gamma/\sigma} > 0 \\ C &= -\frac{p^2\beta - p(1+\beta+\gamma/\sigma)+1}{\gamma/\sigma} < 0. \end{aligned} \tag{A.62}$$

The welfare loss when the current state is z is described as

$$L^{(0)} = \frac{1}{1-p\beta}\left[(\pi^{(0)})^2 + \beta(1-p)(\pi^{(1)})^2\right]. \tag{A.63}$$

The inflation rates under one-period commitment policy are the solutions to minimize the welfare loss subject to equation (A.61). Interest rates are derived from equation (A.41):

$$i^{(1)} = -\frac{1}{\gamma/\sigma}\pi^{(1)} + \bar{r}. \tag{A.64}$$

One-Period Duration Policy When $i^{(1)}$ Is Zero If $i^{(1)}$ calculated above becomes negative, $i^{(1)}$ is zero. Under one-period duration policy, for the period of $k \geq 2$, the policy becomes discretionary. Therefore,

$$\pi^{(2)} = \pi^{(3)} = \cdots = 0. \tag{A.65}$$

From equation (A.41), $\pi^{(1)}$ is given by

$$i^{(1)} = 0 = -\frac{1}{\gamma/\sigma}\pi^{(1)} + \overline{r}.$$

That is, I obtain

$$\pi^{(1)} = \gamma/\sigma\overline{r}. \tag{A.66}$$

Equation (A.61) yields

$$\pi^{(0)} = -\frac{B\gamma/\sigma\overline{r} + \underline{r}}{C}. \tag{A.67}$$

A.4 Simple Interest Rate Rules

For $k \geq 2$, the inertial policy rule is written as

$$i^{(k)} = \rho i^{(k-1)} + (1-\rho)(\overline{r} + \phi_\pi \pi^{(k)}). \tag{A.68}$$

I assume that $i^{(k)} \geq 0$ for $k \geq 1$. Then, for $k = 1$, I have

$$i^{(1)} = \rho i^{*(0)} + (1-\rho)(\overline{r} + \phi_\pi \pi^{(1)}), \tag{A.69}$$

and for $k = 0$,

$$i^{*(0)} = \rho i^{*(0)} + (1-\rho)(\overline{r} + \phi_\pi \pi^{(0)}). \tag{A.70}$$

Thus, $i^{(1)}$ is simplified as

$$\begin{aligned} i^{(1)} &= \rho(\overline{r} + \phi_\pi \pi^{(0)}) + (1-\rho)(\overline{r} + \phi_\pi \pi^{(1)}) \\ &= \overline{r} + \phi_\pi \{\rho \pi^{(0)} + (1-\rho)\pi^{(1)}\}. \end{aligned} \tag{A.71}$$

Other equations are equation (A.41)

$$i^{(k)} = -\frac{1}{\gamma/\sigma}\left(\beta\pi^{(k+2)} - (1+\beta+\gamma/\sigma)\pi^{(k+1)} + \pi^{(k)}\right) + \overline{r}, \tag{A.72}$$

for $k \geq 1$ and equation (A.43),

$$0 = A\pi^{(2)} + B\pi^{(1)} + C\pi^{(0)} + \underline{r}, \tag{A.73}$$

for $k = 0$.

These equations give the path of inflation and interest rates. For $k \geq 2$, equations (A.41) and (A.68) are simplified as

$$\begin{pmatrix} i^{(k)} - \overline{r} \\ \pi^{(k)} \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} (i^{(k-1)} - \overline{r}). \tag{A.74}$$

Matrix D satisfies

$$D_1 = \rho + (1-\rho)\phi_\pi D_2, \tag{A.75}$$

$$D_1 = -\frac{1}{\gamma/\sigma}\left(D_2 D_1^2 - (1+\beta+\gamma/\sigma)D_2 D_1 + D_2\right). \tag{A.76}$$

The other three variables of $\{i^{(1)}, \pi^{(1)}, \pi^{(0)}\}$ are calculated from the following three equations, (A.41), (A.43), and (A.71) respectively:

$$\begin{aligned} i^{(1)} \quad = \quad & -\frac{1}{\gamma/\sigma}\{\beta D_2 D_1 (i^{(1)} - \overline{r}) \\ & -(1+\beta+\gamma/\sigma)D_2(i^{(1)} - \overline{r}) + \pi^{(1)}\} + \overline{r}, \end{aligned} \tag{A.77}$$

$$0 = AD_2(i^{(1)} - \overline{r}) + B\pi^{(1)} + C\pi^{(0)} + \underline{r}, \tag{A.78}$$

$$i^{(1)} = \overline{r} + \phi_\pi\{\rho\pi^{(0)} + (1-\rho)\pi^{(1)}\}. \tag{A.79}$$

Table 1: Parameters for Simulations

Markov process		
p	0.5	Persistence of the adverse shock
$\overline{r}$	0.01	Real interest rate at n
$\underline{r}$	-0.0025	Real interest rate at z

Structured parameters		
β	0.99	Discount factor
γ	0.024	Elasticity of inflation to output
σ	1	Inverse of the elasticity of output to real interest rates

Policy rule parameters

ρ	0 or 0.8	Inertia
ϕ_π	1.5	Coefficient on inflation

Table 2: Welfare Loss

Commitment policy	0.0024
Time-invariant duration policy Two-periods duration policy	0.0026
One-period duration policy	0.0031
Inertial rule	0.0091
Discretionary policy	0.0140
Non-inertial rule	0.0140
Inertial rule 2	0.2748
Measured by permanent changes in inflation (annualized, percent)	

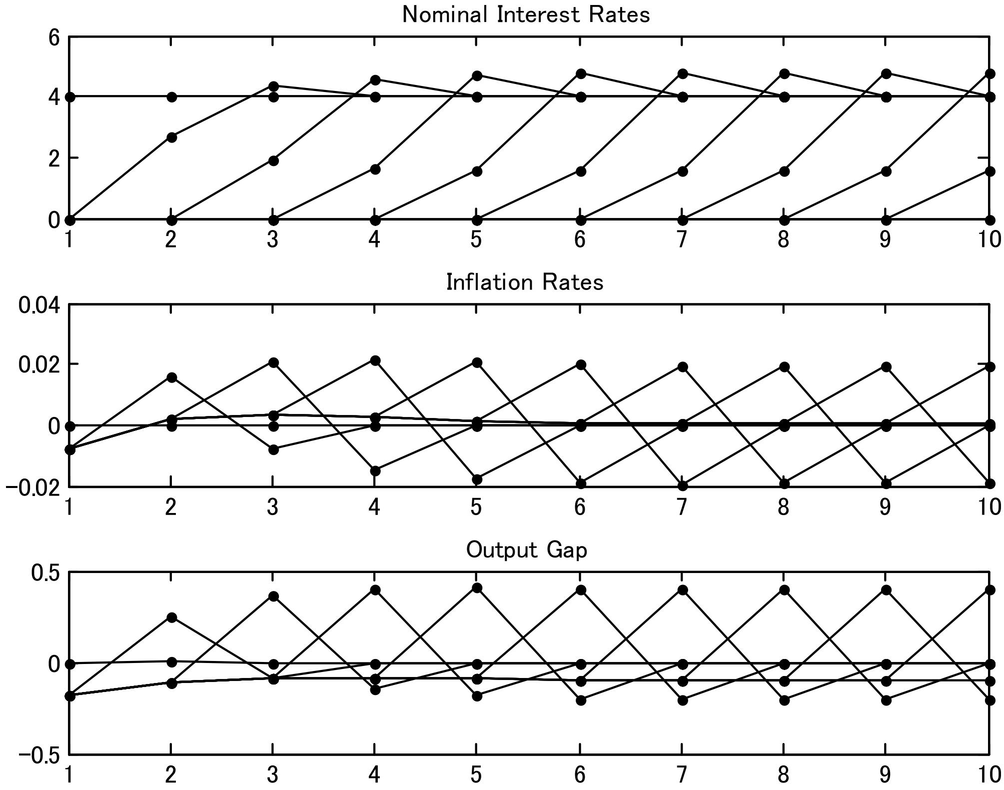

Figure 1: Commitment Policy

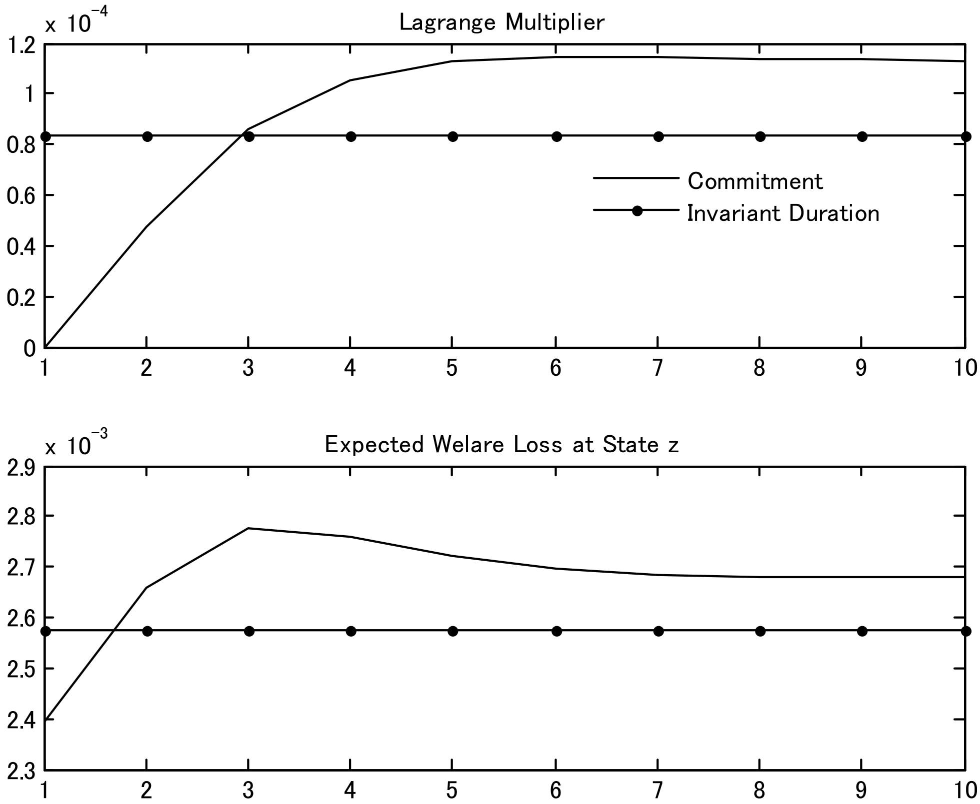

Figure 2: Time-Inconsistency of Commitment Policy and Time-Invariant Duration Policy

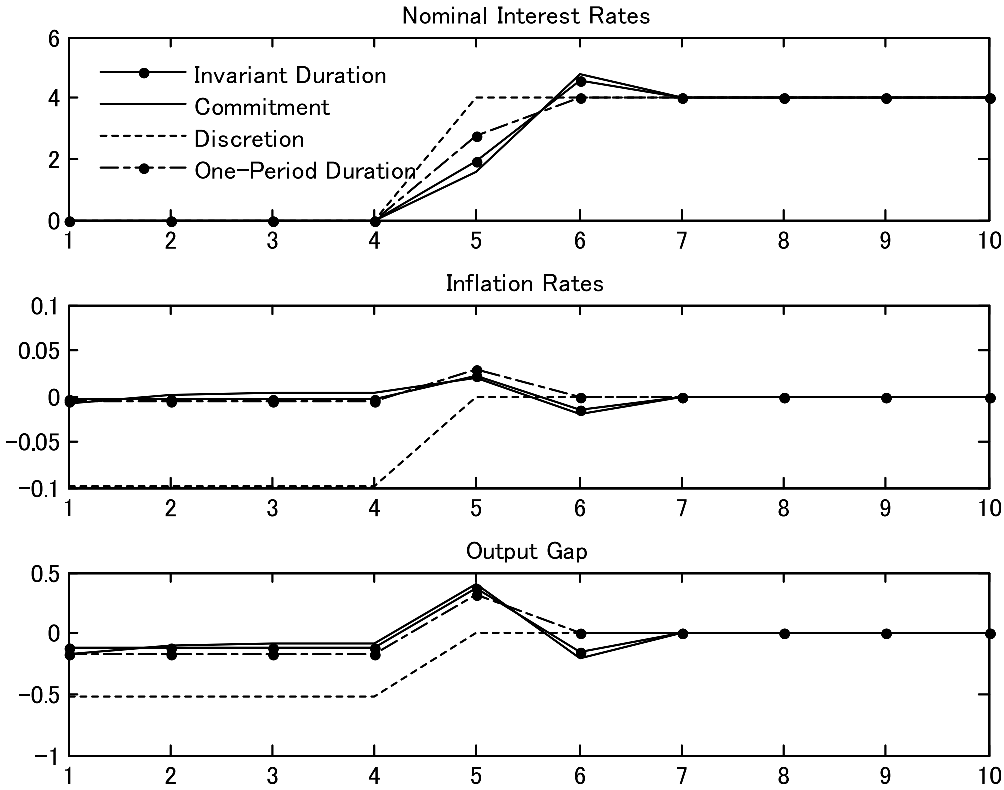

Figure 3: Optimal Policy (1)

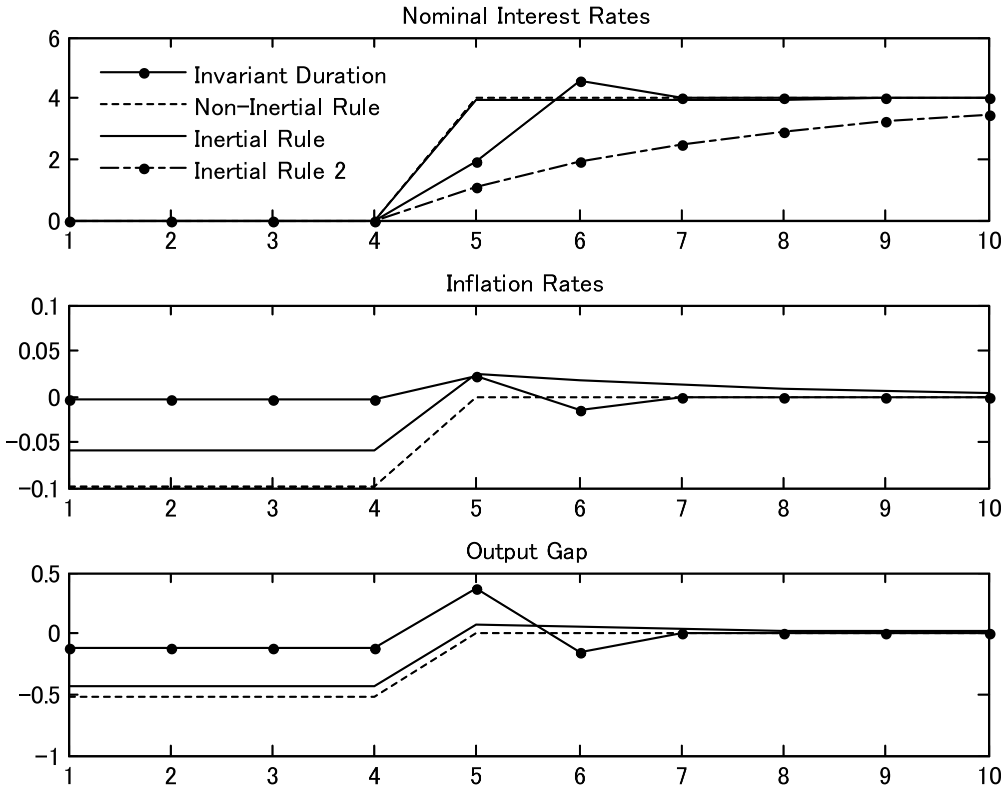

Figure 4: Optimal Policy (2)

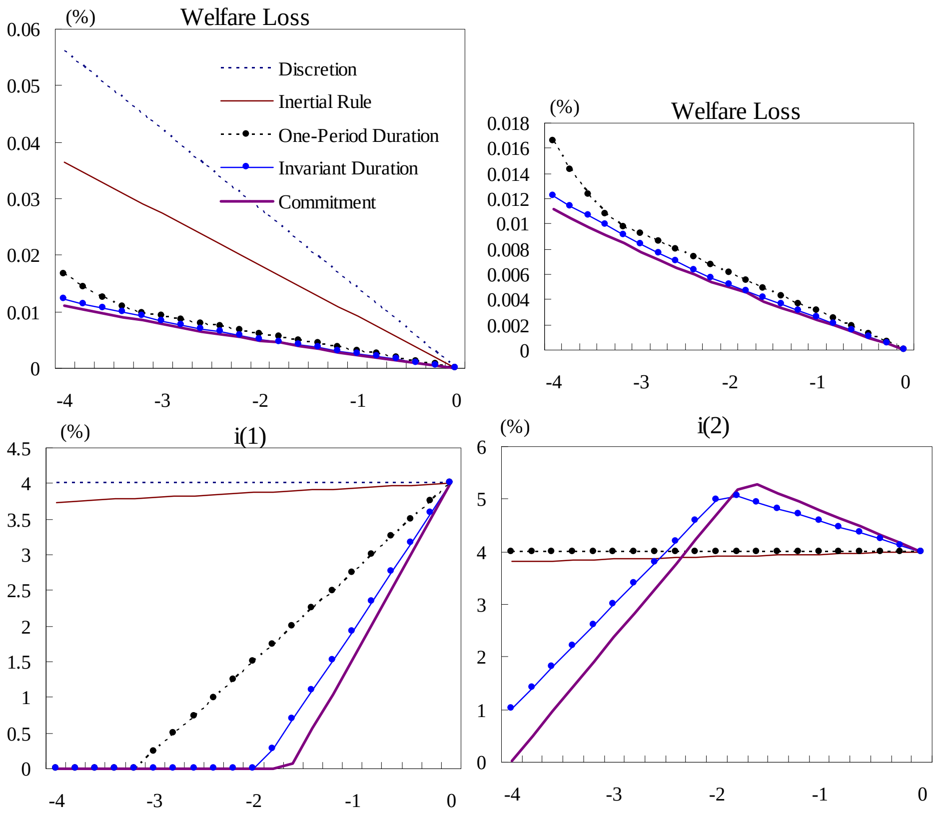

Figure 5: Sensitivity to $\underline{r}$ (Real Interest Rate at z)

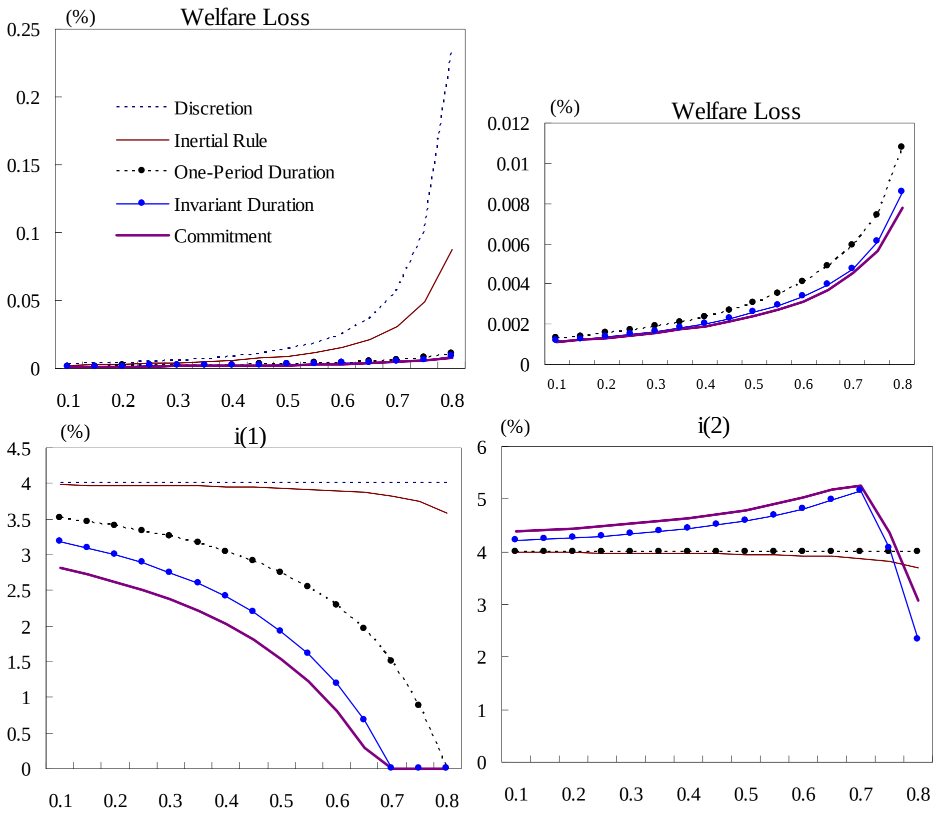

Figure 6: Sensitivity to p (Persistence of the Adverse Shock)

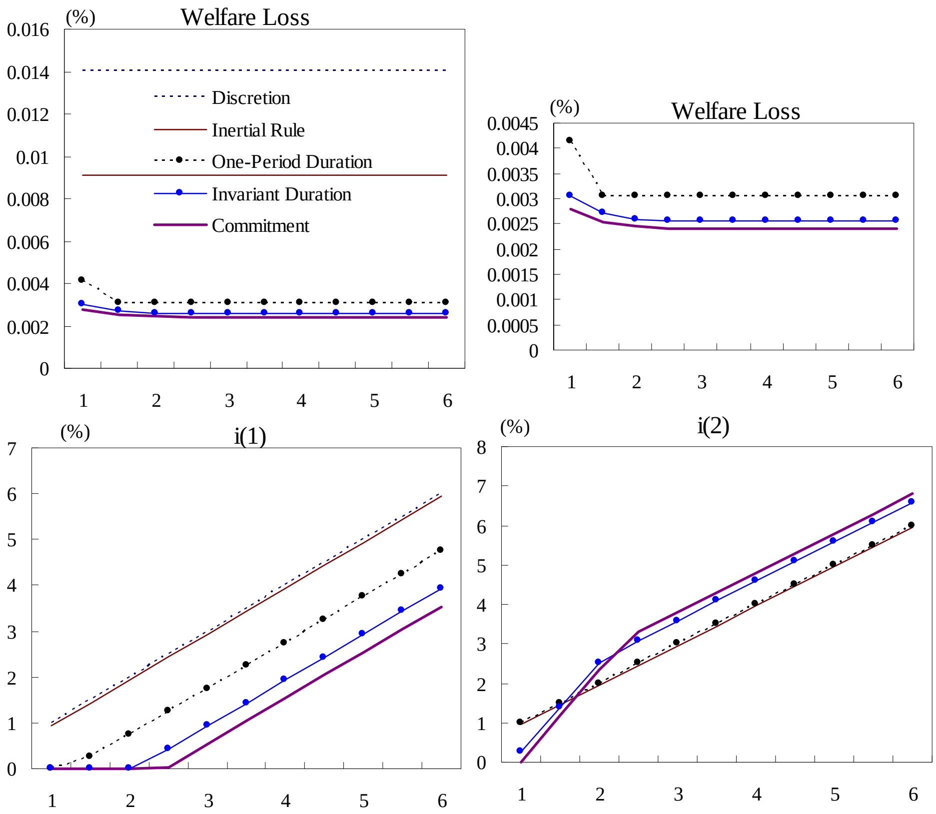

Figure 7: Sensitivity to $\bar{r}$ (Real Interest Rate at n)

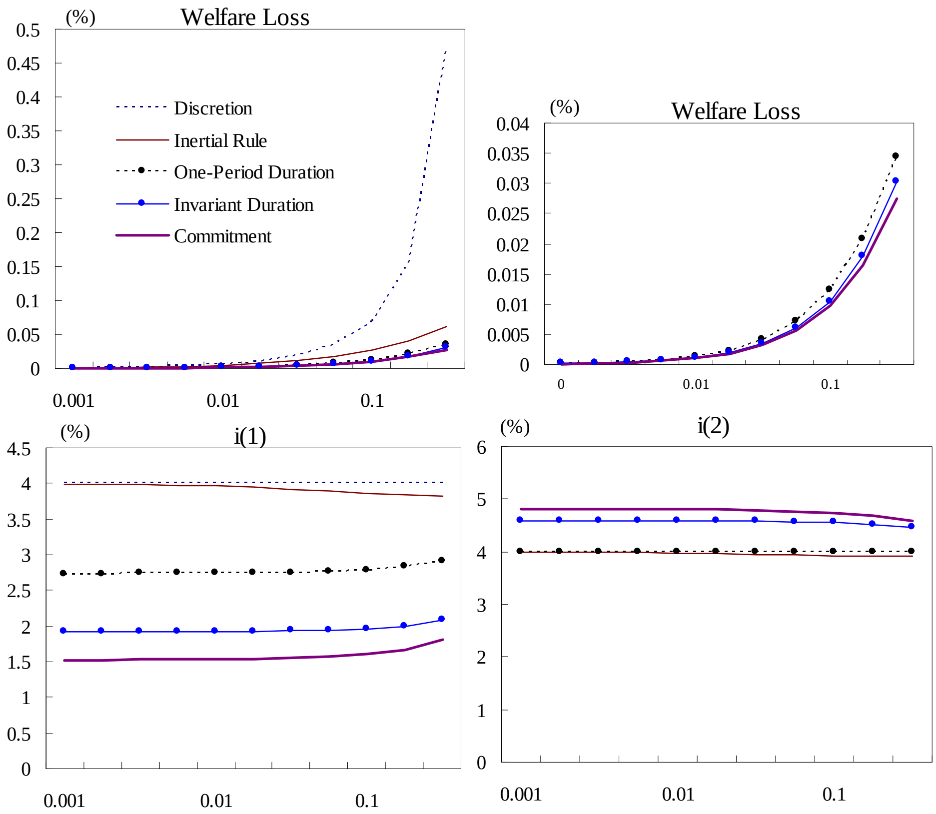

Figure 8: Sensitivity to γ/σ (Elasticity of Inflation to Real Interest Rate)